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Research Article

A Computation Method for a Projection of a Projective Variety

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Abstract: Consider a projective variety $W \subseteq \mathbb{P}^n$ and a point $p \in \mathbb{P}^n \backslash W$. The projection at point p onto $\mathbb{P}^{n-1}$ is represented by the mapping $\pi_p\colon \mathbb{P}^n \backslash \{p\} \to \mathbb{P}^{n-1}$, where $\pi_p(q)$ denotes the point of intersection between the line $\overline{pq}$ and $\mathbb{P}^{n-1}$, for $q \neq p$. In this article, we derive a generator set for the ideal $I(\pi_p(W))$ when W is defined as the set of zero points of certain homogeneous polynomials.

Projektif Variyetinin Bir İzdüşümü İçin Hesaplama Metodu

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Anahtar Kelimeler
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Öz: $\mathbb{P}^n$'de bir projektif variyeti W ve bu variyetinin üzerinde olmayan bir p noktası alalım. p noktasından $\mathbb{P}^{n-1}$' e bir izdüşüm $\pi_p\colon \mathbb{P}^n \backslash \{p\} \to \mathbb{P}^{n-1}$ fonksiyonu ile gösterilir burada $\pi_p(q)$, $\overline{pq}$ doğrusunun $q \neq p$ için $\mathbb{P}^{n-1}$'deki kestiği noktayı ifade etmektedir. Bu makalede, W belirli homojen polinomların sıfır noktalarının kümesi olduğunda, biz $I(\pi_p(W))$ ideali için üreteç kümesi bulacağız.

1. Introduction

Let $\mathbb{k}$ stand for an algebraically closed field. The projective space over $\mathbb{k}$, denoted as $\mathbb{P}^n_{\mathbb{k}}$ or simply $\mathbb{P}^n$, represents the collection of one-dimensional subspaces of $\mathbb{k}^{n+1}$

A point $p = [x_0, x_1, \ldots, x_n]$ in $\mathbb{P}^n$ corresponds to a line passing through the origin and the point $(x_0, x_1, \ldots, x_n)$ in $\mathbb{k}^{n+1}$. Therefore, a polynomial $H \in \mathbb{k}[x_0, x_1, \ldots, x_n]$ defines a function on $\mathbb{P}^n$ if and only if it is a homogeneous polynomial.

Given homogeneous polynomials $H_1, H_2, \ldots, H_s$ in $\mathbb{k}[x_0, x_1, \ldots, x_n]$, a projective variety defined by these polynomials is the set

$$V(H_1, H_2, \ldots, H_s) = \{p \in \mathbb{P}^n \mid H_i(p) = 0 \; where \; 1 \leq i \leq s\}. \tag{1}$$

For a given projective variety W, the ideal of this is defined as

$$I(W) = < h \in \Bbbk[x_0, x_1, \ldots, x_n] \mid \forall p \in W, h(p) = 0 >. \tag{2}$$

It is well-known that $I(W)$ is a homogeneous ideal, meaning it has a generating set containing homogeneous polynomials.

Consider a point p in $\mathbb{P}^n \setminus \mathbb{P}^{n-1}$. For another point $q \in \mathbb{P}^n$, the projective line along p and q is denoted as $\overline{pq}$. The projection map

$$\pi_p: \mathbb{P}^n \setminus \{p\} \to \mathbb{P}^{n-1} \tag{3}$$

is defined by

$$\pi_p(q) = \overline{pq} \cap \mathbb{P}^{n-1}. \tag{4}$$

Following a projective linear transformation, one can set $p = [0,0, \ldots, 1]$.

Assume $W = V(H_1, H_2, \ldots, H_s)$ is a projective variety, and let p be a point not lying on W. Then, $\pi_p(W)$ is termed a projection of V from p to $\mathbb{P}^{n-1}$. It is established that $\pi_p(W)$ is a projective variety in $\mathbb{P}^{n-1}$. (see Harris (1992) , Theorem 3.5).

In the paper, we try to find homogeneous polynomials $G_1, G_2, \ldots, G_t$ such that $I(\pi_p(W)) = < G_1, G_2, \ldots, G_t >$ from given $W = V(H_1, H_2, \ldots, H_s)$. To do this we use the elimination theory and Gröbner bases.

This problem is stated (Harris, 1992). The relation between projection and resultants is also given in (Harris, 1992). The resultant is related to elimination theory in (Cox et al., 1996).

In this paper, we present a constractive method for determining the ideal of $\pi_p(W)$ utilizing elimination theory and Gröbner basis techniques. The structure of the paper is outlined as follows. Section 2 compiles pertinent results on resultants. The primary findings are expounded upon in Section 3. Towards the conclusion of Section 3, there is an elucidation on the generalization of projection from a linear space and the process of identifying its ideal.

2. Materials and Methods

To reach our goal, we obtain a relation between a projection of projective variety and elimination theory via resultants. After that using the computation technique of the elimination ideal with the Gröbner basis, we find a method for to obtain the ideal of $\pi_p(W)$. We explain our methods by examples.

3. Resultant

Resultants are important in the elimination theory. The multipolynomial resultant can be used to eliminate variables from the system of equations and it is also a powerful to for finding solution of polynomial equations.

Definition 3.1. (Harris, 1992) Let two polynomials be H_1 and H_2 in $\Bbbk[x_0, x_1, \ldots, x_n]$, we can express them as

$H_1 = a_0 x_n^r + \cdots + a_r$, $a_0 \neq 0$ and $H_2 = b_0 x_n^s + \cdots + b_s$, $b_0 \neq 0$, where $a_i, b_i \in \Bbbk[x_0, x_1, \ldots, x_{n-1}]$. The resultant of H_1 and H_2 with respect to x_n is defined as

$$Res(H_1, H_2, x_n) = \begin{vmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \ddots & \vdots & b_1 & b_0 & \ddots & \vdots \\ \vdots & a_1 & \ddots & 0 & \vdots & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\ \vdots & \vdots & \ddots & a_1 & \vdots & \vdots & \ddots & b_1 \\ a_s & \vdots & \ddots & \vdots & b_t & \vdots & \ddots & \vdots \\ 0 & a_s & \ddots & \vdots & 0 & b_t & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_s & 0 & \cdots & 0 & b_t \end{vmatrix} \tag{5}$$

In (Harris, 1992), Harris asserts that $\pi_p(W) = V(J)$, where

$$J = < Res(H_1, H_2, x_n) \mid H_1, H_2 \in I(W) \text{ and } H_1, H_2 \text{ are homogeneous} > \tag{6}$$

However, due to the infinite nature of homogeneous polynomials in $I(W)$ this result does not provide a finite set of polynomials that define the projection.

Lemma 3.1. $Res(H_1, H_2, x_n) = A_1 H_1 + A_2 H_2$ for some $A_1, A_2 \in \Bbbk[x_0, x_1, \ldots, x_n]$.

Proof. If $Res(H_1, H_2, x_n) = 0$ then $A_1 = A_2 = 0$. Hence, we may assume $\boldsymbol{R}es(H_1, H_2, x_n) \neq 0$. To find $\overline{A_1}$ and $\overline{A_2}$ in $\Bbbk(x_0, x_1, \ldots, x_{n-1})[x_n]$ such that $\overline{A_1} H_1 + \overline{A_2} H_2 = 1$ where

$$\begin{aligned} H_1 &= a_0 x_n^l + \cdots + a_s \text{ and } H_2 = b_0 x_n^m + \cdots + a_t. \\ \overline{A_1} &= c_0 x_n^{t-1} + \cdots + c^{t-1} \text{ and } \overline{A_2} = d_0 x_n^{s-1} + \cdots + d^{s-1} \end{aligned} \tag{7}$$

and $c_0, c_1, \ldots, c_{t-1}, d_0, d_1, \ldots, d_{s-1}$ are unknowns in $\Bbbk[x_0, x_1, \ldots, x_{n-1}]$. By substituting these formulas into (7) and comparing the coefficients of powers of x_n, we obtain the following system:

$$\begin{aligned} a_0 c_0 + b_0 d_0 &= 0 \\ a_1 c_0 + a_0 c_1 + b_1 d_0 + b_0 d_1 &= 0 \\ \vdots &= \vdots \\ a_s c_{t-1} + b_t d_{s-1} &= 1 \end{aligned} \tag{8}$$

The coefficient matrix corresponds to the matrix in the definition of the resultant of H_1 and H_2. The condition $Res(H_1, H_2, x_n) \neq 0$ ensures that this linear system possesses a unique solution. Applying Cramer's rule,

$$c_0 = \frac{1}{Res(H_1, H_2, x_n)} \begin{vmatrix} 0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ 0 & a_0 & \ddots & \vdots & b_1 & b_0 & \ddots & \vdots \\ \vdots & a_1 & \ddots & 0 & \vdots & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\ \vdots & \vdots & \ddots & a_1 & \vdots & \vdots & \ddots & b_1 \\ a_s & \vdots & \ddots & \vdots & b_t & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & \vdots & 0 & b_t & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 0 & a_s & 0 & \cdots & 0 & b_t \end{vmatrix} \tag{9}$$

Similar formulas can be derived for the other c_i's and d_i's. Given that $\overline{A_1} = c_0 x_n^{t-1} + \cdots + c^{t-1}$, it is possible to factor out the common denominator $Res(H_1, H_2, x_n)$ and express $\overline{A_1}$ in the form $\overline{A_1} = \frac{A_1}{Res(H_1,H_2,x_n)}$ where A_1 in $\mathbb{k}[x_0, x_1, \ldots, x_{n-1}]$. Likewise, we have $\overline{A_2} = \frac{A_2}{Res(H_1,H_2,x_n)}$ where A_1 in $\mathbb{k}[, x_1, \ldots, x_{n-1}]$. As $\overline{A_1}$ and $\overline{A_2}$ satisfy $\overline{A_1}H_1 + \overline{A_2}H_2 = 1$, it follows that $A_1H_1 + A_2H_2 = Res(H_1, H_2, x_n)$. ■

Lemma 3.2. For any $q[x_0, x_1, \ldots, x_{n-1}] \in \mathbb{P}^{n-1}$, $Res(H_1, H_2, x_n)(q) = 0$ if and only if $H_1(q, x_n)$ and $H_2(q, x_n)$ have a common root as polynomials in x_n or leading coefficients of both H_1 and H_2 vanish at q.

Proof. If $a_0 \neq 0$ and $b_0 \neq 0$, then

$$Res(H_1, H_2, x_n)(q) = \begin{vmatrix} a_0(q) & 0 & \cdots & 0 & b_0(q) & 0 & \cdots & 0 \\ a_1(q) & a_0(q) & \ddots & \vdots & b_1(q) & b_0(q) & \ddots & \vdots \\ \vdots & a_1(q) & \ddots & 0 & \vdots & b_1(q) & \ddots & 0 \\ \vdots & \vdots & \ddots & a_0(q) & \vdots & \vdots & \ddots & b_0(q) \\ \vdots & \vdots & \ddots & a_1(q) & \vdots & \vdots & \ddots & b_1(q) \\ a_s(q) & \vdots & \ddots & \vdots & b_t(q) & \vdots & \ddots & \vdots \\ 0 & a_s(q) & \ddots & \vdots & 0 & b_t(q) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_s(q) & 0 & \cdots & 0 & b_t(q) \end{vmatrix} \quad (10)$$

The determinant is equal to $Res(H_1, H_2, x_n)$. If $Res(H_1, H_2, x_n) = 0$, it implies that $H_1(q, x_n)$ and $H_2(q, x_n)$ share a common root as polynomials over $\mathbb{k}$. Consequently, $Res(H_1, H_2, x_n)(q) = 0$ if and only if $H_1(q, x_n)$ and $H_2(q, x_n)$have a common zero.

In cases where both $a_0(q) = 0$ and $b_0(q) = 0$ the condition $Res(H_1, H_2, x_n)(q) = 0$ is evident.

Now, let's consider the scenario where $a_0(q) \neq 0$ and $b_0(q) = 0$. If $b_1(q) \neq 0$, then $H_1(q, x_n) = b_1(q)x_n^{t-1} + \cdots + b_t(q)$ and

$$Res(H_1, H_2, x_n)(q) = \begin{vmatrix} a_0(q) & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_1(q) & a_0(q) & \ddots & \vdots & b_1(q) & 0 & \ddots & \vdots \\ \vdots & a_1(q) & \ddots & 0 & \vdots & b_1(q) & \ddots & 0 \\ \vdots & \vdots & \ddots & a_0(q) & \vdots & \vdots & \ddots & b_0(q) \\ \vdots & \vdots & \ddots & a_1(q) & \vdots & \vdots & \ddots & b_1(q) \\ a_s(q) & \vdots & \ddots & \vdots & b_t(q) & \vdots & \ddots & \vdots \\ 0 & a_s(q) & \ddots & \vdots & 0 & b_t(q) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_s(q) & 0 & \cdots & 0 & b_t(q) \end{vmatrix} \quad (11)$$

The determinant is incorrectly sized to serve as the resultant of $H_1(q, x_n)$ and $H_2(q, x_n)$. Expanding by minors along the first-row yields

$$Res(H_1, H_2, x_n)(q) = a_0(q)Res(H_1(q, x_n), H_2(q, x_n), x_n). \quad (12)$$

In a more general setting, it is reasonable to assume that $H_2(q, x_n)$ has a degree of $m - p$ where $p \geq 1$. In such a scenario, expanding by minors along the first p rows results in

$$Res(H_1, H_2, x_n)(q)^p = a_0(q)Res(H_1(q, x_n), H_2(q, x_n), x_n). \quad (13)$$

Once again, $Res(H_1, H_2, x_n) = 0$ if and only if $H_1(q, x_n)$ and $H_2(q, x_n)$ share a common zero as polynomials in x_n.

The case where $a_0(q) = 0$ and $b_0(q) \neq 0$ can be similarly addressed. ■

4. Theoretical Result

Lemma 4.1. If $q \in \mathbb{P}^n - \{p\}$, the line $l = \overline{pq}$ intersects V if and only if every pair H_1 and H_2 of homogeneous polynomials in $I(V)$ has a common zero on l.

Proof. If the line l and the variety V intersect at a point x, it is evident that x serves as a common zero for every pair of homogeneous polynomials in $I(V)$. Conversely, if l does not intersect V, then there exists a polynomial $H_1 \in I(V)$ that vanishes at a finite number of points on l, denoted as $x_1, x_2, \dots, x_m$. Since $x_i \notin V$, there exists a polynomial $H_2 \in I(V)$ that does not vanish at x_i for $i = 1,2,\dots,m$. Consequently, H_1 and H_2 have no common zero on l. ■

Lemma 4.2. For any $q \in \mathbb{P}^{n-1}$, every pair H_1, H_2 of homogeneous polynomials in $I(V)$ has a common zero of the type $[\alpha q, \beta]$ on $l = \overline{pq}$ if and only if the homogeneous polynomial $Res(H_1, H_2, x_n)$ vanishes at q for every pair of homogeneous polynomials.

Proof. Given $q = [x_0, \dots, x_{n-1}] \in \mathbb{P}^{n-1}$ the line $l = \overline{pq}$ is defined as $[\alpha x_0, \dots, \alpha x_{n-1}, \beta] \in \mathbb{P}^l$. If every pair of homogeneous polynomials $H_1, H_2 \in I(V)$ has a common point on l, according to Lemma 2.1., l intersects V Let $[\bar{\alpha} x_0, \dots, \bar{\alpha} x_{n-1}, \beta] \in V \cap I$. Since p is not on V, $\bar{\alpha} \neq 0$. This implies that $[\bar{\alpha} x_0, \dots, \bar{\alpha} x_{n-1}, \beta]$ is a common zero of H_1 and H_2 on l. Then $\frac{\bar{\beta}}{\bar{\alpha}}$ is a common zero of $H_1(q, x_n)$ and $H_2(q, x_n)$ as polynomials in x_n. Therefore, $Res(H_1(q, x_n), H_2(q, x_n), x_n) = 0$, implying $Res(H_1, H_2, x_n)(q) = 0$.

Conversely, assume $Res(H_1, H_2, x_n) = 0$. Let H_1 be a homogeneous polynomial of degree S and H_2 a homogeneous polynomial of degree T. If both leading terms vanish at q, then $s < S$ and $t < T$ and $p = [0,\dots,0,1] \in l$ is a common zero for H_1 and H_2. Otherwise, $H_1(q, x_n)$ and $H_2(q, x_n)$ have a common zero as polynomials in x_n, denoted as $\bar{\beta}$; then $[q, \bar{\beta}]$ is the common zero of H_1 and H_2. on l. ■

Combining these two lemmas, the image $\bar{V}$ of the projection $\pi: V \to \mathbb{P}^{n-1}$ is the common zero locus of the polynomials $Res(H_1, H_2)$ where H_1 and H_2 range over all pairs of homogeneous elements of $I(V)$. In other words, $\bar{V} = V(J)$, where $J = < Res(H_1, H_2) | H_1, H_2 \in I(V) >$ is a homogeneous ideal. However, it is essential to note that this yields a set of generators with infinitely many elements, and it is not immediately evident why $J = I(\bar{V})$.

In cases where $I(V) = < f_1, \dots, f_m >$, the collection of $Res(f_i, f_j, x_n)$ for $1 \leq i < j \leq m$ does not necessarily form a set of generators for $I(\bar{V})$. In other words, given a basis for $I(V)$, obtaining a basis for $I(\bar{V})$ cannot be achieved using this method in finitely many steps. The following theorem, however, implicitly provides an algorithm for such a purpose:

Theorem 4.1. $I(\bar{V}) = J = I(V) \cap \Bbbk[x_0, x_1, \dots, x_{n-1}]$.

Proof. Let $Res(H_1, H_2, x_n) \in \{Res(H_1, H_2, x_n) : H_1, H_2 \in I(V)\}$ for some $H_1, H_2 \in I(V)$. According to Lemma 2.1.,

$$Res(H_1, H_2, x_n) = A_1 H_1 + A_2 H_2 \text{ for some } A_1, A_2 \in \Bbbk[x_0, x_1, \dots, x_n]. \tag{14}$$

Therefore, $Res(H_1, H_2, x_n) \in I$. As $Res(H_1, H_2, x_n) \in \Bbbk[x_0, x_1, \dots, x_{n-1}]$ by the definition of the resultant, it follows that

$$Res(H_1, H_2, x_n) \in I(V) \cap \Bbbk[x_0, x_1, \dots, x_n]. \tag{15}$$

This implies that $J \subset I(V) \cap \mathbb{k}[x_0, x_1, \ldots, x_{n-1}]$.

Conversely, for any $H_1 \in I(V) \cap \mathbb{k}[x_0, x_1, \ldots, x_{n-1}]$ both H_1 and $x_n H_1$ belong to I. Then,

$$H_1 = Res(H_1, x_n H_1, x_n) \in \{Res(H_1, H_2, x_n), H_1, H_2 \in I\}. \tag{16}$$

Therefore, $I(V) \cap \mathbb{k}[x_0, x_1, \ldots, x_{n-1}] \subset J$. Since $I(V)$ is radical, $I(V) \cap \mathbb{k}[x_0, x_1, \ldots, x_{n-1}]$ is also radical. Therefore,

$$I(V(J)) = I(\bar{V}) = I(V) \cap \mathbb{k}[x_0, x_1, \ldots, x_{n-1}] = J. \tag{17}$$

■

5. Gröbner Bases and Elimination Theory

Definition 5.1. (Cox et al., 1996), A well ordering $<$ on the set of polynomials H is termed a monomial ordering if it satisfies the condition that for any polynomials h_1 and h_2 in H:

$$h_1 > h_2 \Rightarrow x_i h_1 x_j > x_i h_2 x_j \tag{18}$$

This implies that the ordering is consistent with both left and right multiplications by x_i and x_j on polynomials in H.

Definition 5.2 (Cox et al., 1996), Let $I \subset \mathbb{k}[x_0, x_1, \ldots, x_{n-1}]$ be an ideal and consider a fixed monomial order. A finite subset $\{h_1, h_2, \ldots, h_s\} \subset I$, where $< LT(h_1), LT(h_2), \ldots, LT(h_s) > = < LT(I) >$ is referred to as a Gröbner basis.

Definition 5.3. (Cox et al., 1996), Given $I = < f_1, f_2, \ldots, f_s > \subset \mathbb{k}[x_1, x_2 \ldots, x_n]$ the t-th elimination ideal I_t is the ideal of $\mathbb{k}[x_{t+1}, x_1, \ldots, x_n]$ defined by

$$I_t = I \cap \mathbb{k}[x_{t+1}, x_1, \ldots, x_n]. \tag{17}$$

Theorem 5.1. (The Elimination Theorem) (Cox et al., 1996) Let $I \subset \mathbb{k}[\, x_1, x_2 \ldots, x_n]$ be an ideal and let G be a Gröbner basis of I with respect to lex order where $x_1 > x_2 > \cdots > x_n$. Then for every $0 \leq t \leq n$, the set

$$G^t = G \cap \mathbb{k}[x_{t+1}, x_1, \ldots, x_n] \tag{18}$$

is a Gröbner basis of t-th elimination ideal I_t.

Theorem 3.1. provides a method using Gröbner basis techniques to derive a basis for $I(\bar{V})$ from a given basis $I(V)$.

Corollary 5.1. If $G = \{g_1, g_2, \ldots, g_t\}$ is a Gröbner basis for $I(V)$ with respect to an order $<$ such that $x_n > x_i$ for any $i = 1,2, \ldots, n-1$, then $G_1 = G \cap \mathbb{k}[x_0, x_1, \ldots, x_{n-1}]$ forms a Gröbner basis for $I(\bar{V})$.

Proof. After relabeling if necessary, one can assume that $G_1 = \{g_1, g_2, \ldots, g_r\}$. It is evident that $G_1 \subset I(V) \cap \mathbb{k}[x_0, x_1, \ldots, x_{n-1}]$. Consider an arbitrary polynomial $f \in I(V) \cap \mathbb{k}[x_0, x_1, \ldots, x_{n-1}]$. Since G is a Gröbner basis for $I(V)$ and $f \in I(V)$ the remainder on division by G is zero. Furthermore, due to the monomial order with $x_n > x_i$ for any $i = 1,2, \ldots, n-1$, the leading terms of $g_{r+1}, g_{r+2}, \ldots, g_m$ involve x_n and are greater than every monomial in f. Thus, when applying the division algorithm, $g_{r+1}, g_{r+2}, \ldots, g_m$ will not appear. Consequently, the division of f by G results in an equation of the form:

$$f = \sum_{i=1}^{r} h_i g_i + g_{r+1}0 + g_{r+2}0 + \cdots + g_l 0 \qquad (19)$$

which implies $f \in < g_1, g_2, \ldots, g_r >$. This establishes that G_1 is a basis for $I(V) \cap \Bbbk[x_0, x_1, \ldots, x_{n-1}]$. In fact, since the division of any $f \in I(V) \cap \Bbbk[x_0, x_1, \ldots, x_{n-1}]$ by G_1 leaves a zero remainder, G_1 serves as a Gröbner basis for $I(\bar{V})$. ■

Example 5.1. The twisted cubic in $\mathbb{P}^3(\mathbb{C})$ is defined by $V(I)$, where $I = \{I_1, I_2, I_3\}$ and $I_1 = x_0x_2 - x_1^2$, $I_2 = x_0x_3 - x_1x_2$ and $I_3 = x_1x_3 - x_2^2$. Let $p = [0,1,0,0]$. Using lex order with $x_2 > x_0 > x_1 > x_3$, a Gröbner basis for I is $\{x_1^3 - x_0^2x_3, x_1x_2 - x_0x_3, x_1^2 - x_0x_2, x_2^2 - x_1x_3\}$. Therefore, $\pi(V) = V(x_1^3 - x_0^2x_3)$.

Generalizing the result to projection from a linear space is straightforward.

Corollary 5.2. Let L be a linear subspace of $\mathbb{P}^n$, and let L' be the complementary subspace of L. Up to a projective linear transformation, assume $L = V(x_0, x_1, \ldots, x_i)$ and $L' = V(x_{i+1}, x_{i+2}, \ldots, x_n)$. Let V be a variety in $\mathbb{P}^n$ such that $V \cap L = \emptyset$. Let π_L be the projection from L to L', and $\bar{V} = \pi_L$. Then $G \cap \Bbbk[x_0, x_1, \ldots, x_i]$ is a Gröbner basis of $I(\bar{V})$ under an order such that $x_{i_0} > x_{j_0}$ for $i_0 = i+1, \ldots, n$ and $j_0 = 0, \ldots, i$.

Theorem 5.2. (Cox et al., 1996) Let $I \subset \Bbbk[x_0, x_1, \ldots, x_n]$ be an ideal, and let G be a Gröbner basis of I with respect to the lexicographic order where $x_n > x_{n-1} > \cdots > x_1 > x_0$. Then, the polynomial in G that does not contain the variable x_n generates $I \cap \Bbbk[x_0, x_1, \ldots, x_{n-1}]$.

Another application of Gröbner bases is to find a generating set for the radical of an ideal from a given generating set of the ideal (see Decker et al. (1999) and Kemper (2002))

Now, we can describe the method for finding the generating set for the ideal of the projection of a projective variety. Given $W = V(F_1, \ldots, F_s)$ first find a generating set $\{H_1, H_2, \ldots, H_t\}$ for the radical ideal $\sqrt{< F_1, \ldots, F_s >}$. Then, using lexicographic order with $x_n > x_{n-1} > \cdots > x_1 > x_0$, find a Gröbner basis G for the ideal $< H_1, H_2, \ldots, H_t >$.

Since $I(\pi_P(W)) = \sqrt{< F_1, \ldots, F_s >} \cap \Bbbk[x_0, x_1, \ldots, x_n]$ the polynomials in G not containing the variable x_n form a generating set for $I(\pi_P(W))$.

Example 5.2. Consider $W = V(x_0x_3 - x_1^2, x_0x_2 - x_1x_3, x_1x_2 - x_3^2)$ in $\mathbb{P}^3(\mathbb{C})$. It is well-known that $I = < x_0x_3 - x_1^2, x_0x_2 - x_1x_3, x_1x_2 - x_3^2 >$ is a radical ideal. A Gröbner basis for I with respect to the lexicographic order where $x_3 > x_2 > x_1 > x_0$, is $G = \{x_0^2x_2 - x_1^3, x_0x_3 - x_1^2, x_1x_3 - x_0x_2, x_3^2 - x_1x_2\}$. Since $G \cap \mathbb{C}[x_0, x_1, x_2] = \{x_0^2x_2 - x_1^3\}$, $\pi_P(W) = V(x_0^2x_2 - x_1^3)$ where $p = [0,0,0,1]$.

6. Discussion and Conclusion

In the literature, the projection is related to resultant theory (see Harris (1992)). However, the computing the resultant can be very complicated. In this article, using relation between resultant and elimination theory given in (Cox et al., 1996), we give a method finding projection using Gröbner bases. Since Gröbner basis computation is adopted in almost every computer algebra system, the projection can be easily computed using our method. In our examples, we use lexicographic orders.

In some cases, Gröbner bases with respect to lexicographic order can be huge. In this case, other elimination orders can be used. The detail of elimination orders can be form in (Greuel & Pfister, 2008).

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