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Research Article

Skew generalized von Neumann-Jordan constant in Banach spaces

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Abstract

We introduce a new geometric constant $C_{NJ}^{p}(\zeta,\eta,X)$ in Banach spaces, which is called the skew generalized von Neumann-Jordan constant. First, the upper and lower bounds of the new constant are given for any Banach space. Then we calculate the constant values for some particular spaces. On this basis, we discuss the relation between the constant $C_{NJ}^{p}(\zeta,\eta,X)$ and the convexity modules $\delta_X(\varepsilon)$, the James constant $J(X)$. Finally, some sufficient conditions for the uniform normal structure associated with the constant $C_{NJ}^{p}(\zeta,\eta,X)$ are established.

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1. Introduction

Throughout this article, let X be a real Banach space. B_X, S_X, and $ex(B_X)$ are denoted as the unit ball, the unit sphere of X, and the set of extreme points of B_X, respectively.

Geometric constants play an important role in the theory of Banach spaces and have been fully developed. For more papers on geometric constants, refer to ([2–4], [12, 13]). By using geometric constants as a tool to study the geometric properties of Banach spaces in a more detailed way, we can better quantitatively analyze and characterize the geometric properties. By determining the values of the constants in various abstract Banach spaces, we can get a series of properties in the corresponding spaces, and the values of the corresponding geometric constants in some concrete spaces also help us to better study the geometric features of Banach spaces. In recent years, many scholars have generalized some classical constants such as the von Neumann-Jordan constant and the James constant. Now, let's review these two constants.

Let X be a Banach space, the von-Neumann constant [19] is defined as

$$C_{NJ}(X)=\sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2\|x\|^2+2\|y\|^2}: x,y\in X,(x,y)\neq(0,0)\right\},$$

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and the James constant $J(X)$ [7] is defined as

$$J(X) = \sup\left\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\right\}.$$

Cui, Huang, Hudzik and Kaczmarek [5] introduced a new geometric constant $C_{NJ}^{(p)}(X)$ called the generalized von Neumann-Jordan constant, defined by

$$C_{NJ}^{(p)}(X) = \sup\left\{\frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}\left(\|x\|^p + \|y\|^p\right)} : x, y \in X, (x, y) \neq (0, 0)\right\},$$

where $1 \leq p < \infty$. These scholars have shown that $1 \leq C_{NJ}^{(p)}(X) \leq 2$ for any Banach space X and for any $1 \leq p < \infty$. Meanwhile, the following inequality holds

$$J(X) \leq 2^{\frac{p-1}{p}} \sqrt[p]{C_{NJ}^{(p)}(X)},$$

where $1 < p < \infty$.

Zuo and Cui [18] introduced the function $J_{X,p}(t)$, defined by

$$J_{X,p}(t) = \sup\left\{\left(\frac{\|x+ty\|^p + \|x-ty\|^p}{2}\right)^{\frac{1}{p}} : x, y \in S_X\right\},$$

on the interval $[0, \infty)$, where $1 \leq p < \infty$. It considers a generalization of $J(X)$ in the power of p. By the convexity of the function $f(x) = x^p$, the following inequality holds:

$$\frac{\|x+ty\|^p + \|x-ty\|^p}{2} \geq \left(\frac{\|x+ty\| + \|x-ty\|}{2}\right)^p.$$

Further, Yang et al. [17] generalized the constant $C_{NJ}(X)$ in the unit sphere:

$$\tilde{C}_{NJ}^{(p)}(X) = \sup\left\{\frac{\|x+y\|_X^p + \|x-y\|_X^p}{2^p} : x, y \in X, \|x\|_X = \|y\|_X = 1\right\}.$$

They discussed the relationship between $\tilde{C}_{NJ}^{(p)}(X)$ and $C_{NJ}^{(p)}(X)$, and the value of the constant $\tilde{C}_{NJ}^{(p)}(X)$ is estimated.

In new research, Liu et al. [9] introduced a constant with a skew relationship, defined by for $\zeta, \eta > 0$

$$L_{YJ}(\zeta, \eta, X) = \sup\left\{\frac{\|\zeta x + \eta y\|^2 + \|\eta x - \zeta y\|^2}{(\zeta^2 + \eta^2)(\|x\|^2 + \|y\|^2)} : x, y \in X, (x, y) \neq (0, 0)\right\}.$$

Another similar constant

$$L'_{YJ}(\zeta, \eta, X) = \sup\left\{\frac{\|\zeta x + \eta y\|^2 + \|\eta x - \zeta y\|^2}{2(\zeta^2 + \eta^2)} : x, y \in S_X\right\}$$

was also introduced by Liu et al. [10].

Through continuous research on the above constants, we will propose a new constant $C_{NJ}^{p}(\zeta, \eta, X)$.

2. The constant $C_{NJ}^{p}(\zeta, \eta, X)$

We introduce a new constant based on the constant $L_{YJ}(\zeta, \eta, X)$. From now on, we will consider only Banach spaces of dimension at least 2. We begin by introducing the following key definition: for $\zeta, \eta > 0$

$$\begin{aligned} C_{NJ}^{p}(\zeta, \eta, X) &= \sup\left\{\frac{\|\zeta x + \eta y\|^p + \|\eta x - \zeta y\|^p}{2^{p-2}(\zeta^p + \eta^p)(\|x\|^p + \|y\|^p)} : x, y \in X, (x, y) \neq (0, 0)\right\} \\ &= \sup\left\{\frac{\|\zeta x + \eta t y\|^p + \|\eta x - \zeta t y\|^p}{2^{p-2}[(\zeta^p + \eta^p t^p) + (\eta^p + \zeta^p t^p)]} : x, y \in S_X, 0 \leq t \leq 1\right\}, \end{aligned}$$

where $1 \leq p < \infty$.

Remark 2.1. Notice that if $\zeta = \eta$, then $C^p_{NJ}(\zeta, \eta, X) = C^{(p)}_{NJ}(X)$.

Next, we will give a lemma to compute the lower and upper bounds of the constant $C^p_{NJ}(\zeta, \eta, X)$.

Lemma 2.2. *Let $x, y > 0, 1 \leq p < \infty$, then $(x + y)^p \leq 2^{p-1}(x^p + y^p)$.*

Proposition 2.3. *Let X be a Banach space, then*

$$\frac{1}{2^{p-2}} \leq C^p_{NJ}(\zeta, \eta, X) \leq 2.$$

Proof. According to convexity of function $f(x) = \|\cdot\|^p$, we have

$$\begin{aligned}\frac{\|\zeta x + \eta y\|^p}{(\zeta+\eta)^p} + \frac{\|\eta x - \zeta y\|^p}{(\zeta+\eta)^p} &= \left\|\frac{\zeta}{\zeta+\eta}x + \frac{\eta}{\zeta+\eta}y\right\|^p + \left\|\frac{\eta}{\zeta+\eta}x + \frac{\zeta}{\zeta+\eta}(-y)\right\|^p \\ &\leq \frac{\zeta}{\zeta+\eta}\|x\|^p + \frac{\eta}{\zeta+\eta}\|y\|^p + \frac{\eta}{\zeta+\eta}\|x\|^p + \frac{\zeta}{\zeta+\eta}\|-y\|^p \\ &= \|x\|^p + \|y\|^p.\end{aligned}$$

This means that

$$\frac{\|\zeta x + \eta y\|^p + \|\eta x - \zeta y\|^p}{2^{p-2}(\zeta^p + \eta^p)(\|x\|^p + \|y\|^p)} \leq \frac{(\zeta+\eta)^p}{2^{p-2}\,(\zeta^p + \eta^p)}.$$

Combined with Lemma 2.2, we have $\frac{(\zeta+\eta)^p}{2^{p-2}(\zeta^p+\eta^p)} \leq 2$, which implies that $C^p_{NJ}(\zeta, \eta, X) \leq 2$. On the other hand, we assume that $x \neq 0, y = 0$, then

$$\frac{\|\zeta x + \eta y\|^p + \|\eta x - \zeta y\|^p}{2^{p-2}(\zeta^p + \eta^p)(\|x\|^p + \|y\|^p)} = \frac{1}{2^{p-2}}.$$

Hence

$$C^p_{NJ}(\zeta, \eta, X) \geq \frac{1}{2^{p-2}}.$$

□

Remark 2.4. If $p = 1$, then $2 \leq C_{NJ}(\zeta, \eta, X) \leq 2$, we can get $C_{NJ}(\zeta, \eta, X) = 2$.

From the above Remark 2.4, we find that when $p = 1$, the constant $C^p_{NJ}(\zeta, \eta, X) = 2$. Therefore, we will only consider the case where $p > 1$. Then, to investigate the values of the constant $C^p_{NJ}(\zeta, \eta, X)$ in some specific Banach spaces, the following two examples are presented.

Example 2.5. Let X be $\mathbb{R}^2$ endowed with the l_∞ norm defined by

$$\|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}.$$

Let $x = (1, 1), y = (1, -1)$. Thus, we obtain

$$\|x + y\| = 2, \ \|x - y\| = 2.$$

Let $\zeta = \eta = 1$, then $C^p_{NJ}(1, 1, X) = 2$.

Example 2.6. Let $p > 1$, $\zeta \geq \eta \geq \zeta t$, $(\zeta - \eta)(\eta + \zeta)^{p-1} - \zeta^p < 0$, X be the space $\mathbb{R}^2$ with the norm defined by

$$\|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_1, & x_1 x_2 \leq 0, \\ \|(x_1, x_2)\|_\infty, & x_1 x_2 \geq 0. \end{cases}$$

Then

$$C^p_{NJ}(\zeta, \eta, X) = \frac{\zeta^p + (\eta + \zeta t_0)^p}{2^{p-2}(\zeta^p + \eta^p t_0^p + \eta^p + \zeta^p t_0^p)},$$

where $t_0 \in (0, 1)$ is the only solution of the equation

$$1 - \frac{\eta}{\zeta} t^{p-1} = \left(\frac{\zeta t}{\eta + \zeta t}\right)^{p-1}.$$

Proof. Thanks to Minkowski inequality, for any $x, y \in S_X$, we can express this in terms of extreme points in the following form

$$x = \alpha x_1 + (1-\alpha)x_2 \text{ and } y = \beta y_1 + (1-\beta)y_2,$$

where x_1, x_2, y_1 and y_2 are extreme points of the corresponding line segment, $\alpha, \beta \in [0,1]$. We have

$$\begin{aligned}
&\|\zeta x + \eta ty\|^p + \|\eta x - \zeta ty\|^p \\
&= \|\alpha(\zeta x_1 + \eta ty) + (1-\alpha)(\zeta x_2 + \eta ty)\|^p + \|\alpha(\eta x_1 - \zeta ty) + (1-\alpha)(\eta x_2 - \zeta ty)\|^p \\
&\leq \alpha\|\zeta x_1 + \eta ty\|^p + (1-\alpha)\|\zeta x_2 + \eta ty\|^p + \alpha\|\eta x_1 - \zeta ty\|^p + (1-\alpha)\|\eta x_2 - \zeta ty\|^p \\
&= \alpha\|\beta(\zeta x_1 + \eta ty_1) + (1-\beta)(\zeta x_1 + \eta ty_2)\|^p + \alpha\|\beta(\eta x_1 - \zeta ty_1) + (1-\beta)(\eta x_1 - \zeta ty_2)\|^p \\
&+ (1-\alpha)\|\beta(\zeta x_2 + \mu ty_1) + (1-\beta)(\zeta x_2 + \eta ty_2)\|^p \\
&+ (1-\alpha)\|\beta(\eta x_2 - \zeta ty_1) + (1-\beta)(\eta x_2 - \zeta ty_2)\|^p \\
&\leq \alpha\beta\left[\|\zeta x_1 + \eta ty_1\|^p + \|\eta x_1 - \zeta ty_1\|^p\right] + \alpha(1-\beta)\left[\|\zeta x_1 + \eta ty_2\|^p + \|\eta x_1 - \zeta ty_2\|^p\right] \\
&+ (1-\alpha)\beta\left[\|\zeta x_2 + \eta ty_1\|^2 + \|\eta x_2 - \zeta ty_1\|^p\right] \\
&+ (1-\alpha)(1-\beta)\left[\|\zeta x_2 + \eta ty_2\|^2 + \|\eta x_2 - \zeta ty_2\|^p\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\zeta x + \eta ty\|^p + \|\eta x - \zeta ty\|^p \leq \max\{ &\|\zeta x_1 + \eta ty_1\|^p + \|\eta x_1 - \zeta ty_1\|^p, \\
&\|\zeta x_1 + \eta ty_2\|^p + \|\eta x_1 - \zeta ty_2\|^p, \\
&\|\zeta x_2 + \eta ty_1\|^p + \|\eta x_2 - \zeta ty_1\|^p, \\
&\|\zeta x_2 + \eta ty_2\|^p + \|\eta x_2 - \zeta ty_2\|^p\}.
\end{aligned}$$

Then we only need to consider the values of the constant $C_{NJ}^p(\zeta, \eta, X)$ at the extreme points. Here are all the extreme points:

$$\text{ext}(B_X) = \{(1,0), (1,1), (0,1), (-1,0), (-1,-1), (0,-1)\}.$$

Since it is possible to replace x with $-x$ and y with $-y$, it is only necessary to consider when the extreme points are

$$\{(1,0), (0,1), (1,1)\}.$$

Obviously, for these sets of x, y points, we can easily obtain $\|\zeta x + \eta ty\|^p + \|\eta x - \zeta ty\|^p \leq \zeta^p + (\eta + \zeta t)^p$ for every $t \in [0,1]$. Therefore,

$$C_{NJ}^p(\zeta, \eta, X) \leq \sup_{t\in[0,1]} \left\{ \frac{\zeta^p + (\eta + \zeta t)^p}{2^{p-2}(\zeta^p + \eta^p t^p + \eta^p + \zeta^p t^p)} \right\}.$$

Let $f(t) = \frac{\zeta^p + (\eta+\zeta t)^p}{\zeta^p + \eta^p t^p + \eta^p + \zeta^p t^p}$, then

$$f'(t) = \frac{\zeta p(\eta + \zeta t)^{p-1}(\zeta^p + \eta^p)}{(\zeta^p + \eta^p t^p + \eta^p + \zeta^p t^p)^2} \left[1 - \frac{\eta}{\zeta} t^{p-1} - \left(\frac{\zeta t}{\eta + \zeta t}\right)^{p-1}\right].$$

Let $s(t) = 1 - \frac{\eta}{\zeta} t^{p-1} - \left(\frac{\zeta t}{\eta+\zeta t}\right)^{p-1}$, $s(t)$ is decreasing from 1 to $1 - \frac{\eta}{\zeta} - \left(\frac{\zeta}{\eta+\zeta}\right)^{p-1}$ on $[0,1]$. Therefore, there is an only $t_0 \in (0,1)$ such that $s(t_0) = 0$. Then, we have

$$C_{NJ}^p(\zeta, \eta, X) \leq \frac{\zeta^p + (\eta + \zeta t_0)^p}{2^{p-2}(\zeta^p + \eta^p t_0^p + \eta^p + \zeta^p t_0^p)}.$$

On the other hand, let $x_0 = (0,1), y_0 = (t_0, 0)$, we have

$$C_{NJ}^p(\zeta, \eta, X) \geq \frac{\zeta^p + (\zeta + \eta t_0)^p}{2^{p-2}(\zeta^p + \eta^p t_0^p + \eta^p + \zeta^p t_0^p)}.$$

Hence,

$$C_{NJ}^{p}(\zeta,\eta,X)=\frac{\zeta^p+(\eta+\zeta t_0)^p}{2^{p-2}(\zeta^p+\eta^p t_0^p+\eta^p+\zeta^p t_0^p)},$$

where $t_0\in(0,1)$ is the only solution of $1-\frac{\zeta}{\eta}t^{p-1}=\left(\frac{\zeta t}{\zeta+\eta t}\right)^{p-1}$. □

Remark 2.7. If $p=2,\zeta=\eta=1$, then $C_{NJ}^{2}(1,1,X)=C_{NJ}(X)$. Now, we show that the value of the constant $C_{NJ}^{2}(1,1,X)$ is the same as the von Neumann-Jordan constant in this norm space.

For $p=2,\zeta=\eta=1$, we have $t^2+t-1=0$, then $t=\frac{1+\sqrt{5}}{2}$. Hence, we can get $C_{NJ}^{2}(1,1,X)=\frac{3+\sqrt{5}}{4}$.

Corollary 2.8. *In this norm space, we have*

$$C_{NJ}^{\frac{3}{2}}(2,1,X)\approx 1.55 \text{ and } C_{NJ}^{\frac{4}{3}}(1,2,X)\approx 1.60.$$

Proof. (1) For $p=\frac{3}{2},\zeta=2,\eta=1$, we have $1-\frac{1}{2}t^{\frac{1}{2}}=(\frac{2t}{1+2t})^{\frac{1}{2}}$, $t\approx 0.42$, hence $C_{NJ}^{\frac{3}{2}}(2,1,X)\approx 1.55$.

(2) For $p=\frac{4}{3},\zeta=1,\eta=2$, we have $1-2t^{\frac{1}{3}}=(\frac{t}{2+t})^{\frac{1}{3}}$, $t\approx 0.05$, hence $C_{NJ}^{\frac{4}{3}}(1,2,X)\approx 1.60$. □

3. The relations with $\delta_X(\varepsilon)$ and $J(X)$

In this section, we will compare the connection between the constant $C_{NJ}^{p}(\zeta,\eta,X)$ and the modulus of convexity $\delta_X(\varepsilon)$, the James constant $J(X)$. The James constant $J(X)$ has been introduced in the introduction. Now, we recall the definition of the convexity modules $\delta_X(\varepsilon)$.

Definition 3.1. Clarkson [8] introduced the concept of convexity modules, defined by

$$\delta_X(\varepsilon)=\inf\left\{1-\frac{\|x+y\|}{2}:x,y\in S_X,\|x-y\|=\varepsilon\right\},\ 0\le\varepsilon\le 2.$$

Next, restricting $\varepsilon\in(1,2]$, we derive the property that the convexity module is greater than 0 if the constant $C_{NJ}^{p}(\zeta,\eta,X)$ satisfies certain conditions. Further, we compare the inequality between the constant $C_{NJ}^{p}(\zeta,\eta,X)$ and convexity modules $\delta_X(\varepsilon)$.

Theorem 3.2. *Let $\zeta\ge\eta$, $\varepsilon\in(1,2]$, X be a Banach space. If*

$$C_{NJ}^{p}(\zeta,\eta,X)<\frac{(\zeta+\eta)^p(1+(\varepsilon-1)^p)}{2^{p-1}(\zeta^p+\eta^p)}$$

for $1\le p<\infty$, then $\delta_X(\varepsilon)>0$.

Proof of Theorem 3.2. Suppose $\delta_X(\varepsilon)=0$, then there exist $x_n,y_n\in S_X$ such that $\|x_n-y_n\|=\varepsilon$ for all $n\in\mathbb{N}$ and $\lim\limits_{n\to\infty}\|x_n+y_n\|=2$. According to the following elementary inequality:

$$\begin{aligned}\|\zeta x_n+\eta y_n\|&=\|\zeta(x_n+y_n)-(\zeta-\eta)y_n\|\\&\ge\zeta\|x_n+y_n\|-(\zeta-\eta)\|y_n\|\\&=2\zeta-(\zeta-\eta)=\zeta+\eta\end{aligned}$$

and

$$\begin{aligned}\|\eta x_n-\zeta y_n\|&=\|(\zeta+\eta)(x_n-y_n)+\eta y_n-\zeta x_n\|\\&\ge(\zeta+\eta)\|x_n-y_n\|-\eta\|y_n\|-\zeta\|x_n\|\\&=(\zeta+\eta)(\varepsilon-1).\end{aligned}$$

We can deduce that

$$\begin{aligned}\frac{(\zeta+\eta)^p(1+(\varepsilon-1)^p)}{2^{p-1}(\zeta^p+\eta^p)} &= \frac{(\zeta+\eta)^p+[(\zeta+\eta)(\varepsilon-1)]^p}{2^{p-1}(\zeta^p+\eta^p)} \\ &\leq \lim_{n\to\infty}\inf\frac{\|\zeta x+\eta y\|^p+\|\eta x-\zeta y\|^p}{2^{p-2}(\zeta^p+\eta^p)(\|x\|^p+\|y\|^p)} \\ &\leq C_{NJ}^p(\zeta,\eta,X) \\ &< \frac{(\zeta+\eta)^p(1+(\varepsilon-1)^p)}{2^{p-1}(\zeta^p+\eta^p)},\end{aligned}$$

a contradiction. This completes the proof. □

Theorem 3.3. *Let $\zeta \geq \eta$, $\varepsilon \in (1,2]$. For any Banach space X, we have*

$$C_{NJ}^p(\zeta,\eta,X) \geq \frac{[(\zeta+\eta)(\varepsilon-1)]^p+(\zeta+\eta-2\zeta\delta_X(\varepsilon))^p}{2^{p-1}(\zeta^p+\eta^p)},$$

where $1 \leq p < \infty$.

Proof of Theorem 3.3. Suppose that there exists $x, y \in B_X$ such that $\|x-y\| \geq \varepsilon$. Then

$$\begin{aligned}C_{NJ}^p(\zeta,\eta,X) &\geq \frac{\|\zeta x+\eta y\|^p+\|\eta x-\zeta y\|^p}{2^{p-2}(\zeta^p+\eta^p)(\|x\|^p+\|y\|^p)} \\ &\geq \frac{(\zeta\|x_n+y_n\|-(\zeta-\eta)\|y_n\|)^p+[(\zeta+\eta)(\varepsilon-1)]^p}{2^{p-2}(\zeta^p+\eta^p)(\|x\|^p+\|y\|^p)}.\end{aligned}$$

Obviously equivalent to

$$1-\frac{\|x+y\|}{2} \geq 1+\frac{\eta-\zeta-(C_{NJ}^p(\zeta,\eta,X)\cdot 2^{p-1}(\zeta^p+\eta^p)-[(\zeta+\eta)(\varepsilon-1)]^p)^{1/p}}{2\zeta}.$$

Combined with the definition of convexity modules $\delta_X(\varepsilon)$, we have

$$\delta_X(\varepsilon) \geq \frac{1}{2}+\frac{\eta-(C_{NJ}^p(\zeta,\eta,X)\cdot 2^{p-1}(\zeta^p+\eta^p)-[(\zeta+\eta)(\varepsilon-1)]^p)^{1/p}}{2\zeta}.$$

Then

$$C_{NJ}^p(\zeta,\eta,X) \geq \frac{[(\zeta+\eta)(\varepsilon-1)]^p+(\zeta+\eta-2\zeta\delta_X(\varepsilon))^p}{2^{p-1}(\zeta^p+\eta^p)}.$$

□

In addition, we find that the constant $C_{NJ}^p(\zeta,\eta,X)$ is somewhat related to the James constant $J(X)$.

Proposition 3.4. *Let X be a Banach space and for any $1 < p < \infty$, then the following inequality holds:*

$$\left[2^{p-2}\left(\zeta^p+\eta^p\right)C_{NJ}^p(\zeta,\eta,X)\right]^{1/p} \geq \max\{\zeta,\eta\}J(X)-|\zeta-\eta|.$$

Proof. For any $x, y \in S_X$, we have

$$\begin{aligned}2(\min\{\|\zeta x+\eta y\|,\|\eta x-\zeta y\|\})^p &\leq \|\zeta x+\eta y\|^p+\|\eta x-\zeta y\|^p \\ &\leq 2^{p-2}(\zeta^p+\eta^p)(\|x\|^p+\|y\|^p)C_{NJ}^p(\zeta,\eta,X) \\ &= 2^{p-1}(\zeta^p+\eta^p)C_{NJ}^p(\zeta,\eta,X),\end{aligned}$$

so

$$\min\{\|\zeta x+\eta y\|,\|\eta x-\zeta y\|\} \leq [2^{p-2}(\zeta^p+\eta^p)C_{NJ}^p(\zeta,\eta,X)]^{1/p}.$$

Then we can obtain

$$\begin{aligned}\left[2^{p-2}\left(\zeta^p+\eta^p\right)C_{NJ}^p(\zeta,\eta,X)\right]^{1/p} &\geq \min\{\|\zeta x+\eta y\|,\|\eta x-\zeta y\|\} \\ &\geq \min\{\zeta\|x+y\|-|\zeta-\eta|,\zeta\|x-y\|-|\zeta-\eta|\}.\end{aligned}$$

Thus,

$$\left[2^{p-2}\left(\zeta^p+\eta^p\right)C_{NJ}^{p}(\zeta,\eta,X)\right]^{1/p}\geq \zeta J(X)-|\zeta-\eta|.$$

Similarly, we can also obtain the following inequality

$$\left[2^{p-2}\left(\zeta^p+\eta^p\right)C_{NJ}^{p}(\zeta,\eta,X)\right]^{1/p}\geq \eta J(X)-|\zeta-\eta|.$$

□

Now, combined Proposition 3.4 with the fact that X is uniformly non-square if and only if $J(X)<2$ (see[6]), we have a simple corollary.

Corollary 3.5. *Let X be a Banach space. The following conclusions are equivalent:*
(i) X is uniformly non-square.
(ii) $C_{NJ}^{p}(\zeta,\eta,X)<\frac{(\zeta+\eta)^p}{2^{p-2}(\zeta^p+\eta^p)}$ holds for all $\zeta,\eta>0$.
(iii) $C_{NJ}^{p}(\zeta,\eta,X)<\frac{(\zeta+\eta)^p}{2^{p-2}(\zeta^p+\eta^p)}$ holds for some $\zeta,\eta>0$.

4. The constant $C_{NJ}^{p}(\zeta,\eta,X)$ and uniform normal structure

In this section, we will show the sufficient conditions for Banach space X to have a uniform normal structure [11] under the filter framework. Now, we recall the content about the filter.

An ultrafilter $\mathfrak{U}$ is defined on index set I. Let $\{x_i\}_{i\in I}$ be a subset of the Hausdorff topological space X and $\{x_i\}_{i\in I}$ converges to x with respect to $\mathfrak{U}$, denoted by $\lim_{\mathfrak{U}} x_i=x$. Let $\{X_i\}_{i\in I}$ be a family of Banach spaces. The ultraproduct of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I,X_i)/N_{\mathfrak{U}}$ equipped with the quotient norm:

$$l_\infty(I,X_i)=\left\{\|(x_i)\|=\sup_{i\in I}\|x_i\|<\infty\right\},$$

and

$$N_{\mathfrak{U}}=\left\{(x_i)\in l_\infty(I,X_i):\lim_{\mathfrak{U}}\|x_i\|=0\right\}.$$

For more research on filters in Banach spaces see [1, 15]. Then, combining the ideas in reference [16], we use a theorem to show that Banach space X has a uniform normal structure.

Theorem 4.1. *Let $\zeta\geq\eta\geq\zeta t$, X be a Banach space and the inequality*

$$C_{NJ}^{p}(\zeta,\eta,X)<\frac{\left(\zeta\sqrt{4\zeta^2+\eta^2t^2}+\zeta\eta t\right)^p+\left(\eta\sqrt{4\zeta^2+\eta^2t^2}+2\zeta^2t-\eta^2t\right)^p}{2^{2p-2}\zeta^p\left(\zeta^p+\eta^pt^p+\eta^p+\zeta^pt^p\right)}$$

holds for some $t\in(0,1]$. Then X has uniform normal structure.

Proof of Theorem 4.1. It's easy to see that X is uniformly non-square, hence X is super-reflexive (see [18]). Therefore, this is enough to prove that X has a normal structure. Applying lemma 2 in [14], there exist $\tilde{x}_1,\tilde{x}_2,\tilde{x}_3\in S_{\tilde{X}}$ and $\tilde{f}_1,\tilde{f}_2,\tilde{f}_3\in S_{\tilde{X}^*}$ satisfying the following three properties:
(i) $\|\tilde{x}_i-\tilde{x}_j\|=1$ and $\tilde{f}_i\left(\tilde{x}_j\right)=0$ for all $i\neq j$.
(ii) $\tilde{f}_i\left(\tilde{x}_i\right)=1$ for $i=1,2,3$.
(iii) $\|\tilde{x}_3-(\tilde{x}_2+\tilde{x}_1)\|\geq\|\tilde{x}_2+\tilde{x}_1\|$.
Next, let $g(t)=\frac{\left(\sqrt{4\zeta^2+\eta^2t^2}+2\zeta-\eta t\right)}{2\zeta}$ and consider three possible cases.
Case 1: If $\|\tilde{x}_1+\tilde{x}_2\|\leq g(t)$. Let $\tilde{x}=\tilde{x}_1-\tilde{x}_2$ and $\tilde{y}=\frac{(\tilde{x}_1+\tilde{x}_2)}{g(t)}$. Then $\tilde{x},\tilde{y}\in B_{\tilde{X}}$, we have

$$\begin{aligned}\|\zeta\tilde{x}+\eta t\tilde{y}\| &= \left\|\left(\zeta+\frac{\eta t}{g(t)}\right)\tilde{x}_1-\left(\zeta-\frac{\eta t}{g(t)}\right)\tilde{x}_2\right\| \\ &\geq \left(\zeta+\frac{\eta t}{g(t)}\right)\tilde{f}_1(\tilde{x}_1)-\left|\zeta-\frac{\eta t}{g(t)}\right|\tilde{f}_1(\tilde{x}_2) \\ &= \zeta+\frac{\eta t}{g(t)},\end{aligned}$$

and

$$\begin{aligned}\|\eta\tilde{x}-\zeta t\tilde{y}\| &= \left\|\left(\eta-\frac{\zeta t}{g(t)}\right)\tilde{x}_1-\left(\eta+\frac{\zeta t}{g(t)}\right)\tilde{x}_2\right\| \\ &\geq \left(\eta+\frac{\zeta t}{g(t)}\right)\tilde{f}_2(\tilde{x}_2)-\left|\eta-\frac{\zeta t}{g(t)}\right|\tilde{f}_2(\tilde{x}_1) \\ &= \eta+\frac{\zeta t}{g(t)}.\end{aligned}$$

Case 2: If $\|\tilde{x}_1+\tilde{x}_2\|\geq g(t)$ and $\|\tilde{x}_3+\tilde{x}_2-\tilde{x}_1\|\leq g(t)$. Let $\tilde{x}=\tilde{x}_2-\tilde{x}_3$ and $\tilde{y}=\frac{(\tilde{x}_3+\tilde{x}_2-\tilde{x}_1)}{g(t)}$. Then $\tilde{x},\tilde{y}\in B_{\tilde{X}}$, we have

$$\begin{aligned}\|\zeta\tilde{x}+\eta t\tilde{y}\| &= \left\|\left(\zeta+\frac{\eta t}{g(t)}\right)\tilde{x}_2-\left(\zeta-\frac{\eta t}{g(t)}\right)\tilde{x}_3-\frac{\eta t}{g(t)}\tilde{x}_1\right\| \\ &\geq \left(\zeta+\frac{\eta t}{g(t)}\right)\tilde{f}_2(\tilde{x}_2)-\left|\zeta-\frac{\eta t}{g(t)}\right|\tilde{f}_2(\tilde{x}_3)-\frac{\eta t}{g(t)}\tilde{f}_2(\tilde{x}_1) \\ &= \zeta+\frac{\eta t}{g(t)},\end{aligned}$$

and

$$\begin{aligned}\|\eta\tilde{x}-\zeta t\tilde{y}\| &= \left\|\left(\eta+\frac{\zeta t}{g(t)}\right)\tilde{x}_3+\left(\eta-\frac{\zeta t}{g(t)}\right)\tilde{x}_2-\frac{\zeta t}{g(t)}\tilde{x}_1\right\| \\ &\geq \left(\eta+\frac{\zeta t}{g(t)}\right)\tilde{f}_3(\tilde{x}_3)-\left|\eta-\frac{\zeta t}{g(t)}\right|\tilde{f}_3(\tilde{x}_2)-\frac{\zeta t}{g(t)}\tilde{f}_3(\tilde{x}_1) \\ &= \eta+\frac{\zeta t}{g(t)}.\end{aligned}$$

Case 3: If $\|\tilde{x}_1+\tilde{x}_2\|\geq g(t)$ and $\|\tilde{x}_3+\tilde{x}_2-\tilde{x}_1\|\geq g(t)$. Let $\tilde{x}=\tilde{x}_3-\tilde{x}_1$ and $\tilde{y}=\tilde{x}_2$. Then $\tilde{x},\tilde{y}\in B_{\tilde{X}}$, we have

$$\begin{aligned}\|\zeta\tilde{x}+\eta t\tilde{y}\| &= \|\zeta(\tilde{x}_3-\tilde{x}_1)+\eta t\tilde{x}_2\| \\ &= \|\zeta(\tilde{x}_3+\tilde{x}_2-\tilde{x}_1)+(\eta t-\zeta)\tilde{x}_2\| \\ &\geq \zeta(g(t)-1)+\eta t,\end{aligned}$$

and

$$\begin{aligned}\|\eta\tilde{x}-\zeta t\tilde{y}\| &= \|\eta(\tilde{x}_3-\tilde{x}_1)-\zeta t\tilde{x}_2\| \\ &= \|\eta(\tilde{x}_3-(\tilde{x}_2+\tilde{x}_1))+(\eta-\zeta t)\tilde{x}_2\| \\ &\geq \eta(g(t)-1)+\zeta t.\end{aligned}$$

Then, according to the definition of $C^p_{NJ}(\zeta,\eta,X)$ and $C^p_{NJ}(\zeta,\eta,X)=C^p_{NJ}(\zeta,\eta,\tilde{X})$, we obtain

$$\begin{aligned}C^p_{NJ}(\zeta,\eta,X) &\geq \max\left\{\frac{\|\zeta+\frac{\eta t}{g(t)}\|^p+\|\eta+\frac{\zeta t}{g(t)}\|^p}{2^{p-2}\left(\zeta^p+\eta^pt^p+\eta^p+\zeta^pt^p\right)},\frac{\|\zeta(g(t)-1)+\eta t)\|^p+\|\eta(g(t)-1)+\zeta t)\|^p}{2^{p-2}\left(\zeta^p+\eta^pt^p+\eta^p+\zeta^pt^p\right)}\right\} \\ &= \frac{\left(\zeta\sqrt{4\zeta^2+\eta^2t^2}+\zeta\eta t\right)^p+\left(\eta\sqrt{4\zeta^2+\eta^2t^2}+2\zeta^2t-\eta^2t\right)^p}{2^{2p-2}\zeta^p\left(\zeta^p+\eta^pt^p+\eta^p+\zeta^pt^p\right)}.\end{aligned}$$

This is a contradiction. The proof is complete. □

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