Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Published online: 09.05.2025
DOI : 10.15672/hujms.1451185

RESEARCH ARTICLE

Symplectic embeddings into cylinders for certain symplectic manifolds

Nil İpek Şirikçi

Department of Economics, Middle East Technical University, 06800 Ankara, Türkiye

Abstract

We present a proof of a result on displaceability of subsets of symplectic manifolds satisfying certain conditions one of which is that the subset is precompact in a connected neighborhood that symplectically embeds into $\mathbb{R}^{2n}$. The proof utilizes an inequality between the displacement energy and the cylindrical capacity for subsets of $\mathbb{R}^{2n}$ to obtain an inequality for subsets of the symplectic manifold. We also state a corollary which utilizes other results on nondisplaceable Lagrangians.

Mathematics Subject Classification (2020). 53D05

Keywords. symplectic embeddings, symplectic manifolds, Lagrangian submanifolds , symplectic cylinder, displacement energy, cylindrical capacity

1. Introduction

A foundational theorem in symplectic topology is Gromov's nonsqueezing theorem which states that a ball cannot be squeezed by a symplectomorphism into a cylinder with smaller radius [11,12]. This is stated as the following:

Define the symplectic cylinder $(Z^{2n}(r), \omega_0)$ as

$$Z^{2n}(r) := B^2(r) \times \mathbb{R}^{2n-2}$$

where

$$B^{2n}(r) = \{z \in \mathbb{R}^{2n} : \ |z| \leq r\}.$$

Here ω_0 denotes the standard symplectic form on $\mathbb{R}^{2n}$ and in the following we will always assume that $\mathbb{R}^{2n}$ is equipped with the standard symplectic form.

Theorem 1.1 (Gromov, [3,12])**.** *If there exists a symplectic embedding of* $(B^{2n}(r), \omega_0) \hookrightarrow (Z^{2n}(R), \omega_0)$*, then* $r \leq R$*.*

Email address: sirikci@metu.edu.tr
Received: 11.03.2024; Accepted: 14.04.2025

This result was generalized to symplectic embeddings into $M \times B^2$ for an arbitrary manifold M by Lalonde and McDuff [8,9]. Many results have been obtained on symplectic embedding problems for various manifolds some of which are stated in [6, 10, 12, 16] and in [4, 5, 7, 13, 14, 17].

In this work, we consider the case when a connected symplectic manifold has a subset L with a symplectic embedding of a connected neighborhood of L in which L is precompact into $(\mathbb{R}^{2n}, \omega_0)$ such that a displacement energy condition is satisfied. We obtain an inequality between the displacement energy of L in M and the cylindrical capacity of the image of L under the symplectic embedding. We also obtain that L is displaceable in M under these assumptions. This is stated as Theorem 1.4. We also state Lemmas 1.2 and 1.3 about Hofer's norm. The proof of Theorem 1.4 uses Lemma 1.3:

Lemma 1.2. *Let ψ be a symplectic embedding of a neighborhood U into $\mathbb{R}^{2n}$. Then any compactly supported Hamiltonian H on $\mathbb{R}^{2n}$ with support in $\psi(U)$ has Hofer's norm satisfying $\rho(1, \phi_H) \leq \rho(1, \phi_{H\circ\psi})$.*

Proof. For the definition of Hofer distance, see p. 466 of [12]. The Hofer distance between identity and ϕ_H is called Hofer's norm in [15].

We have

$$\rho(1, \phi_H) = \inf_{\phi_G = \phi_H} \int_0^1 \|G_t\| dt = \inf_{\phi_G = \phi_H} \Big(\max_{\mathbb{R}^{2n}} G_t - \min_{\mathbb{R}^{2n}} G_t \Big) dt$$

where $G : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ is a compactly supported smooth Hamiltonian that generates the Hamiltonian symplectomorphism ϕ_G as the time-1 map and the infimum is taken over all compactly supported Hamiltonians G such that $\phi_G = \phi_H$.

For an embedding ψ of a neighborhood $U \subset M$ into $\mathbb{R}^{2n}$ and for H a compactly supported Hamiltonian on $\mathbb{R}^{2n}$ with support in $\psi(U)$, we have

$$\rho(1, \phi_{H\circ\psi}) = \inf_{\substack{\phi_{G\circ\psi} = \phi_{H\circ\psi} \\ supp(G) \subset \psi(U)}} \int_0^1 \|(G\circ\psi)_t\| dt = \inf_{\substack{\phi_{G\circ\psi} = \phi_{H\circ\psi} \\ supp(G) \subset \psi(U)}} \int_0^1 \Big(\max_U (G\circ\psi)_t - \min_U (G\circ\psi)_t \Big) dt$$

where

$$\max_U (G \circ \psi)_t = \max_{\psi(U)} G_t$$

and

$$\min_U (G \circ \psi)_t = \min_{\psi(U)} G_t.$$

Hence we have $\rho(1, \phi_H) \leq \rho(1, \phi_{H\circ\psi})$. □

Lemma 1.3. *Hofer's norm is invariant under symplectomorphism.*

Theorem 1.4. *Assume that (M, ω) is a connected symplectic manifold with a subset L and that there is a symplectic embedding ψ of a connected neighborhood U of L in which L is precompact into $(\mathbb{R}^{2n}, \omega_0)$ such that the displacement energies satisfies $e_{\psi(U)}(\psi(L)) = e_{\mathbb{R}^{2n}}(\psi(L))$. Then the displacement energy of L in M is less than or equal to the cylindrical capacity of $\psi(L)$ and L is displaceable in M.*

We also obtain the following statement which is a corollary of results by Gromov (for parts a) and c)), Frauenfelder and Schlenk (for part b)), and Buhovsky (for part d)) and of Theorem 1.4:

Corollary 1.5. *Assume that (M, ω) is a connected symplectic manifold and that any of the following set of conditions hold:*

a) (M,ω) *is without boundary, is convex at infinity and has a compact Lagrangian submanifold* L *such that* $\omega|_{\pi_2(M,L)} = 0$ *is satisfied.*

b) (M,ω) *is a weakly exact and convex, and* $L \subset M \setminus \partial M$ *is a closed Lagrangian submanifold such that the inclusion of* L *into* M *induces an injection* $\pi_1(L) \to \pi_1(M)$ *and* L *admits a metric none of whose closed geodesics is contractible.*

c) (M,ω) *is geometrically bounded and has a closed Lagrangian submanifold* L *and* $\omega|_{\pi_2(M,L)} = 0$ *is satisfied.*

d) $L = \mathbb{RP}^n \hookrightarrow M$ *is a monotone Lagrangian embedding into tame* (M,ω) *and* $N_L \geq 3$ *where* N_L *is the minimal Maslov number of* L*.*

Then no connected neighborhood of L *symplectically embeds into* $\mathbb{R}^{2n}$ *in such a way that* $e_{\psi(U)}(\psi(L)) = e_{\mathbb{R}^{2n}}(\psi(L))$*.*

2. Proofs

2.1. Proof of Theorem 1.4

Let L be a subset of a connected symplectic manifold (M,ω). By assumption, there is a connected neighborhood U of L in which L is precompact and a symplectic embedding ψ of $(U,\omega|_U)$ to $(\mathbb{R}^{2n},\omega_0)$ such that the condition on the displacement energy specified in the theorem is satisfied.

Let $e_M(A)$ denote the displacement energy of a subset A of M defined in Section 12.3 of [12] as the following:

If A is compact,

$$e_M(A) = inf\{\rho(1,\phi) | \phi \in Ham(M,\omega),\ \phi(A) \cap A = \emptyset\}$$

where $\rho(1,\phi)$ is Hofer's distance between identity and ϕ and $Ham(M,\omega)$ is the set of compactly supported Hamiltonian symplectomorphisms of (M,ω).

If A is not compact,

$$e_M(A) = sup\{e_M(K) | K \subset A, K \text{ is compact}\}.$$

By definition, $e_M(L) \leq e_U(L)$.

Note that for any $\phi_H \in Ham(\psi(U),\omega_0|_{\psi(U)})$, we have $\phi_{H\circ\psi} \in Ham(U,\omega|_U)$. Also $\phi_H(\psi(L)) \cap \psi(L) = \emptyset$ implies $\phi_{H\circ\psi}(L) \cap L = \emptyset$. By Lemma 1.3, $\rho(1,\phi_H) = \rho(1,\phi_{H\circ\psi})$. Hence we have

$$e_M(L) \leq e_U(L) = e_{\psi(U)}(\psi(L)).$$

By assumption, $e_{\psi(U)}(\psi(L)) = e_{\mathbb{R}^{2n}}(\psi(L))$.

Since $\psi(L) \subset \mathbb{R}^{2n}$, by Theorem 12.3.4 of [12], we have

$$e_{\mathbb{R}^{2n}}(\psi(L)) \leq \bar{w}_G(\psi(L)),$$

where $\bar{w}_G(A)$ is the cylindrical capacity of A and is defined for any subset $A \subset \mathbb{R}^{2n}$ as

$$\bar{w}_G(A) = \inf\left\{\pi r^2 \;\middle|\; A \text{ embeds symplectically in } Z^{2n}(r) \text{ by a symplectomorphism of } \mathbb{R}^{2n}\right\}.$$

Then, by the above inequalities,

$$e_M(L) \leq \bar{w}_G(\psi(L)).$$

By assumption, U symplectically embeds into $\mathbb{R}^{2n}$. Since L is precompact, there exits r such that $\psi(L)$ symplectically embeds in $Z^{2n}(r)$ by a symplectomorphism of $\mathbb{R}^{2n}$. Hence the cylindrical capacity of $\psi(L)$, $\bar{w}_G(\psi(L))$ is finite. This implies that $e_M(L)$ is finite. Hence L is displaceable in M.

2.2. Proof of Corollary 1.5

By the following results, we conclude that L is nondisplaceable:

For a) : By Gromov's theorem stated in p.297 of [11].
For b) : By Theorem 5 stated in [2].
For c) : The explanation in the paragraph following Theorem 5 in [2] states that the conclusion follows by Gromov's theorem.
For d) : By Theorem 3 stated in [1].

Then the statement follows from Theorem 1.4.

References

[1] L. Buhovsky, *The Maslov class of Lagrangian tori and quantum products in Floer cohomology*, Journal of Topology and Analysis, **2**(1), 57-75, 2010.

[2] U. Frauenfelder and F. Shlenk, *Hamiltonian Dynamics on Convex Symplectic Manifolds*, Israel Journal of Mathematics, **159**, 1-56, 2007.

[3] M. Gromov, *Pseudo-holomorphic curves in symplectic manifolds*, Inventiones Mathematicae, **82**, 307-347, 1985.

[4] L. Guth, *Symplectic embeddings of polydisks*, Invent. Math. **172** (3), 477-489, 2008.

[5] R. Hind and E. Kerman, *New obstructions to symplectic embeddings*, Invent. Math. **196**(2), 383-452, 2014.

[6] M. Hutchings, *Recent progress on symplectic embedding problems in four dimensions*, Proc. Natl. Acad. Sci. USA **108**(20), 8093-8099, 2011.

[7] M. Hutchings, *Beyond ECH capacities* Geom. Topol. **20**(2), 1085-1126, 2016.

[8] F. Lalonde and D. McDuff, *The Geometry of Symplectic Energy*, Annals of Mathematics, **141**(2), 349-371, 1995.

[9] F. Lalonde, and D. McDuff, *Local Non-squeezing Theorems and Stability*, Geometric and Functional Analysis, **5**(2), 364-386, 1995.

[10] D. McDuff, *Symplectic Topology Today*, AMS Joint Mathematics Meeting, Baltimore, Colloquium Lectures, 2014. Available online: http://jointmathematicsmeetings.org/meetings/national/jmm2014/colloqnov2.pdf

[11] D. McDuff and D. Salamon, *J-holomorphic Curves and Symplectic Topology*, Colloquium Publications Vol. 52, AMS, Providence RI, 2004.

[12] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford University Press, 2017.

[13] Y. Ostrover and G.B. Vinicius, *Symplectic embeddings of the ℓ_p-sum of two discs* J. Topol. Anal. **14**(4), 793-821, 2022.

[14] Á. Pelayo and S.V. Ngoc, *Sharp symplectic embeddings of cylinders* Indag. Math. (N.S.) **27**(1), 307-317, 2016.

[15] L. Polterovich, *The Geometry of the Group of Symplectic Diffeomorphisms*, Birkhäuser Basel, 2001.

[16] F. Schlenk, *Symplectic embedding problems, old and new*, Bull. Amer. Math. Soc. (N.S.) **55**(2), 139-182, 2018.

[17] J. Swoboda, and F. Ziltener, *A symplectically non-squeezable small set and the regular coisotropic capacity*, J. Symplectic Geom. **11**(4), 509-523, 2013.