http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 3, Pages 787–801 (2024)
DOI:10.31801/cfsuasmas.1452069
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: March 15, 2024; Accepted: May 13, 2024

APPROXIMATION PROPERTIES OF THE UNIVARIATE AND BIVARIATE BERNSTEIN-STANCU OPERATORS OF MAX-PRODUCT KIND

Ayşe Kübra YEŞİLNACAR BİNMAR[1], Ecem ACAR[2] and Sevilay KIRCI SERENBAY[3]

[1,3]Department of Mathematics, Harran University, Şanlıurfa, TÜRKİYE
[2]Department of Mathematics and Science Education, Harran University, Şanlıurfa, TÜRKİYE

Abstract. This paper presents the nonlinear maximum product type of univariate and bivariate Bernstein–Stancu operators and uses new definitions to investigate the approximation properties. The order of approximation obtained with the nonlinear maximum product type of operator sequences would be better than the degree of approximation of the known linear operator sequences.

1. Introduction

In 1969, Stancu [3] introduced the Bernstein-Stancu polynomials as follows

$$B_n^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k}x^k(1-x)^k, \tag{1}$$

where $n \in \mathbb{N}$, $f \in C[0,1]$, which is the space of all real valued continuous functions defined on $[0,1]$, real numbers α and β are fixed, indicating that $0 \leq \alpha \leq \beta$. The classical Bernstein polynomials are obtained in the condition $\alpha = \beta = 0$.

The approximation of a continuous function by a series of linear positive operators is the main topic of Korovkin-type approximation theory (see [1], [2], [18]- [21]). Recently, Bede et al., [4] have introduced nonlinear positive operators in place of linear positive operators. Bede et al., introduced the max-product version of families of linear approximation operators (see [7], [8]), which generated a new field of

2020 *Mathematics Subject Classification.* 41A10, 41A25, 41A36.
Keywords. Bernstein–Stancu operators, nonlinear operators, degree of approximation.
[1] aysekubrabinmar@gmail.com; 0000-0001-7861-2742;
[2] karakusecem@harran.edu.tr-Corresponding author; 0000-0002-2517-5849;
[3] sevilaykirci@gmail.com; 0000-0001-5819-9997.

approximation theory. Up to now researchers have investigated and explored many aspects of max product operators (see [13]- [17]). It is concluded that they have an even better order of approximation for certain subclasses of functions and the same order as in the case of positive linear operators.

Lemma 1. *([5]) Let $I \subset \mathbb{R}$ be a bounded or unbounded interval, $CB_+(I)$ be the space of real valued continuous and bounded functions defined on I, and $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the next requirements:*

(i) (Monotonicity) If $f, g \in CB_+(I)$ provide $f \leq g$ then $L_n(f) \leq L_n(g)$ for all $n \in \mathbb{N}$;

(ii) (Sublinearity) For all $f, g \in CB_+(I)$ $L_n(f+g) \leq L_n(f) + L_n(g)$.

In [11], the function of two real variables function f be given over the unit square $s : [0,1][0,1]$ then the bivariate Bernstein polynomial of degree (n,m), corresponding to the function f, is defined by means of the formula

$$B_{n,m}(f)(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} p_{n,i}(x) p_{m,j}(y) f(i/n, j/m).$$

The square interval of bivariate Bernstein polynomials connected to a function of $f(x,y)$ given by

$$\begin{aligned} B_{n,m}(f)(x,y) =& \sum_{i=0}^{n} \sum_{j=0}^{m} p_{n,i}(x) p_{m,j}(y) f(i/n, j/m) \\ =& \frac{\sum_{i=0}^{n} \sum_{j=0}^{m} p_{n,i}(x) p_{m,j}(y) f(i/n, j/m)}{\sum_{i=0}^{n} \sum_{j=0}^{m} p_{n,i}(x) p_{m,j}(y)}, \end{aligned}$$

where $f : [0,1] \times [0,1] \to \mathbb{R}$, $(x,y) \in [0,1]^2$, $n, m \in \mathbb{N}$ (see Hildebrandt–Schoenberg [9], Butzer [10]). Also, in [6] max-product Bernstein operators of two variables defined by

$$B_{n,m}^{(M)}(f)(x,y) = \frac{\bigvee_{i=0}^{n} \bigvee_{j=0}^{m} p_{n,i}(x) p_{m,j}(y) f(i/n, j/m)}{\bigvee_{i=0}^{n} \bigvee_{j=0}^{m} p_{n,i}(x) p_{m,j}(y)}, \ (x,y) \in [0,1]^2, n, m \in \mathbb{N}.$$

In this work, we study the max-product Bernstein-Stancu operators. Firstly, we give the definition of the bivariate Bernstein-Stancu max product operators and investigate approximation properties of these operators. Then, the bivariate Bernstein-Stancu max product operators will be defined and approximation properties for bivariate Bernstein-Stancu max product operators will be investigated.

2. Construction of the Bernstein–Stancu Operators of Max-Product Kind and the Approximation Properties

We describe the max-product type nonlinear Bernstein-Stancu operators as follows

$$P_z^{(M)}(\kappa;x)=\frac{\bigvee_{m=0}^{z} p_{z,m}(x)\kappa\left(\frac{m+\rho}{z+\theta}\right)}{\bigvee_{m=0}^{z} p_{z,m}(x)}, z\in\mathbb{N} \tag{2}$$

with $p_{z,m}(x)=\binom{z}{m}x^m(1-x)^{z-m}$ where $\kappa:[0,1]\to\mathbb{R}^+,\ x\in[0,1]\ ;\ \rho,\theta\in\mathbb{R}^+$ $0\le\rho\le\theta$ and

$$lim_{z\to\infty}\frac{1}{z+\theta}=0.$$

Here, $P_z^{(M)}(\kappa;x)$ is positive and continuous on $[0,1]$ for the continuous function κ (see [12]). We also know that for $\forall z\in\mathbb{N}$, $P_z^{(M)}(\kappa;0)-\kappa(0)=0$ and the following definitions will be given for the arbitrary $0\le x\le 1$.

Now, the approximation rate will be calculated with the help of the modulus of continuity for maximum product type Bernstein-Stancu operators. To prove the main results, we need the following notations and auxiliary results. For each $m,\varpi\in\{0,1,...,z\}$ and $x\in\left[\frac{\varpi+\rho}{z+\theta+1}\frac{\varpi+\rho+1}{z+\theta+1}\right]$, $z\in\mathbb{N}$ and $0\le\rho\le\theta$, let us indicate

$$N_{m,z,\varpi}(x)=\frac{p_{z,m}(x)|\frac{m+\rho}{z+\theta}-x|}{p_{z,\varpi}(x)},\ n_{m,z,\varpi}(x)=\frac{p_{z,m}(x)}{p_{z,\varpi}(x)}.$$

Let $m,\varpi\in\{0,1,...,z\}$, $x\in\left[\frac{\varpi+\rho}{z+\theta+1},\frac{\varpi+\rho+1}{z+\theta+1}\right]$, $z\in\mathbb{N}$ and for $\rho,\theta\in\mathbb{R}^+$, we have $0\le\rho\le\theta$. Hence, it is obvious that

i. If $\varpi+1\le m$ then $N_{m,z,\varpi}(x)=\frac{p_{z,m}(x)(\frac{m+\rho}{z+\theta}-x)}{p_{z,\varpi}(x)}$,
ii. if $m\le\varpi-1$ then $N_{m,z,\varpi}(x)=\frac{p_{z,m}(x)(x-\frac{m+\rho}{z+\theta})}{p_{z,\varpi}(x)}$.

Additionally, for each $m,\varpi\in\{0,1,...,z\}$, $x\in\left[\frac{\varpi+\rho}{z+\theta+1},\frac{\varpi+\rho+1}{z+\theta+1}\right]$, $z\in\mathbb{N}$ and for $\rho,\theta\in\mathbb{R}^+$, we get $0\le\rho\le\theta$. Let us indicate

i. If $\varpi+2\le m$, then $\overline{N}_{m,z,\varpi}(x)=\frac{p_{z,m}(x)(\frac{m+\rho}{z+\theta+1}-x)}{p_{z,\varpi}(x)}$,
ii. if $m\le\varpi-2$ then $\underline{N}_{m,z,\varpi}(x)=\frac{p_{z,m}(x)(x-\frac{m+\rho}{z+\theta+1})}{p_{z,\varpi}(x)}$.

Lemma 2. *Let* $x\in\left[\frac{\varpi+\rho}{z+\theta+1},\frac{\varpi+\rho+1}{z+\theta+1}\right]$ *and for all* $m,\varpi\in\{0,1,...,z\}$,

i. *If* $\varpi+2\le m$, *then* $\overline{N}_{m,z,\varpi}(x)\le N_{m,z,\varpi}(x)\le 3\overline{N}_{m,z,\varpi}(x)$.
ii. *If* $m\le\varpi-2$, *then* $N_{m,z,\varpi}(x)\le\underline{N}_{m,z,\varpi}(x)\le 6N_{m,z,\varpi}(x)$.

Proof. (i) This inequality $\overline{N}_{m,z,\varpi}(x) \leq N_{m,z,\varpi}(x)$ is immediate. Besides,

$$\begin{aligned}\frac{N_{m,z,\varpi}(x)}{\overline{N}_{m,z,\varpi}(x)} &= \frac{\frac{m+\rho}{z+\theta}-x}{\frac{m+\rho}{z+\theta+1}-x} \leq \frac{\frac{m+\rho}{z+\theta}-\frac{\varpi+\rho}{z+\theta+1}}{\frac{m+\rho}{z+\theta+1}-\frac{\varpi+\rho+1}{z+\theta+1}} \\ &\leq \frac{mz+m\theta+m+\rho-z\varpi-\theta\varpi}{(z+\theta)(m-\varpi-1)} \\ &\leq \frac{m-\varpi}{m-\varpi-1}.\frac{\rho}{(z+\theta)(m-\varpi-1)} \leq 3\end{aligned} \tag{3}$$

which proves (i).

(ii) The inequality $N_{m,z,\varpi}(x) \leq \underline{N}_{m,z,\varpi}(x)$ is immediate. Also

$$\begin{aligned}\frac{\underline{N}_{m,z,\varpi}(x)}{N_{m,z,\varpi}(x)} &= \frac{x-\frac{m+\rho}{z+\theta+1}}{x-\frac{m+\rho}{z+\theta}} \leq \frac{\frac{\varpi+\rho+1}{z+\theta+1}-\frac{m+\rho}{z+\theta+1}}{\frac{\varpi+\rho}{z+\theta+1}-\frac{m+\rho}{z+\theta}} \\ &= \frac{(z+\theta+1)(\varpi+\rho+1-m-\rho)}{(z+\theta)(\varpi+\rho-m-\rho)-m-\rho} \\ &\leq \frac{(z+\theta+1)(\varpi+1-m)}{(z+\theta)(\varpi+\rho-m-\rho-1)} \\ &= \frac{z+\theta+1}{z+\theta}.\frac{\varpi+1-m}{\varpi-m-1} \leq 2\frac{\varpi+1-m}{\varpi-m-1} \\ &= 2\left(1+\frac{1}{\varpi-m-1}\right) \leq 6\end{aligned}$$

Therefore, we get the following inequality

$$N_{m,z,\varpi}(x) \leq \underline{N}_{m,z,\varpi}(x) \leq 6N_{m,z,\varpi}(x),$$

and it is the proof of the lemma. □

Lemma 3. *For all* $m,\varpi \in \{0,1,...,z\}$ *and* $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$, *we have*

$$n_{m,z,\varpi}(x) \leq 1$$

Proof. We have two states 1)$\varpi \leq m$, 2)$m \leq \varpi$

1)It is obvious that the function $g(x) = \frac{1-x}{x}$ is nonincreasing on $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$ it means that

$$\frac{n_{m,z,\varpi}(x)}{n_{m+1,z,\varpi}(x)} = \frac{m+1}{z-m}.\frac{1-x}{x} \geq \frac{m+1}{z-m}.\frac{1-\frac{\varpi+\rho+1}{z+\theta}}{\frac{\varpi+\rho+1}{z+\theta}} \geq 1,$$

which implies that

$$... \leq n_{m+2,z,\varpi}(x) \leq n_{m+1,z,\varpi}(x) \leq n_{m,z,\varpi}(x).$$

2)For $m \leq \varpi$, we obtain

$$\frac{n_{m,z,\varpi}(x)}{n_{m-1,z,\varpi}(x)} = \frac{z-m+1}{m}.\frac{x}{1-x} \geq \frac{z-m+1}{m}.\frac{\frac{\varpi+\rho}{z+\theta}}{1-\frac{\varpi+\rho}{z+\theta}} \geq 1,$$

which immediately implies

$$n_{0,z,\varpi}(x) \leq ... \leq n_{m-2,z,\varpi}(x) \leq n_{m-1,z,\varpi}(x) \leq n_{m,z,\varpi}(x)$$

Since for $\forall m, \varphi \in \{1,2,...,z\}$, $n_{m,z,\varphi}(x) = 1$ the conclusion of the lemma is obvious. □

Lemma 4. *Let* $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$, $z \in \mathbb{N}$ *ve* $\rho, \theta \in \mathbb{R}^+$ $0 \leq \rho \leq \theta$ *olmak üzere,*

i) *If* $m \in \{\varpi+2, \varpi+3, ..., z-1\}$ *is such that* $\varpi \leq (m+\rho) - \sqrt{m+\rho+1}$, *then* $\overline{N}_{m+1,z,\varpi}(x) \leq \overline{N}_{m,z,\varpi}(x)$.

ii) *If* $m \in \{1,2,...,\varpi-2\}$ *is such that* $(m+\rho)+\sqrt{m+\rho} \leq \varpi$, *then* $\underline{N}_{m-1,z,\varpi}(x) \leq \underline{N}_{m,z,\varpi}(x)$.

Proof. (i) We observe that

$$\frac{\overline{N}_{m,z,\varpi}(x)}{\overline{N}_{m+1,z,\varpi}(x)} = \frac{m+\rho+1}{z+\theta-m-\rho}.\frac{1-x}{x}.\frac{\frac{m+\rho}{z+\theta+1}-x}{\frac{m+\rho+1}{z+\theta+1}-x}.$$

Since the function $\psi(x) = \frac{1-x}{x}.\frac{\frac{m+\rho}{z+\theta+1}-x}{\frac{m+\rho+1}{z+\theta+1}-x}$ is nonincreasing on $[0,1]$, it means that

$$\psi(x) \geq \psi(\frac{\varpi+\rho+1}{z+\theta+1}) = \frac{z+\theta-\varpi-\rho}{\varpi+\rho+1}\frac{m+\rho-\varpi-\rho-1}{m+\rho-\varpi-\rho} = \frac{z+\theta-\varpi-\rho}{\varpi+\rho+1}\frac{m-\varpi-1}{m-\varpi},$$

for all $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1},\right]$. Then since the contidion $\varpi \leq (m+\rho) - \sqrt{m+\rho+1}$ implies

$$\begin{aligned}(m+\rho+1)(m+\rho-\varpi-\rho-1) \geq& (\varpi+\rho+1)(m+\rho-\varpi-\rho),\\ (m+\rho+1)(m-\varpi-1) \geq& (\varpi+\rho+1)(m-\varpi),\end{aligned}$$

we obtain

$$\frac{\overline{N}_{m,z,\varpi}(x)}{\overline{N}_{m+1,z,\varpi}(x)} \geq \frac{m+\rho+1}{z+\theta-m-\rho}.\frac{z+\theta-\varpi-\rho}{\varpi+\rho+1}.\frac{m-\varpi-1}{m-\varpi} \geq 1.$$

(ii) We observe that

$$\frac{\underline{N}_{m,z,\varpi}(x)}{\underline{N}_{m+1,z,\varpi}(x)} = \frac{z+\theta-m-\rho+1}{m+\rho}.\frac{x}{1-x}.\frac{x-\frac{m+\rho}{z+\theta+1}}{x-\frac{m+\rho-1}{z+\theta+1}}.$$

Since the function $g(x) = \frac{1-x}{x}\frac{x-\frac{m+\rho}{z+\theta+1}}{x-\frac{m+\rho-1}{z+\theta+1}}$ is nondecreasing on $[0,1]$ it means that $g(x) \geq g(\frac{\varpi+\rho}{z+\theta+1}) = \frac{\varpi+\rho}{z+\theta+1-\varpi-\rho}\frac{\varpi-m}{\varpi-m+1}$, for all $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$. Then,since the condition $(m+\rho)+\sqrt{m+\rho} \leq \varpi+\rho$ implies $(\varpi+\rho)(\varpi-m) \geq (m-\rho)(\varpi-m-1)$, we obtain

$$\frac{\underline{N}_{m,z,\varpi}(x)}{\underline{N}_{m+1,z,\varpi}(x)} \geq \frac{z+\theta-m-\rho+1}{m+\rho}.\frac{\varpi+\rho}{z+\theta+1-\varpi-\rho}.\frac{\varpi-m}{\varpi-m+1} \geq 1,$$

which proves the lemma. □

Lemma 5. *We have*

$$\bigvee_{m=0}^{z} p_{z,m}(x) = p_{z,\varpi}(x),\ \forall x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right],\ \varpi \in \{0,1,...,z\}$$

which $p_{z,m}(x) = C_z^m x^m (1-x)^{z-m}$.

Proof. Firstly, we demonstrate that for fixed $z \in \mathbb{N}$ and $0 \leq m+\rho \leq m+\rho+1 \leq z$, we have

$$0 \leq p_{z,m+1}(x) \leq p_{z,m}(x) \text{ if and only if } x \in \left[0, \frac{\varpi+\rho+1}{z+\theta+1}\right].$$

Actually, the inequality one reduces to

$$0 \leq \binom{z}{m+1} x^{m+1}(1-x)^{z-m-1} \leq \binom{z}{m} x^m (1-x)^z - m,$$

which, after being simplified, is equivalent to

$$0 \leq x\left[\binom{z}{m+1} + \binom{z}{m}\right] \leq \binom{z}{m}.$$

Regarding the equality $\binom{z}{m+1} + \binom{z}{m} = \binom{z+1}{m+1}$, The inequality mentioned above straight away equals $0 \leq x \leq \frac{m+1}{z+1}$. Using $m = 0, ..., z$ in the inequality that was just demonstrated above, we obtain

$$p_{z,\rho+1}(x) \leq p_{z,\rho}(x), \text{ if and only if } x \in \left[0, \frac{\rho+1}{z+\theta+1}\right],$$

$$p_{z,\rho+2}(x) \leq p_{z,\rho+1}(x), \text{ if and only if } x \in \left[0, \frac{\rho+2}{z+\theta+1}\right],$$

$$p_{z,\rho+3}(x) \leq p_{z,\rho+2}(x), \text{ if and only if } x \in \left[0, \frac{\rho+3}{z+\theta+1}\right],$$

so on,

$$p_{z,m+1}(x) \leq p_{z,m}(x), \text{ if and only if } x \in \left[0, \frac{m+\rho+1}{z+\theta+1}\right],$$

so on,

$$p_{z,z-2}(x) \leq p_{z,z-3}(x), \text{ if and only if } x \in \left[0, \frac{z+\theta-2}{z+\theta+1}\right],$$

$$p_{z,z-1}(x) \leq p_{z,z-2}(x), \text{ if and only if } x \in \left[0, \frac{z+\theta-1}{z+\theta+1}\right],$$

$$p_{z,z}(x) \leq p_{z,z-1}(x), \text{ if and only if } x \in \left[0, \frac{z+\theta}{z+\theta+1}\right].$$

From all these inequalities, we easily get:

If $x \in \left[0, \frac{\rho+1}{z+\theta+1}\right]$, then $p_{z,m+\rho}(x) \leq p_{z,\rho}(x)$, for all $m = 0, ..., z$,

If $x \in \left[\frac{\rho+1}{z+\theta+1}, \frac{\rho+2}{z+\theta+1}\right]$, then $p_{z,m+\rho}(x) \leq p_{z,\rho+1}(x)$, for all $m = 0, ..., z$,

If $x \in \left[\frac{\rho+2}{z+\theta+1}, \frac{\rho+3}{z+\theta+1}\right]$, then $p_{z,m+\rho}(x) \leq p_{z,\rho+2}(x)$, for all $m = 0, ..., z$,

and so on finally

$$\text{if } x \in \left[\frac{\rho+z}{z+\theta+1}, 1\right] \text{ then } p_{z,m}(x) \leq p_{z,z}(x), \text{ for all } m = 0, ..., z,$$

that proves the lemma. □

Theorem 1. *Let* $\kappa : [0,1] \to [0,1]$, κ, *be a continuous function on* $[0,1]$. *Then, we obtain*

$$|P_z^{(M)}(\kappa)(x) - \kappa(x)| \leq 12\omega_1\left(\kappa, \frac{\sqrt{x+\rho}}{\sqrt{z+\theta+1}}\right), x \in [0,1], \forall n \in \mathbb{N}.$$

Here, $\omega_1(\kappa, \delta) = sup(|\kappa(x) - \kappa(t)| : x, t \in [0,1], |x-t| \leq \delta)$.

Proof. Checking that the max-product Bernstein-Stancu operators satisfy the requirements in Lemma 1 is easy.

$$|P_z^{(M)}(\kappa)(x) - \kappa(x)| \leq \left(1 + \frac{1}{\delta_z} P_z^{(M)}(\varrho_x)(x)\right)\omega_1(\kappa; \delta_z), \tag{4}$$

which $\varrho_x(t) = |t - x|$. So, it is enough to estimate

$$E_z(x) = P_z^{(M)}(\varrho)(x) = \frac{\bigvee_{m=0}^{z} p_{z,m}(x)\kappa\left|\frac{m+\rho}{z+\theta+1} - x\right|}{\bigvee_{m=0}^{z} p_{z,m}(x)}.$$

Let $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$ where $\varpi \in \{0,1,...,z\}$ is fixed, arbitrary. By Lemma 5 we easily obtain

$$E_z(x) = \max_{m=0,1,\ldots,z}\{N_{m,z,\varpi}(x)\}, x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right].$$

Now, we can assume $\varpi = 0,1,...,z$, since simple calculation for $\varpi = 0$ shows that in this case we get $E_z(x) \leq \frac{1}{z+\theta+1}$, for all $x \in \left[0, \frac{1}{z+\theta+1}\right]$. Consequently, an upper estimate for each $N_{m,z,\varpi}(x)$ must still be obtained, when $m = 0,1,...$ and $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$. Actually, we will demonstrate that

$$N_{m,z,\varpi}(x) \leq 6\frac{\sqrt{x+\rho}}{z+\theta+1}, \tag{5}$$

which immediately implies that

$$E_z(x) \leq 6\frac{\sqrt{x+\rho}}{z+\theta+1},\ x \in \left[0, \frac{1}{z+\theta+1}\right],$$

and taking $\delta_z = 6\frac{\sqrt{x+\rho}}{z+\theta+1}$ in (4) we immediately get the estimate in the statement. To demonstrate (5), we consider the subsequent circumstances: i)$m \in \{\varpi-1, \varpi, \varpi+1\}$; ii)$m \geq \varpi+2$ and iii)$m \leq \varpi-2$

Case i). If $m = \varpi$, then $N_{\varpi,z,\varpi}(x) = \left|\frac{\varpi+\rho}{z+\theta} - x\right|$. Since $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$ it means that $N_{\varpi,z,\varpi}(x) \leq \frac{1}{z+\theta+1}$.

If $m = \varpi+1$, then $N_{\varpi+1,z,\varpi}(x) = n_{\varpi+1,z,\varpi}(x)\left(\frac{\varpi+\rho+1}{z+\theta} - x\right)$. Since by Lemma 3 we have $n_{\varpi+1,z,\varpi}(x) \leq 1$, we obtain $N_{\varpi+1,z,\varpi}(x) \leq \frac{\varpi+\rho+1}{z+\theta} - x \leq \frac{\varpi+\rho+1}{z+\theta} - \frac{\varpi+\rho}{z+\theta+1} \leq \frac{3}{z+\theta+1}$.

If $m = \varpi-1$, then $N_{\varpi-1,z,\varpi}(x) = n_{\varpi-1,z,\varpi}(x)\left(x - \frac{\varpi+\rho-1}{z+\theta}\right) \leq \frac{\varpi+\rho+1}{z+\theta+1} - \frac{\varpi+\rho-1}{z+\theta} \leq \frac{2}{z+\theta+1}$

Case ii). Subcase a) Suppose first that $m + \sqrt{m+1} \leq \varpi$, we get

$$\begin{aligned}\overline{N}_{m,z,\varpi}(x) &= n_{m,z,\varpi}(x)\left(\frac{m+\rho}{z+\theta+1} - x\right) \leq \frac{m+\rho}{z+\theta+1} - x \\ &\leq \frac{m+\rho}{z+\theta+1} - \frac{\varpi+\rho}{z+\theta+1} \leq \frac{m+\rho}{z+\theta+1} - \frac{m+\sqrt{m+1}}{z+\theta+1} \\ &= \frac{\sqrt{m+1}}{z+\theta+1} \leq \frac{1}{\sqrt{z+\theta+1}}.\end{aligned}$$

Subcase b) Assume that $m - \sqrt{m+1} \geq \varpi$. Since the function $\vartheta(x) = x - \sqrt{x+1}$ is nondecreasing on $x \in [0,1]$ it follows that there exists $\overline{m} \in \{0,...,z\}$,of maximum value, such that $\overline{m} - \sqrt{\overline{m}+1} < \varpi$. Then for $m_1 = \overline{m}+1$ we obtain $m_1 - \sqrt{m_1+1} \geq$

ϖ and

$$\overline{N}_{\overline{m},z,\varpi}(x) = n_{\overline{m},z,\varpi}(x)\left(\frac{\overline{m}+\rho+1}{z+\theta+1} - x\right) \leq \frac{\overline{m}+\rho+1}{z+\theta+1} - x$$
$$\leq \frac{\overline{m}+\rho+1}{z+\theta+1} - \frac{\varpi+\rho}{z+\theta+1} \leq \frac{\overline{m}+\rho+1}{z+\theta+1} - \frac{\overline{m}-\sqrt{\overline{m}+1}}{z+\theta+1}$$
$$= \frac{\sqrt{\overline{m+1}}+1}{z+\theta+1} \leq \frac{2}{\sqrt{z+\theta+1}}.$$

Also, we have $m_1 \geq \varpi + 2$. Indeed, this is a consequence of the fact that ϑ is nondecreasing on the interval $x \in [0,1]$ and because it is easy to see that $\vartheta(\varpi+1) \leq \varpi$. By Lemma 4 it follows that $\overline{N}_{\overline{m}+1,z,\varpi}(x) \geq \overline{N}_{\overline{m}+2,z,\varpi}(x) \geq ... \geq \overline{N}_{z,z,\varpi}$. We thus obtain $\overline{N}_{\overline{m},z,\varpi}(x) = \frac{2}{\sqrt{z+\theta+1}}$ for any $m \in \{\overline{m}+1, \overline{m}+2, ..., z\}$. Therefore, in both subcases, we get $N_{m,z,\varpi}(x) \leq \frac{6}{\sqrt{z+\theta+1}}$.

Case iii). Subcase a) Assume first that $m + \sqrt{m} \geq \varpi$. Then we get

$$\underline{N}_{m,z,\varpi}(x) = n_{m,z,\varpi}(x)\left(x - \frac{m+\rho}{z+\theta+1}\right)$$
$$\leq x - \frac{m+\rho}{z+\theta+1} \leq \frac{\varpi+\rho+1}{z+\theta+1} - \frac{m+\rho}{z+\theta+1}$$
$$\leq \frac{m+\sqrt{m}+\rho+1}{z+\theta+1} - \frac{m+\rho}{z+\theta+1}$$
$$= \frac{\sqrt{m}+\rho+1}{z+\theta+1} \leq \frac{2}{\sqrt{z+\theta+1}}.$$

Subcase b) Assume now that $m + \sqrt{m} \leq \varpi$. Let $\underline{m} \in \{0, ..., z\}$ be the minimum value such that $\underline{m} - \sqrt{\underline{m}} < \varpi$. Then $m_2 = \underline{m} - 1$ satisfies $m_2 - \sqrt{m_2} \geq \varpi$ and

$$\underline{N}_{\underline{m},z,\varpi}(x) = n_{\underline{m},z,\varpi}(x)\left(x - \frac{\underline{m}+\rho-1}{z+\theta+1}\right)$$
$$\leq x - \frac{\underline{m}+\rho+1}{z+\theta+1} \leq \frac{\varpi+\rho+1}{z+\theta+1} - \frac{\underline{m}+\rho-1}{z+\theta+1}$$
$$\leq \frac{\underline{m}+\sqrt{\underline{m}+1}}{z+\theta+1} - \frac{\underline{m}+\rho-1}{z+\theta+1} = \frac{\sqrt{\underline{m}}+2+\rho}{z+\theta+1} \leq \frac{3}{\sqrt{z+\theta+1}}.$$

Additionally, since in this case we have $\varpi \geq 2$ it is immediate that $m_2 \geq \varpi - 2$. By Lemma 4, it follows that $\underline{N}_{\underline{m}-1,z,\varpi}(x) \geq \underline{N}_{\underline{m}-2,z,\varpi}(x) \geq ... \geq \underline{N}_{0,z,\varpi}$. We obtain $\underline{N}_{m,z,\varpi}(x) \leq \frac{3}{\sqrt{z+\theta+1}}$ for $x \in \left[\frac{\varpi+\rho}{z+\theta+1}, \frac{\varpi+\rho+1}{z+\theta+1}\right]$ and for any $m \leq \varpi - 2$. In both subcases, we get $N_{m,z,\varpi}(x) \leq \frac{3}{\sqrt{z+\theta+1}}$.

As a result, by collecting all the predictions in the above cases and subcases, we easily obtain the relation (5) that completes the proof.

□

3. Nonlinear Max-Product Type Bivariate Bernstein-Stancu Operators

We introduce nonlinear bivariate Bernstein-Stancu operators of max-product type in this section.

Let us $\kappa : [0,1] \times [0,1] \to \mathbb{R}^+$ be a continuous function and for $k = 1, 2$, $\rho_k, \theta_k \in \mathbb{R}_+$, with $0 \le \rho_k \le \theta_k$. Then the nonlinear maximum product type of bivariate Bernstein-Stancu operators is defined as follows:

$$P^{(M)}_{z,h,\varpi_k,\iota_k}(\kappa : x, y) = \frac{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y) \kappa\left(\frac{m+\rho_1}{z+\theta_1}, \frac{j+\rho_2}{h+\theta_2}\right)}{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)}, \tag{6}$$

with

$$p_{z,m}(x) = C_z^m x^m (1-x)^{z-m} \text{ and } p_{h,j}(y) = C_h^j y^j (1-y)^{h-j}, \tag{7}$$

for all $x, y \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi, \iota \in \mathbb{N}$.

The subsequent sections provide an error estimate the nonlinear maximum product type of bivariate Bernstein-Stancu operators in terms of modulus of continuity, along with some features of the $P^{(M)}_{z,h,\varpi_k,\iota_k}$ operators.

We require the following notations and auxiliary results for the main result proofs. Now, some definitions for the x and y variables and lemmas will be given.

For each $m = \{0, 1, ..., z\}$, $j = \{0, 1, ..., h\}$ and $x, y \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi, \iota \in \mathbb{N}$, let us denote

$$N_{m,z,\varpi}(x) = \frac{p_{z,m}(x)\left|\frac{m+\rho_1}{z+\theta_1} - x\right|}{p_{z,\varpi}(x)}, \qquad n_{m,z,\varpi}(x) = \frac{p_{z,m}(x)}{p_{z,\varpi}(x)}$$

$$N_{j,h,\iota}(y) = \frac{p_{h,j}(y)\left|\frac{j+\rho_2}{h+\theta_2} - y\right|}{p_{h,\iota}(y)}, \qquad n_{j,h,\iota}(y) = \frac{p_{h,j}(y)}{p_{h,\iota}(y)}.$$

Let $x, y \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi, \iota \in \mathbb{N}$, $m = \{0, 1, ..., z\}$, $j = \{0, 1, ..., h\}$, and for $k = 1, 2$, $\rho_k, \theta_k \in \mathbb{R}_+$, with $0 \le \rho_k \le \theta_k$. Hence, it is obvious that

i) If $\varpi + 1 \le m$, then $N_{m,z,\varpi}(x) = \frac{p_{z,m}(x)(\frac{m+\rho_1}{z+\theta_1} - x)}{p_{z,\varpi}(x)}$,

ii) If $m \le \varpi - 1$, then $N_{m,z,\varpi}(x) = \frac{p_{z,m}(x)(x - \frac{m+\rho_1}{z+\theta_1})}{p_{z,\varpi}(x)}$,

iii) If $\iota + 1 \le j$, then $N_{j,h,\iota}(y) = \frac{p_{h,j}(y)(\frac{j+\rho_2}{h+\theta_2} - y)}{p_{h,\iota}(y)}$,

iv) If $j \le \iota - 1$, then $N_{j,h,\iota}(y) = \frac{p_{h,j}(y)(y - \frac{j+\rho_2}{h+\theta_2})}{p_{h,\iota}(y)}$.

Additionally, let $x, y \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi, \iota \in \mathbb{N}$, $m = \{0,1,...,z\}$, $j = \{0,1,...,h\}$, and for $k = 1,2$, $\rho_k, \theta_k \in \mathbb{R}_+$, with $0 \leq \rho_k \leq \theta_k$. Then, we can denote

$$\underline{N}_{m,z,\varpi}(x) = \frac{p_{z,m}(x)(x - \frac{m+\rho_1}{z+\theta_1+1})}{p_{z,\varpi}(x)}, \text{ for } m \leq \varpi - 2$$

$$\underline{N}_{j,h,\iota}(y) = \frac{p_{h,j}(y)(y - \frac{j+\rho_2}{h+\theta_2+1})}{p_{h,\iota}(y)}, \text{ for } j \leq \iota - 2$$

Lemma 6. *Let* $x, y \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi, \iota \in \mathbb{N}$, *and for* $k = 1,2$, $\rho_k, \theta_k \in \mathbb{R}_+$, *with* $0 \leq \rho_k \leq \theta_k$.

i) *For all* $m, \varpi = \{0,1,...,z\}$ *and* $\varpi+2 \leq m$, *we have* $\overline{N}_{m,z,\varpi}(x) \leq N_{m,z,\varpi}(x) \leq 3\overline{N}_{m,z,\varpi}(x)$.
ii) *For all* $j, \iota = \{0,1,...,h\}$ *and* $\iota + 2 \leq j$, *we have* $\overline{N}_{j,h,\iota}(y) \leq N_{j,h,\iota}(y) \leq 3\overline{N}_{j,h,\iota}(y)$.
iii) *For all* $m, \varpi = \{0,1,...,z\}$ *and* $m \leq \varpi - 2$, *we have* $N_{m,z,\varpi}(x) \leq \underline{N}_{m,z,\varpi}(x) \leq 6N_{m,z,\varpi}(x)$
iv) *For all* $j, \iota = \{0,1,...,h\}$ *and* $\iota + 2 \leq j$, *we have* $j \leq \iota - 2$, $N_{j,h,\iota}(y) \leq \underline{N}_{j,h,\iota}(y) \leq 6N_{j,h,\iota}(y)$

The proof is in a similar way to the univariate given in Lemma 2.

Lemma 7. *For all* $m, \varpi = \{0,1,...,z\}$, $j, \iota = \{0,1,...,h\}$ *and* $x, y \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, *we have*

$$n_{m,z,\varpi}(x) \leq 1 \text{ and } n_{j,h,\iota}(y) \leq 1. \tag{8}$$

The proof is in a similar way to the univariate given in Lemma 3.

Lemma 8. *Let* $p_{z,m}(x)$ *and* $p_{h,j}(y)$ *defined as given in (7). Then we have*

$$\bigvee_{m=0}^{z} p_{z,m}(x). \bigvee_{j=0}^{h} p_{h,j}(y) = p_{z,\varpi}(x).p_{h,\iota}(y),$$

for all $x, y \in [0,1]^2$ *ve* $(x,y) \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi = 0,...z$, $\iota = 0,...h$.

Proof. Since we have $\bigvee_{m=0}^{z} p_{z,m}(x) > 0$, $\bigvee_{j=0}^{h} p_{h,j}(y) > 0$ for all $x, y \in [0,1]$ Firstly, we claim that for $\varpi = 0,...z, \iota = 0,...h$, we have

$$0 \leq p_{z,m+1}(x) \leq p_{z,m}(x),$$
$$0 \leq p_{h,j+1}(y) \leq p_{h,j}(y),$$

if and only if $x \in \left[0, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right]$ and $y \in \left[0, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$. Writing the following inequality is simple:

$$0 \leq \binom{z}{\varpi+1} x^{\varpi+1}(1-x)^{z-\varpi-1} \leq \binom{z}{\varpi} x^{\varpi}(1-x)^{z-\varpi},$$

$$0 \leq \binom{h}{\iota+1} y^{\iota+1}(1-y)^{h-\iota-1} \leq \binom{h}{\iota} y^{\iota}(1-y)^{h-\iota}$$

and after simplifcation,

$$0 \leq x\left[\binom{z}{\varpi+1} + \binom{z}{\varpi}\right] \leq \binom{z}{\varpi},$$

$$0 \leq y\left[\binom{z}{\iota+1} + \binom{h}{\iota}\right] \leq \binom{h}{\iota}.$$

From the equality $\binom{z}{\varpi+1} + \binom{z}{\varpi} = \binom{z+1}{\varpi+1}$, we get $0 \leq x \leq \frac{\varpi+\rho_1+1}{z+\theta 1+1}$, $0 \leq y \leq \frac{\iota+\rho_2+1}{h+\theta_2+1}$. Also, denoting

$$A_{m,z,\varpi}(x) = \frac{p_{z,m}(x)}{p_{z,\varpi}(x)} = \frac{\binom{z}{m}}{\binom{z}{\varpi}} \left(\frac{x}{1-x}\right)^{m-\varpi}, \tag{9}$$

$$A_{j,h,\iota}(y) = \frac{p_{h,m}(y)}{p_{h,\iota}(y)} = \frac{\binom{h}{j}}{\binom{h}{\iota}} \left(\frac{y}{1-y}\right)^{j-\iota}, \tag{10}$$

so, we can write $A_{m,z,\varpi,j,h,\iota}(x,y) = A_{m,z,\varpi}(x).A_{j,h,\iota}(y)$. Therefore, we can use the following formula to prove the approximation results

$$P^{(M)}_{z,h,\varpi_k,\iota_k}(\kappa : x,y) = \bigvee_{m=0}^{z} \bigvee_{j=0}^{h} p_{h,j}(y) A_{m,z,\varpi,j,h,\iota}(x,y) \kappa\left(\frac{m+\rho_1}{z+\theta_1}, \frac{j+\rho_2}{h+\theta_2}\right), \tag{11}$$

for all $(x,y) \in \left[\frac{\varpi+\rho_1}{z+\theta_1+1}, \frac{\varpi+\rho_1+1}{z+\theta_1+1}\right] \times \left[\frac{\iota+\rho_2}{h+\theta_2+1}, \frac{\iota+\rho_2+1}{h+\theta_2+1}\right]$, $\varpi = 0, ...z$, $\iota = 0, ...h$.

It easily follows that we can write

$$P^{(M)}_{z,h,\varpi_k,\iota_k}(\kappa : x,y) = P^{(M)}_{z,\varpi_k,x}\left[P^{(M)(\kappa)}_{h,\iota_k,y}\right](x,y)$$

where, if $F = F(x,y)$ then the notations $P^{(M)}_{z,x}(F)$ means that the univariate max-product Bernstein operator $P^{(M)}_{z}(F)$ is applied to F considered as function of x while $P^{(M)}_{z,y}(F)$ means that the univariate max-product Bernstein operator $P^{(M)}_{z}(F)$ is applied to F considered as function of y. In other words, the bivariate max-product Bernstein operators are tensor products of the univariate max- product Bernstein operators.

□

Definition 1. *Suppose that* $\kappa : I : [0,1] \times [0,1] \to \mathbb{R}^+$ *olmak üzere,*

(i) *Let for all* $y, x, x+\phi \in [0,1], \phi > 0$, $\kappa(x+\phi, y) - \kappa(x,y) \geq 0(resp., \leq 0)$. *Then, the function* κ *is increasing with respect to* x *on* I *(resp., decreasing).*

(ii) *Let for all* $x, y, y+\Phi \in [0,1], \Phi > 0,\ \kappa(x, y+\Phi) - \kappa(x, y) \geq 0(resp., \leq 0)$. *Ten, the function* κ *is increasing with respect to* y *on* I *(resp., decreasing).*

(iii) *Let for all* $x, x+\phi, y, y+\Phi \in [0,1], \phi, \Phi > 0\ \Delta_2\kappa(x,y) = \kappa(x+phi, y+\Phi) - \kappa(x, y+\Phi) - \kappa(x+\phi, y) + \kappa(x,y) \geq 0(resp., \leq 0)$ *Ten, the function* κ *is upper bidimensional monotone on* I *(resp., lower bidimensional monotone). (Bede, Coroianu ve Gal, 2016).*

Theorem 2. *Let* $\kappa : [0,1] \times [0,1] \to \mathbb{R}_+$ *be a continuous function. We have*

$$|P_{z,h}^{(M)}(\kappa)(x,y) - \kappa(x,y)| \leq 18\omega_1\left(\kappa; \frac{\sqrt{x+\rho_1}}{\sqrt{z+\theta_1+1}}, \frac{\sqrt{y+\rho_2}}{\sqrt{h+\theta_2+1}}\right),$$

for all $x, y \in [0,1]$ *and* $z, h \in \mathbb{N}$. *Here*

$$\omega_1(\kappa;\gamma,\delta) = \sup\{|\ \kappa)(x,y) - \kappa)(z,t)\ |; x,y,z,t[0,1]\times[0,1], |\ x-z\ |\leq\gamma, |\ y-t\ |\leq\delta\}.$$

Proof. Taking into account the inequality valid for the positive numbers A_k, B_k, $k \in \{0,1,...,s\}$,

$$|max_{k\in\{0,1,\ldots,s\}}\{A_k\} - max_{k\in\{0,1,\ldots,s\}}\{B_k\}| \leq max_{k\in\{0,1,\ldots,s\}}\{|A_k - B_k|\}$$

we obtain

$$\begin{aligned}
&|P_{z,h}^{(M)}(\kappa)(x,y) - \kappa(x,y)| \\
&= \left| \frac{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)\kappa\left(\frac{m+\rho_1}{z+\theta_1}, \frac{j+\rho_2}{h+\theta_2}\right)}{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)} - \frac{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)\kappa(x,y)}{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)} \right| \\
&\leq \frac{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y) \left|\kappa\left(\frac{m+\rho_1}{z+\theta_1}, \frac{j+\rho_2}{h+\theta_2}\right) - \kappa(x,y)\right|}{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)} \\
&\leq \frac{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)\omega_1\left(\kappa; |m+\rho_1/z+\theta_1-x|, |j+\rho_2/h+\theta_2-y)\,|\right)}{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)} \\
&= \frac{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)\omega_1\left(\kappa; \delta\frac{|m+\rho_1/z+\theta_1-x|}{\delta}, \nu\frac{|j+\rho_2/h+\theta_2-y|)}{\nu}\right)}{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)}
\end{aligned}$$

$$= \frac{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)\omega_1 \left(1 + \frac{|m+\rho_1/z+\theta_1-x|}{\delta} + \frac{|j+\rho_2/h+\theta_2-y|)}{\nu}\right)\omega_1(\kappa,\delta,\nu)}{\bigvee_{m=0}^{z} p_{z,m}(x) \bigvee_{j=0}^{h} p_{h,j}(y)}$$

$$= \omega_1(\kappa,\delta,\nu)\left(1 + \frac{1}{\delta}\frac{\bigvee_{m=0}^{z} p_{z,m}(x)\left(\frac{m+\rho_1}{z+\theta_1-x}\right)}{\bigvee_{m=0}^{z} p_{z,m}(x)} + + \frac{1}{\nu}\frac{\bigvee_{j=0}^{h} p_{h,j}(y)\left(\frac{m+\rho_2}{h+\theta_2-y}\right)}{\bigvee_{j=0}^{h} p_{h,j}(y)}\right)$$

Burada, $\delta = \frac{6\sqrt{x+\rho_1}}{\sqrt{z+\theta_1+1}}$ ve $\nu = \frac{6\sqrt{y+\rho_2}}{\sqrt{h+\theta_2+1}}$

$$\begin{aligned} |P_{z,h}^{(M)}(\kappa)(x,y) - \kappa(x,y)| &\leq 3\omega_1\left(\kappa; \frac{6\sqrt{x+\rho_1}}{\sqrt{z+\theta_1+1}}, \frac{6\sqrt{y+\rho_2}}{\sqrt{h+\theta_2+1}}\right) \\ &\leq 18\omega_1\left(\kappa; \frac{\sqrt{x+\rho_1}}{\sqrt{z+\theta_1+1}}, \frac{\sqrt{y+\rho_2}}{\sqrt{h+\theta_2+1}}\right) \end{aligned}$$

□

Author Contribution Statements All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Declaration of Competing Interests Authors state no conflict of interest.

Acknowledgements This article is extracted from A. K. Yeşilnacar Binmar's master thesis ([22]) dissertation entitled "Aproximation properties two biviariate maximum product type operators", supervised by Sevilay Kırcı Serenbay (Master's Thesis Dissertation, Harran University, Şanlıurfa, Türkiye, 2023).

References

[1] Altomare, F., Campiti, M., Korovkin-Type Approximation Theory and Its Applications, Walter de Gruyter, Berlin, 1994.

[2] Korovkin, P. P., Linear Operators and Approximation Theory, Hindustan Publ. Corp., India, 1960.

[3] Stancu, D. D., Asupra unei generalizări a polinoamelor lui Bernstein, *Studia Universitatis Babeş-Bolyai*, 14(2) (1969), 31-45 (in Romanian).

[4] Bede, B., Coroianu, L., Gal, S. G., Approximation and shape preserving properties of the Bernstein operator of max-product kind, Intern, *J. Math. and Math. Sci.*, 26 pages (2009). doi:10.1155/2009/590589

[5] Bede, B., Gal, S. G., Approximation by nonlinear Bernstein and Favard-Szasz- Mirakjan operators of max-product kind, *Journal of Concrete and Applicable Mathematics*, 8(2) (2010), 193-207.

[6] Bede, B., Coroianu, L., Gal, S. G., Approximation by Max-Product Type Operators, Heidelberg, Springer, 2016.

[7] Coroianu, L., Gal, S. G,. Approximation by nonlinear generalized sampling operators of max-product kind, *Sampl. Theory Signal Image Process*, 9 (2010), 59-75. https://doi.org/10.1007/BF03549524

[8] Coroianu, L., Gal, S. G., Approximation by max-product sampling operators based on sinc-type kernels, *Sampl. Theory Signal Image Process*, 10 (2011), 211-230. https://doi.org/10.1007/BF03549542

[9] Hildebrandt, T. H., Schoenberg, I. J., On linear functional operations and the moment problem, *Ann. Math.*, 34(2) (1933), 317-328.

[10] Butzer, P. L., On two-dimensional Bernstein polynomials, *Can. J. Math.*, 5 (1953), 107-113.

[11] Martinez, F. L., Some properties of two-dimensional Bernstein polynomials, *Journal of approximation theory*, 59(3) (1989), 300-306. https://doi.org/10.1016/0021-9045(89)90095-6

[12] Kırcı Serenbay, S., Yavuz, H., Approximation Of Modified Bernstein-Stancu Operators Of Maximum-Product Type, presented at the İzdaş Kongre, Ankara, Turkey, 2021.

[13] Acar, E., Kırcı Serenbay, S., Approximation by nonlinear q-Bernstein-Chlodowsky operators, *TWMS J. App. and Eng. Math.*, 14(1) (2024), 42-51.

[14] Acar, E., Özalp Guller, Ö., Kırcı Serenbay, S., Approximation by nonlinear Meyer-König and Zeller operators based on q-integers, *International Journal of Mathematics and Computer in Engineering*, 2(2) (2024), 71-82.

[15] Acar, E., Kırcı Serenbay, S., Özalp Guller, Ö., Approximation by nonlinear Bernstein-Chlodowsky operators of Kantorovich type, *Filomat*, 37(14) (2023), 4621-4627. https://doi.org/10.2298/FIL2314621A

[16] Özalp Guller, Ö., Acar, E., Kırcı Serenbay, S., Nonlinear bivariate Bernstein-Chlodowsky operators of maximum product type, *Journal of Mathematics*, (2022). https://doi.org/10.1155/2022/4742433

[17] Acar, E., Holhoş, A., Kırcı Serenbay, S., Polynomial weighted approximation by Szasz-Mirakyan operators of max-product type, *Kragujevac Journal of Mathematics*, 49(3) (2022), 365-373. 10.46793/KgJMat2503.365A

[18] Gairola, A. R., Singh, A., Rathour, L., Mishra, V. N., Improved rate of approximation by modification of Baskakov operator, *Operators and Matrices*, 16(4), (2022), 1097-1123. dx.doi.org/10.7153/oam-2022-16-72

[19] Gairola, A. R., Maindola, S., Rathour, L., Mishra, L. N., Mishra, V. N., Better uniform approximation by new Bivariate Bernstein Operators, *International Journal of Analysis and Applications*, 20(60) (2022), 1-19. https://doi.org/10.28924/2291-8639-20-2022-60

[20] Gairola, A. R., Bisht, N., Rathour, L., Mishra, L. N., Mishra, V. N., Order of approximation by a new univariate Kantorovich Type Operator, *International Journal of Analysis and Applications*, 21 (2023), 1-17. https://doi.org/10.28924/2291-8639-21-2023-106

[21] Mishra, V. N., Khatri, K., Mishra, L. N., Deepmala, Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators, *Journal of Inequalities and Applications*, 586 (2013). https://doi.org/10.1186/1029-242X-2013-586

[22] Yeşilnacar Binmar, A. K., Aproximation properties two biviariate maximum product type operators, Master Thesis, Harran University, Şanlıurfa, Türkiye, 2023.