Turk. J. Math. Comput. Sci.
16(2)(2024) 426–437
© MatDer
DOI : 10.47000/tjmcs.1452547

On Fixed Point Results for Some Multi-valued Operators with respect to Extended wt-Distance on Extended b-Metric Spaces

Irem Eroğlu

Department of Mathematics, Faculty of Science and Arts, Ordu University, 52200, Ordu, Turkey.

Received: 13-03-2024 • Accepted: 25-10-2024

Abstract. In this paper, we prove some new fixed point results for multi-valued mappings via extended wt-distance on complete extended b-metric spaces. We obtain a generalization of Feng and Liu [8] fixed point theorem on multi-valued operators. Moreover, we give a homotopy application to show the applicability of our results.

2020 AMS Classification: 47H09, 47H10, 54H25

Keywords: Extended b-metric, contraction, extended wt-distance, wt-distance, fixed point, multi-valued mapping.

1. Introduction and Preliminaries

The notion of b-metric space was introduced by Bakhtin [4] and later on studied and developed by many researchers [3, 5, 13, 15, 17–20, 24]. The concept of b-metric space was given by the following:

Definition 1.1 ([4]). Let X be a non empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if it satisfies the following properties for each $x, y, z \in X$:

(i) $d(x, y) = 0 \Leftrightarrow x = y$
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, z) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d, s) is called a b-metric space and every metric space is a b-metric space, where $s = 1$. Morever, each b-metric d induces a topology τ_d such that for each open set G of τ_d and for each $x \in G$ there is an $r > 0$ such that $x \in B_d(x, r) \subseteq G$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$. Contrary to the metric case, b-metrics are not necessarily continuous functions and the set $B_d(x, r)$ is not necessarily τ_d-open set. Hussain et al [10] extended the concept of w-distance given by Kada [11] to the b-metric spaces and introduced the following concept of wt-distance.

Definition 1.2 ([10]). Let (X, d, s) be a b-metric space with $s \geq 1$. Then, a function $p : X \times X \to [0, \infty)$ is called a wt-distance on X if the following conditions are satisfied:

(i) $p(x, z) \leq s[p(x, y) + p(y, z)]$
(ii) for any $x \in X$, $p(x, .) : X \to [0, \infty)$ is s-lower semi-continuous
(iii) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

The notion of s-lower semi-continuity means that if either $\lim_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leq \lim_{x_n \to x_0} sf(x_n)$, whenever $x_n \in X$ for each $n \in \mathbb{N}$ and $x_n \to x_0$.

Email address: iremeroglu@odu.edu.tr (I. Eroğlu)

Demma et al [6] proved some fixed point theorems for multi-valued operators by using wt-distance in the complete b-metric spaces and they gave the following result which is a generalization of Feng and Liu [8] on multi-valued operators.

According to [6], let (X, d, s) be a b-metric space and p be a wt-distance on X. For a positive constant $b \in (0, 1)$ define the set I_b^x as follows:

$$I_b^x = \{y \in Tx : bp(x, y) \leq p(x, Tx)\}.$$

Theorem 1.3 ([6]). *Let (X, d, s) be a complete b-metric space. Let $C(X)$ be the set of all nonempty closed subsets of X and $T : X \to C(X)$ a multi-valued operator, $p : X \times X \to [0, \infty)$ a wt-distance on X and $b \in (0, 1)$. Suppose that there exits, $c \in (0, 1)$ with $cb^{-1} \in [0, s^{-1})$, such that for any $x \in X$ there is $y \in I_b^x$ satisfying*

$$p(y, Ty) \leq cp(x, y).$$

If one of the following assertions holds:

(i) *$p(x, Tx) = 0$ if there exists a sequence $(x_n) \subset X$ such that $p(x_n, Tx_n) \to 0$, as $n \to \infty$;*
(ii) *the function $f : X \to \mathbb{R}$ as $f(x) = p(x, T(x))$ is s-lower semi-continuous;*
(iii) *for every $y \in X$ with $y \notin T(y)$, we have*

$$\inf_{x \in X}\{p(x, y) + p(x, Tx)\} > 0$$

(iv) *T is a closed operator,*

then T has a fixed point in X.

In [12], Kamran et al. introduced the concept of extended b-metric space as follows:

Definition 1.4 ([12]). Let X be a nonempty set and $\theta : X \times X \to [1, \infty)$. The function $d_\theta : X \times X \to [0, \infty)$ is called an extended b-metric space if for all $x, y, z \in X$, it satisfies:

(i) $d_\theta(x, y) = 0 \Leftrightarrow x = y$
(ii) $d_\theta(x, y) = d_\theta(y, x)$
(iii) $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, z) + d_\theta(z, y)]$.

The pair (X, d_θ) is called extended b-metric space. The definitions of convergence, Cauchy sequence and completeness was given in [12] as follows:

Definition 1.5 ([12]). Let (X, d_θ) be an extended b-metric space.

(i) A sequence $(x_n) \in X$ is said to converge to $x \in X$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_n, x) < \epsilon$ for all $n \geq N$.
(ii) A sequence $(x_n) \in X$ is said to be Cauchy, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_m, x_n) < \epsilon$ for all $m, n \geq \mathbb{N}$.

An extended b-metric space (X, d_θ) is complete if every Cauchy sequence in X is convergent.

In [2], Alamgir et al. generalized Nadler's fixed point theorem by using extended b-metric spaces. Morever, they obtained Mizoguchi-Takahashi's type fixed point theorem for a multi-valued mapping from a complete extended b-metric space X into the non empty closed and bounded subsets of X. The aformentioned main result is as follows:

Theorem 1.6 ([2]). *Let us consider a multi-valued mapping $T : X \to CB(X)$, where (X, d_θ) is a complete extended b-metric space and $CB(X)$ denotes all nonempty closed and bounded subsets of X. Furthermore, let us consider that the following two conditions hold:*

(i) *The mapping $f : X \to \mathbb{R}$ defined by $f(x) = d_\theta(x, Tx)$, $x \in X$, is lower semi-continuous.*
(ii) *There exists $b, c \in (0, 1), b < c$ such that for all $x \in X$ there exists $y \in I_c^x$ satisfying*

$$d_\theta(y, Ty) \leq bd_\theta(x, y).$$

Morever, for each $x_0 \in X$, $lim_{m,n\to\infty}\theta(x_n, x_m) < \frac{1}{\alpha}$ for all $\alpha \in (0, 1)$. Then, T has a fixed point in X.

Morever, they extended Mizoguchi-Takahashi's [14] type fixed point theorem to the concept of extended b-metric spaces by the following result:

Theorem 1.7 ([14]). *Let (X, d_θ) be a complete extended b-metric space. Let us consider a multivalued mapping $T : X \to CB(X)$. Suppose that there exists $c \in (0, 1)$ and $\Omega : [0, \infty) \to [0, c)$ such that*

$$\limsup_{s \to t^+} \Omega(s) < c, \text{ for all } t \in [0, \infty)$$

and for all $x \in X$, there exists $y \in I_c^x$ satisfying

$$d_\theta(y, Ty) \le \Omega(d_\theta(x, y)) d_\theta(x, y)$$

and for each $x_0 \in X$, $\lim_{m,n\to\infty} \theta(x_n, x_m) < \frac{1}{\alpha}$ for any $\alpha \in (0, 1)$, where $x_n = T^n x_0$. If the map $f : X \to \mathbb{R}$ defined by $f(x) = d_\theta(x, Tx), x \in X$, is lower semi-continuous. Then, T has a fixed point in X.

In [7] extended wt-distance definition introduced as follows:

Definition 1.8 ([7]). Let (X, d_θ) be an extended b-metric space, where θ is a function such that $\theta : X \times X \to [1, \infty)$. A function $p_\theta : X \times X \to [0, \infty)$ is called extended wt-distance if the following are satisfied:

(i) $p_\theta(x, z) \le \theta(x, z)[p_\theta(x, y) + p_\theta(y, z)]$;
(ii) for any $x \in X$, $p_\theta(x, .) : X \to [0, \infty)$ is θ-lower semicontinuous;
(iii) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p_\theta(z, x) \le \delta$ and $p_\theta(z, y) \le \delta$ imply $d_\theta(x, y) \le \epsilon$.

The notion of θ-lower semi-continuity means that if either $\lim_{x_n \to x_0} p_\theta(x, x_n) = \infty$ or $p_\theta(x, x_0) \le \lim_{x_n \to x_0} \theta(x, x_0) p_\theta(x, x_n)$, whenever $x_n \in X$ for each $n \in \mathbb{N}$ and $x_n \to x_0$ according to τ_{d_θ} .

Example 1.9 ([7]). Let $X = [0, \infty)$. Define the mappings $\theta : X \times X \to [1, \infty)$ and $d_\theta : X \times X \to [0, \infty)$ as follows: $\theta(x, y) = 1 + x + y$ and

$$d_\theta(x, y) = \begin{cases} x^2 + y^2, & x, y \in X, x \ne y \\ 0, & x = y. \end{cases}$$

Then, (X, d_θ) is an extended b-metric space (see [9]). Let consider the mapping $p_\theta : X \times X \to [0, \infty)$ defined by $p_\theta(x, y) = y^2$. Then, p_θ is an extended wt-distance on (X, d_θ).

Example 1.10 ([7]). Let $X = [0, \infty)$. Define the mappings $\theta : X \times X \to [1, \infty)$ and $d_\theta : X \times X \to [0, \infty)$ as follows: $\theta(x, y) = 1 + x + y$ and

$$d_\theta(x, y) = \begin{cases} x + y, & x, y \in X, x \ne y \\ 0, & x = y \end{cases}$$

Then, (X, d_θ) is an extended b-metric space (see [9]). Let consider the mapping $p_\theta : X \times X \to [0, \infty)$ defined by $p_\theta(x, y) = y$. It is clear that p_θ is an extended wt-distance on (X, d_θ).

Morever, the following fundemental lemma was given in [7].

Lemma 1.11 ([7]). *Let (X, d_θ) be an extended b-metric space and p_θ be an extended wt-distance on X. Let (x_n) and (y_n) be sequences in X, let (α_n) and (β_n) be sequences in $[0, \infty)$ converging to zero and let $x, y, z \in X$. Then, the following hold:*

(i) *If $p_\theta(x_n, y) \le \alpha_n$ and $p_\theta(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p_\theta(x, y) = 0$ and $p_\theta(x, z) = 0$, then $y = z$.*
(ii) *If $p_\theta(x_n, y_n) \le \alpha_n$ and $p_\theta(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $d_\theta(y_n, z) \to 0$*
(iii) *If $p_\theta(x_n, x_m) \le \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.*
(iv) *If $p_\theta(y, x_n) \le \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.*

Throughout this paper, we denote by $C(X)$ the set of all nonempty and closed subsets of X, by $CB(X)$ the set of all nonempty closed and bounded subsets of X and by $K(X)$ nonempty compact subsets of X, where (X, d_θ) is an extended b-metric space.

In this paper, we will extend some results in [2] and [6] to the setting of extended b-metric spaces endowed with an extended wt-distance and we obtain generalizations of Theorem 1.3 and Theorem 1.6 . Morever, we present an application of our results to homotopy theory.

2. Main Results

In this section, we start with the following lemma which is fundemental in some of our results. In the following lemma, we extend Lemma 2.2 in [22] to the concept of extended wt-distance.

Lemma 2.1. *Let (X, d_θ) be an extended b-metric space and F be a closed subset of X. Let p_θ be an extended wt-distance on X. Suppose that there exists $a \in X$ such that $p_\theta(a, a) = 0$. Then, $a \in F$ iff $p_\theta(a, F) = 0$.*

Proof. Let $a \in F$ such that $p_\theta(a, a) = 0$. Then, $p_\theta(a, F) = inf\{p_\theta(a, x) : x \in F\} = 0$. Conversely, let $p_\theta(a, F) = 0$. Then, there exists a sequence $(x_n) \subseteq X$ such that $p_\theta(a, x_n) \to 0$, as $n \to \infty$. Since $p_\theta(a, a) = 0$, by Lemma 1.11, we get $x_n \to a$. Since F is closed, we obtain that $a \in F$. □

Theorem 2.2. *Let (X, d_θ) be a complete extended b-metric space and p_θ be an extended wt-distance on X. Let us consider a multi-valued mapping $T : X \to CB(X)$ such that the following two conditions hold:*

(i) *The mapping $f : X \to \mathbb{R}$ defined by $f(x) = p_\theta(x, Tx)$, $x \in X$, is lower semi-continuous.*
(ii) *There exists $b, c \in (0, 1), b < c$ such that for all $x \in X$ there exists $y \in I_c^x$ satisfying*

$$p_\theta(y, Ty) \leq b p_\theta(x, y)$$

for each $x_0 \in X$, $lim_{m,n\to\infty}\theta(x_n, x_m) < \frac{1}{\alpha}$ for all $\alpha \in (0, 1)$, where $x_n \in T^n x_0$.

Then, there exists $z \in X$ such that $f(z) = 0$. Further, if $p_\theta(z, z) = 0$, then z is a fixed point of T.

Proof. As $Tx \in CB(X)$ for any $x \in X$ and X is complete, I_c^x is nonempty for any constant $c \in (0, 1)$. For some arbitray initial point $x_0 \in X$, there exists $x_1 \in I_c^{x_0}$ such that

$$p_\theta(x_1, Tx_1) \leq b p_\theta(x_0, x_1).$$

And for $x_1 \in X$, there exists $x_2 \in I_c^{x_1}$ such that

$$p_\theta(x_2, Tx_2) \leq b p_\theta(x_1, x_2).$$

Continuing this process, we can obtain an iterative sequence (x_n), where $x_{n+1} \in I_c^{x_n}$ and

$$p_\theta(x_{n+1}, Tx_{n+1}) \leq b p_\theta(x_n, x_{n+1}), n = 0, 1, 2, ... \tag{2.1}$$

Since $x_{n+1} \in I_c^{x_n}$, we can write

$$c p_\theta(x_n, x_{n+1}) \leq p_\theta(x_n, Tx_n), n = 0, 1, 2, ... \tag{2.2}$$

By equations (2.1) and (2.2), we get

$$p_\theta(x_{n+1}, x_{n+2}) \leq \frac{b}{c} p_\theta(x_n, x_{n+1}), n = 0, 1, 2, ... \tag{2.3}$$

$$p_\theta(x_{n+1}, Tx_{n+1}) \leq \frac{b}{c} p_\theta(x_n, Tx_n), n = 0, 1, 2, ... \tag{2.4}$$

By (2.3) and (2.4), it is obvious that

$$p_\theta(x_n, x_{n+1}) \leq \frac{b^n}{c^n} p_\theta(x_0, x_1), n = 0, 1, 2, ...$$

$$p_\theta(x_n, Tx_n) \leq \frac{b^n}{c^n} p_\theta(x_0, Tx_0), n = 0, 1, 2, ... \tag{2.5}$$

Let $\alpha = \frac{b}{c}$. Since $b < c$, we get $\alpha = \frac{b}{c} < 1$. Letting $n \to \infty$ in (2.5), we obtain that

$$\lim_{n\to\infty} p_\theta(x_n, Tx_n) = 0.$$

Now, we will prove that (x_n) is a Cauchy sequence. Let $m, n \in \mathbb{N}$ with $n < m$. Then, by the triangle inequality, we have the following:

$$\begin{aligned} p_\theta(x_n, x_m) &\leq \theta(x_n, x_m)\alpha^n p_\theta(x_0, x_1) + \theta(x_n, x_m)\alpha^{n+1} p_\theta(x_0, x_1) \\ &+ \theta(x_n, x_m)\theta(x_{n+1}, x_m)...\theta(x_{m-1}, x_m)\alpha^{m-1} p_\theta(x_0, x_1) \\ &\leq p_\theta(x_0, x_1)[\theta(x_1, x_m)\theta(x_2, x_m)...\theta(x_n, x_m)\alpha^n + \\ &+ \theta(x_1, x_m)\theta(x_2, x_m)...\theta(x_n, x_m)\theta(x_{n+1}, x_m)\alpha^{n+1} + ... \\ &+ \theta(x_1, x_m)\theta(x_2, x_m)...\theta(x_n, x_m)\theta(x_{n+1}, x_m)\theta(x_{m-1}, x_m)\alpha^{m-1}. \end{aligned}$$

Since $\lim_{m,n\to\infty} \theta(x_n, x_m) < \dfrac{1}{\alpha}$, the series $\sum_{n=1}^{\infty} \alpha^n \prod_{r=1}^{n} \theta(x_r, x_m)$ converges by ratio test for each $m \in \mathbb{N}$.

Let $S = \sum_{n=1}^{\infty} \alpha^n \prod_{r=1}^{n} \theta(x_r, x_m)$ and $S_n = \sum_{i=1}^{n} \alpha^i \prod_{r=1}^{i} \theta(x_r, x_m)$. Thus we have for $n < m$, $p_\theta(x_n, x_m) \leq p_\theta(x_0, x_1)[S_{m-1} - S_n]$, from Lemma 1.11, we get that (x_n) is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. By the Assumption (i), we get that

$$0 \leq f(z) \leq \liminf_{n\to\infty} f(x_n) = 0.$$

Hence, $f(z) = p_\theta(z, Tz) = 0$. If $p_\theta(z, z) = 0$, by Lemma 2.1, we obtain that $z \in Tz$ and hence z is a fixed point of T. □

Example 2.3. Let us consider the complete extended b-metric space and extended wt-distance given in Example 1.10. Now, define the multi-valued mapping $T : X \to CB(X)$ by $T(x) = [0, x]$. Consider the function $f : X \to \mathbb{R}$ defined by $f(x) = p_\Theta(x, Tx)$ and from the definition of p_Θ, we obtain that

$$\begin{aligned} f(x) &= inf\{p_\Theta(x, a) : a \in Tx\} \\ &= inf\{a : a \in Tx\} = inf\{a : a \in [0, x]\} = 0. \end{aligned}$$

Thus, f is a constant function and it is lower semi-continuous. For any $b, c \in (0, 1)$ with $b < c$ and for all $x \in X$, there exists $y = 0 \in Tx = [0, x]$ such that

$$cp_\Theta(x, y) = c.0 = 0 \leq p_\Theta(x, Tx)$$

and

$$p_\Theta(y, Ty) = 0 \leq bp_\Theta(x, y).$$

Consider a sequence $(x_n) = \frac{1}{2n}$ such that $x_n \in T^n(x_0)$ for $x_0 = 1$. Morever, $lim_{m,n\to\infty}\Theta(x_n, x_m) = lim_{m,n\to\infty}(1 + x_n + x_m) = 1 < \frac{1}{\alpha}$, for each $\alpha \in (0, 1)$.

Therefore, it is fullfiled all of the conditions of Theorem 2.2 and T has fixed point.

Remark 2.4. It is obvious that if $\sup_{x,y\in X} \theta(x, y) < \infty$, then extended b-metric is an extended wt-distance. If we choose the function $\theta : X \times X \to [1, \infty)$ to satisfy that $\sup_{x,y\in X} \theta(x, y) < \infty$, then Theorem 1.6 can be given as a corollary of Theorem 2.2 as follows.

Corollary 2.5. *Let us consider a multi-valued mapping $T : X \to CB(X)$, where (X, d_θ) complete extended b-metric space. Furthermore, let us consider that the following two conditions hold:*

(i) *The map $f : X \to \mathbb{R}$ defined by $f(x) = d_\theta(x, Tx)$ is lower semi-continuous.*
(ii) *There exists $b, c \in (0, 1), b < c$ such that for all $x \in X$ there exists $y \in I_c^x$ satisfying*

$$d_\theta(y, Ty) \leq bd_\theta(x, y)$$

for each $x_0 \in X$, $lim_{m,n\to\infty}\theta(x_n, x_m) < \frac{1}{\alpha}$ for all $\alpha \in (0, 1)$, where $x_n \in T^n x_0$.

Then, T has a fixed point in X.

Theorem 2.6. *Let (X, d_θ) be a complete extended b-metric space and p_θ be an extended wt-distance on X. Let us consider a multi-valued mapping $T : X \to C(X)$. Suppose that there exists $c \in (0, 1)$ and $\Omega : [0, \infty) \to [0, c)$ such that*

$$\limsup_{s\to t^+} \Omega(s) < c, \ \textit{for all } t \in [0, \infty)$$

and for all $x \in X$, there exists $y \in I_c^x$ satisfying

$$p_\theta(y, Ty) \leq \Omega(p_\theta(x, y))p_\theta(x, y)$$

and for each $x_0 \in X$ and any sequence $(y_m) \subseteq X$, $\lim_{m,n\to\infty} \theta(x_n, y_m) < \frac{1}{\alpha}$ for any $\alpha \in (0, 1)$, where $x_n \in T^n x_0$. If one of the following conditions holds:

(i) *$p_\theta(x, Tx) = 0$ if there exists a sequence $(x_n) \subset X$ converging to x such that $p_\theta(x_n, Tx_n) \to 0$.*
(ii) *The map $f : X \to \mathbb{R}$ defined by $f(x) = p_\theta(x, Tx), x \in X$, is lower semi-continuous.*
(iii) *T is a closed operator.*
(iv) *For every $y \in X$ with $y \notin T(y)$, we have*

$$\inf_{x\in X}\{p_\theta(x, y) + p_\theta(x, Tx)\} > 0.$$

Then, T has a fixed point in X.

Proof. Since $T(x) \in C(X)$ for any $x \in X$, I_c^x is nonempty for any constant $c \in (0,1)$. Thus, for any initial point $x_0 \in X$, there is $x_1 \in I_c^{x_0}$ such that

$$p_\theta(x_1, Tx_1) \leq \Omega(p_\theta(x_0, x_1))p_\theta(x_0, x_1), \Omega(p_\theta(x_0, x_1)) < c \tag{2.6}$$

and

$$cp_\theta(x_0, x_1) \leq p_\theta(x_0, Tx_0). \tag{2.7}$$

From equations (2.6) and (2.7), we have

$$p_\theta(x_0, Tx_0) - p_\theta(x_1, Tx_1) \geq [c - \Omega(p_\theta(x_0, x_1))]p_\theta(x_0, x_1).$$

Morever, for $x_1 \in X$, there exists $x_2 \in Tx_1, x_1 \neq x_2$ such that

$$p_\theta(x_2, Tx_2) \leq \Omega(p_\theta(x_1, x_2))p_\theta(x_1, x_2), \Omega(p_\theta(x_1, x_2)) < c \tag{2.8}$$

and

$$cp_\theta(x_1, x_2) \leq p_\theta(x_1, Tx_1). \tag{2.9}$$

From (2.8) and (2.9), we have

$$p_\theta(x_1, Tx_1) - p_\theta(x_2, Tx_2) \geq [c - \Omega(p_\theta(x_1, x_2))]p_\theta(x_1, x_2).$$

From (2.6) and (2.9) we get

$$\begin{aligned} p_\theta(x_1, x_2) &\leq \frac{1}{c}p_\theta(x_1, Tx_1) \\ &\leq \frac{1}{c}\Omega(p_\theta(x_0, x_1))p_\theta(x_0, x_1) \\ &< p_\theta(x_0, x_1). \end{aligned}$$

Continuing this process, for $x_n, n > 1$, there exists $x_{n+1} \in Tx_n$, $x_{n+1} \neq x_n$ satisfying

$$cp_\theta(x_n, x_{n+1}) \leq p_\theta(x_n, Tx_n) \tag{2.10}$$

and

$$p_\theta(x_{n+1}, Tx_{n+1}) \leq \Omega(p_\theta(x_n, x_{n+1}))p_\theta(x_n, x_{n+1}), \Omega(p_\theta(x_n, x_{n+1})) < c. \tag{2.11}$$

From (2.10) and (2.11), we have

$$p_\theta(x_n, Tx_n) - p_\theta(x_{n+1}, Tx_{n+1}) \geq [c - \Omega(p_\theta(x_n, x_{n+1}))]p_\theta(x_n, x_{n+1}) > 0 \tag{2.12}$$

and

$$p_\theta(x_n, x_{n+1}) < p_\theta(x_{n-1}, x_n). \tag{2.13}$$

From (2.12) and (2.13), $\{p_\theta(x_n, Tx_n)\}$ and $\{p_\theta(x_n, x_{n+1})\}$ are decreasing and hence convergent. By assumption, there exists $c' \in [0, c)$ such that

$$\limsup_{n\to\infty} \Omega(p_\theta(x_n, x_{n+1}) = c'.$$

Therefore, for any $c_0 \in (c', c)$, there exists $n_0 \in \mathbb{N}$ such that

$$\Omega(p_\theta(x_n, x_{n+1})) < c_0, \text{ for all } n > n_0.$$

From (2.10) and (2.11), we get that

$$p_\theta(x_n, x_{n+1}) < \alpha p_\theta(x_{n-1}, x_n), \text{ where } \alpha = \frac{c_0}{c} \text{ and } n > n_0.$$

Then, for $n > n_0$, we have the following

$$\begin{aligned} p_\theta(x_n, Tx_n) &\leq \Omega(p_\theta(x_{n-1}, x_n))p_\theta(x_{n-1}, x_n) \\ &\leq \frac{1}{c}\Omega(p_\theta(x_{n-1}, x_n))p_\theta(x_{n-1}, Tx_{n-1}) \\ &\leq \frac{\Omega(p_\theta(x_{n-1}, x_n))...\Omega(p_\theta(x_0, x_1))}{c^n} p_\theta(x_0, Tx_0) \\ &= \frac{\Omega(p_\theta(x_{n-1}, x_n))...\Omega(p_\theta(x_{n_0+1}, x_{n_0+2}))}{c^{n-n_0}} \\ &\times \frac{\Omega(p_\theta(x_{n_0}, x_{n_0+1}))...\Omega(p_\theta(x_0, x_1))}{c^{n_0}} p_\theta(x_0, Tx_0) \\ &< (\frac{c_0}{c})^{n-n_0} \frac{\Omega(p_\theta(x_{n_0}, x_{n_0+1}))...\Omega(p_\theta(x_0, x_1))}{c^{n_0}} p_\theta(x_0, Tx_0). \end{aligned}$$

Since $c_0 < c$, $\lim_{n\to\infty}(\frac{c_0}{c})^{n-n_0} = 0$. Thus, we have that

$$\lim_{n\to\infty} p_\theta(x_n, Tx_n) = 0.$$

Let $m > n > n_0$, we have the following inequalities

$$\begin{aligned} p_\theta(x_n, x_m) &\leq \theta(x_n, x_m)p_\theta(x_n, x_{n+1}) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)p_\theta(x_{n+1}, x_{n+2}) + ... \\ &+ \theta(x_n, x_m)\theta(x_{n+1}, x_m)...\theta(x_{m-1}, x_m)p_\theta(x_{m-1}, x_m) \\ &\leq \theta(x_n, x_m)\alpha^n p_\theta(x_0, x_1) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\alpha^{n+1} p_\theta(x_0, x_1) + ... \\ &+ \theta(x_n, x_m)\theta(x_{n+1}, x_m)...\theta(x_{m-1}, x_m)\alpha^{m-1} p_\theta(x_0, x_1). \end{aligned}$$

By following the similar procedure in Theorem 2.2, there exists a Cauchy sequence (x_n) such that $x_{n+1} \in Tx_n, x_{n+1} \neq x_n$. Since X is complete, there exists $z \in X$ such that $x_n \to z$ and we have the following inequality

$$p_\theta(x_n, x_m) \leq p_\theta(x_0, x_1)[S_{m-1} - S_n], \tag{2.14}$$

where $S = \sum_{n=1}^{\infty} \alpha^n \prod_{r=1}^{n} \theta(x_r, x_m)$ and $S_n = \sum_{i=1}^{n} \alpha^i \prod_{r=1}^{i} \theta(x_r, x_m)$.

Case I: The assertion (i) holds: Since $p_\theta(x_n, Tx_n)$ converges to 0 and $x_n \to z$ we have that $p_\theta(z, Tz) = 0$. Then, there exists a sequence $(y_n) \subset Tz$ such that $p_\theta(z, y_n) \to 0$. From (2.14) and θ-lower semicontinuity of p_θ, we obtain the followings:

$$\begin{aligned} p_\theta(x_n, z) &\leq \liminf_{m\to\infty} \theta(x_n, z)p_\theta(x_n, x_m) \\ &\leq \liminf_{m\to\infty} \theta(x_n, z)p_\theta(x_0, x_1)[S_{m-1} - S_n]. \end{aligned}$$

From the triangle inequality, we have

$$\begin{aligned} p_\theta(x_n, y_n) &\leq \theta(x_n, y_n)[p_\theta(x_n, z) + p_\theta(z, y_n)] \\ &\leq \theta(x_n, y_n)[\liminf_{m\to\infty} \theta(x_n, z)p_\theta(x_0, x_1)[S_{m-1} - S_n] + p_\theta(z, y_n)]. \end{aligned} \tag{2.15}$$

Since $\lim_{m,n\to\infty} \theta(x_n, y_m) < \frac{1}{c}$, letting $n \to \infty$ in (2.15) we obtain that $\lim_{n\to\infty} p_\theta(x_n, y_n) = 0$. Hence, $p_\theta(x_n, y_n) \to 0$ and $p_\theta(x_n, z) \to 0$, by lemma 1.11, we get that $d_\theta(y_n, z) \to 0$. Since Tz is a closed, $z \in Tz$.

Case II: If $f(x) = p_\theta(x, Tx)$ is lower semicontinuous, then the assumption (i) holds.

Case III: Let us suppose that the assumption (iii) holds. Then, from $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{N} \cup 0$ and $(x_n, x_{n+1}) \to (z, z)$, we get that $z \in Tz$.

Case IV: Suppose to the contrary that $z \notin Tz$. Then,

$$\begin{aligned} 0 &< \inf_{x\in X}\{p_\theta(x,z) + p_\theta(x,Tx)\} \\ &\le \inf_{n\in\mathbb{N}}\{p_\theta(x_n,z) + p_\theta(x_n,Tx_n)\} \\ &\le \inf_{n\in\mathbb{N}}\{p_\theta(x_n,z) + p_\theta(x_{n-1},x_n)\} \\ &\le \inf_{n\in\mathbb{N}}\{\liminf_{m\to\infty} \theta(x_n,z)p_\theta(x_0,x_1)[S_{m-1} - S_n] + \alpha^{n-1}p_\theta(x_0,x_1)\} = 0, \end{aligned}$$

which is a contradiction thus $z \in Tz$. □

Remark 2.7. If we choose the function $\Omega : [0,\infty) \to [0,c)$ defined by $\Omega(t) = b < c$ for all $t \in [0,\infty)$ in Theorem 2.6, we can give the following result as a corollary.

Corollary 2.8. *Let (X, d_θ) be a complete extended b-metric space and p_θ be an extended wt-distance on X. Let us consider a multivalued mapping $T : X \to C(X)$. Suppose that there exists $b, c \in (0,1), b < c$ such that for all $x \in X$ there exists $y \in I_c^x$ satisfying*

$$p_\theta(y,Ty) \le b p_\theta(x,y)$$

for each $x_0 \in X$, where $x_n = T^n x_0$ and for any sequence $(y_m) \subset X$ with $\lim_{m,n\to\infty}\theta(x_n,y_m) < \frac{1}{\alpha}$ for all $\alpha \in (0,1)$. If one of the following conditions holds:

- (i) *$p_\theta(x,Tx) = 0$ if there exists a sequence $(x_n) \subset X$ converging to x such that $p_\theta(x_n,Tx_n) \to 0$;*
- (ii) *The map $f : X \to \mathbb{R}$ defined by $f(x) = p_\theta(x,Tx), x \in X$, is lower semi-continuous;*
- (iii) *T is a closed operator;*
- (iv) *for every $y \in X$ with $y \notin T(y)$, we have*

$$\inf_{x\in X}\{p_\theta(x,y) + p_\theta(x,Tx)\} > 0.$$

Then, T has a fixed point in X.

Remark 2.9. In [16], the following lemma was proved by considering the following condition

$$\limsup_{m,n\to\infty} \theta(x_n,x_m) < \infty \tag{2.16}$$

rather than the condition $\lim_{m,n\to\infty}\theta(x_n,x_m) < \frac{1}{\alpha}$ for all $\alpha \in (0,1)$ in Theorem 2.2.

Lemma 2.10 ([16]). *Let (X, d_θ) be an extended b-metric space such that the condition (2.16) is satisfied and let (x_n) be a sequence in X. Suppose that there exits $\lambda \in [0,1)$ such that $d_\theta(x_{n+1},x_n) \le \lambda d_\theta(x_n,x_{n-1})$ for all $n \in \mathbb{N}$. Then, (x_n) is a Cauchy sequence in X.*

Morever, from the condition (iii) of Lemma 1.11 and Lemma 2.10, we can easily give the following lemma which helps to prove Theorem 2.6 in a shorter and easier way.

Lemma 2.11. *Let (X, d_θ) be an extended b-metric space and p_θ be an extended wt-distace and the condition (2.16) is satisfied. Let (x_n) be a sequence in X and suppose that there exits $\lambda \in [0,1)$ such that $p_\theta(x_{n+1},x_n) \le \lambda p_\theta(x_n,x_{n-1})$ for all $n \in \mathbb{N}$. Then, (x_n) is a Cauchy sequence in X*

Now, we can give the Theorem 2.6 by using the condition (2.16). Since proof is obvious from Lemma 2.11 and Theorem 2.6, we omitted it.

Theorem 2.12. *Let (X, d_θ) be a complete extended b-metric space and p_θ be an extended wt-distance on X. Let us consider a multi-valued mapping $T : X \to C(X)$. Suppose that there exists $c \in (0,1)$ and $\Omega : [0,\infty) \to [0,c)$ such that*

$$\limsup_{s\to t^+} \Omega(s) < c, \text{ for all } t \in [0,\infty)$$

and for all $x \in X$, there exists $y \in I_c^x$ satisfying

$$p_\theta(y,Ty) \le \Omega(p_\theta(x,y))p_\theta(x,y)$$

and for each $x_0 \in X$ and any sequence $(y_m) \subseteq X$, $\limsup_{m,n\to\infty}\theta(x_n,y_m) < \infty$, where $x_n \in T^n x_0$. If one of the following conditions holds:

(i) $p_\theta(x, Tx) = 0$ *if there exists a sequence* $(x_n) \subset X$ *converging to* x *such that* $p_\theta(x_n, Tx_n) \to 0$;
(ii) *The map* $f : X \to \mathbb{R}$ *defined by* $f(x) = p_\theta(x, Tx), x \in X$, *is lower semi-continuous;*
(iii) T *is a closed operator;*
(iv) *for every* $y \in X$ *with* $y \notin T(y)$, *we have*

$$\inf_{x\in X}\{p_\theta(x, y) + p_\theta(x, Tx)\} > 0.$$

Then, T *has a fixed point in* X.

Notice that if we take $\theta(x, y) = s$ for all $x, y \in X$ in Theorem 1.3, the wt-distance p is actually an extended wt-distance. Moreover, if we choose the function $\Omega : [0,\infty) \to [0, c)$ defined by $\Omega(t) = b < c$ for all $t \in [0,\infty)$ in Theorem 2.12, we can obtain Theorem 1.3 as a result of Theorem 2.12 by the following:

Corollary 2.13. *Let* (X, d, s) *be a complete b-metric space. Let* $C(X)$ *be the set of all nonempty closed subsets of* X *and* $T : X \to C(X)$ *a multi-valued operator,* $p : X \times X \to [0,\infty)$ *a wt-distance on* X *and* $b \in (0, 1)$. *Suppose that there exits,* $c \in (0, 1)$ *with* $cb^{-1} \in [0, s^{-1})$, *such that for any* $x \in X$ *there is* $y \in I^x_b$ *satisfying*

$$p(y, Ty) \le cp(x, y).$$

If one of the following assertions holds:
(i) $p(x, Tx) = 0$ *if there exists a sequence* $(x_n) \subset X$ *such that* $p(x_n, Tx_n) \to 0$, *as* $n \to \infty$;
(ii) *the function* $f : X \to \mathbb{R}$ *as* $f(x) = p(x, T(x))$ *is s-lower semi-continuous;*
(iii) *for every* $y \in X$ *with* $y \notin T(y)$, *we have*

$$\inf_{x\in X}\{p(x, y) + p(x, Tx)\} > 0.$$

(iv) T *is a closed operator,*
then T *has a fixed point in* X.

In [21], the notion of compactness in extended b-metric spaces was given by the following.

Definition 2.14 ([21]). Let (X, d_θ) be an extended b-metric space. A subset C of X is compact if and only if for every sequence of elements of C there exists a subsequence that converges to an element of C.

Remark 2.15. From the uniqueness of the limit in the extended b-metric spaces and the previous definition, we can easily say that every compact subset of an extended b-metric space is a closed subset.

Now, we can give the following result for compact valued multifunctions in extended b-metric spaces via extended wt-distance.

Corollary 2.16. *Let* (X, d_θ) *be a complete extended b-metric space and* p_θ *be an extended wt-distance on* X. *Let us consider a multivalued mapping* $T : X \to K(X)$. *Suppose that there exists* $c \in (0, 1)$ *and* $\Omega : [0,\infty) \to [0, 1)$ *such that*

$$\limsup_{s\to t^+} \Omega(s) < 1, \text{ for all } t \in [0,\infty)$$

and for all $x \in X$, *there exists* $y \in I^x_1$ *satisfying*

$$p_\theta(y, Ty) \le \Omega(p_\theta(x, y))p_\theta(x, y)$$

and for each $x_0 \in X$ *and any sequence* $(y_m) \subseteq X$, $\lim_{m,n\to\infty} \theta(x_n, y_m) < \frac{1}{\alpha}$ *for any* $\alpha \in (0, 1)$, *where* $x_n \in T^n x_0$. *If the map* $f : X \to \mathbb{R}$ *defined by* $f(x) = p_\theta(x, Tx), x \in X$, *is lower semi-continuous, then there exists* $z \in X$ *such that* $f(z) = 0$. *Further, if* $p_\theta(z, z) = 0$, *then* z *is a fixed point of* T.

Proof. Since $Tx \in K(X)$ for any $x \in X$, I^x_1 is nonempty. Hence, for all $x \in X$, there exists $y \in Tx$ such that $p_\theta(x, y) \le p_\theta(x, Tx)$. Let us choose an arbitrary point $x_0 \in X$. By following the similar process as in Theorem 2.6, we obtain a Cauchy sequence (x_n) such that $x_{n+1} \in Tx_n, x_{n+1} \ne x_n$, satisfying the followings:

$$p_\theta(x_n, x_{n+1}) = p_\theta(x_n, Tx_n)$$

and

$$p_\theta(x_n, Tx_n) \le \Omega(p_\theta(x_{n-1}, x_n))p_\theta(x_{n-1}, x_n), \Omega(p_\theta(x_{n-1}, x_n)) < 1.$$

Since X is complete, there exists $z \in X$ such that $x_n \to z$. By the lower semi-continuity of the function f, we have

$$0 \leq p_\theta(z, Tz) \leq \liminf_{n\to\infty} p_\theta(x_n, Tx_n) = 0.$$

Then, $p_\theta(z, Tz) = 0$. Further, if $p_\theta(z, z) = 0$, then z is a fixed point of T. □

3. An Application to Homotopy Theory

In this section, we give an application of our main theorem to homotopy theory inspired by the results given in [1] and [23]. Firstly, we introduce some definitions which are necessary for the homotopy result.

Let (X, d_θ) be an extended b-metric space and p_θ be an extended wt-distance on X. We define the following mapping:

$$H_{p_\theta} : CB(X) \times CB(X) \to [0, \infty)$$

as follows

$$H_{p_\theta}(A, B) = max\{\sup_{a\in A} p_\theta(a, B), \sup_{b\in B} p_\theta(A, b)\}$$

for all $A, B \in CB(X)$.

Now, we give the following proposition. The proof is similar to the proof of Theorem 4.4 in [16], so we omit the proof.

Proposition 3.1. *Let (X, d_θ) be an extended b-metric and p_θ be an extended wt-distance on X. The mapping H_{p_θ} : $CB(X) \times CB(X) \to [0, \infty)$ satisfies the following inequality:*

$$H_{p_\theta}(U, V) \leq \theta_H(U, V)[H_{p_\theta}(U, W) + H_{p_\theta}(W, V)],$$

where the mapping $\theta_H : CB(X) \times CB(X) \to [1, \infty)$ given as follows:

$$\theta_H(U, V) = max\{\sup_{u\in U} \inf_{v\in V} \theta(u, v), \sup_{v\in V} \inf_{u\in U} \theta(u, v)\}.$$

Definition 3.2. Let (X, d_θ) be an extended b-metric space and p_θ be an extended wt-distance on X. Let $T : X\times[0, 1] \to CB(X)$ be a multi-valued operator. The graph of T is defined by the set $G(T) = \{(x, t, y) : x \in X, t \in [0, 1], y \in T(x, t)\}$. If the graph of T is closed in $(X \times [0, 1] \times X, \overline{d}_\theta)$, where

$$\overline{d}_\theta((x, t, y), (x', t', y')) = d_\theta(x, x') + \mid t - t' \mid + d_\theta(y, y'),$$

then T is called a closed multi-valued operator.

Theorem 3.3. *Let (X, d_θ) be a complete extended b-metric space such that d_θ is a continuous functional in first variable and p_θ be an extended wt-distance on X, where there exists a real number $M > 1$ such that $\theta(x, y) < M$ for all $x, y \in X$. Let K be a closed subset of X and O be an open open subset of X with $O \subseteq K$. Assume that $T : K \times [0, 1] \to CB(X)$ is a closed multi-valued mapping satisfying the following conditions:*

(i) *$x \notin T(x, t)$ for each $x \in K\backslash O$ and each $t \in [0, 1]$.*

(ii) *The mapping $f : K \to \mathbb{R}$ given by $f(x) = p_\theta(x, T(x, t))$ is lower semicontinuous for all $t \in [0, 1]$.*

(iii) *There exist $b, c \in (0, 1)$ with $b < c$ and $Mb < 1$ such that for all $x \in X$, there exists $y \in I_c^{(x,t)}$, where*

$$I_c^{(x,t)} = \{y \in T(x, t) : cp_\theta(x, y) \leq p_\theta(x, T(x, t))\},$$

satisfying the following:

$$H_{p_\theta}(T(x, t), T(y, t)) \leq bp_\theta(x, y).$$

(iv) *There exists $L > 0$ such that*

$$H_{p_\theta}(T(x, t_1), T(x, t_2)) \leq L \mid t_1 - t_2 \mid .$$

for all $t_1, t_2 \in [0, 1]$ and for every $x \in K$.

(v) *If $x \in T(x, t)$, then $T(x, t) = \{x\}$.*

(vi) *Let $x \in K$ and $r > 0, t \in [0, 1]$. Then, for each $x^\star \in \overline{B}_{d_\theta}(x, r)$ and $y^\star \in K \cup T(x^\star, t)$, the equality $d_\theta(x, y^\star) = p_\theta(x, y^\star)$ holds.*

(vii) *For each* $x_0 \in X, \lim_{m,n\to\infty} \theta(x_m, x_n) \leq \frac{1}{\alpha}$ *for all* $x_n \in T(x_0, .)$.

If $T(., 0)$ *has a fixed point in* K*, then* $T(., 1)$ *has a fixed point in* K.

Proof. Let us define the set

$$\Phi = \{t \in [0, 1] : x \in T(x, t) \text{ for some } x \in O\}.$$

Since $T(., 0)$ has a fixed point, then Φ is nonempty. Now, we show that Φ is open and closed in $[0, 1]$. Due to the connectedness of $[0, 1]$, we will obtain that $\Phi = [0, 1]$. For this aim, we first show that Φ is closed in $[0, 1]$. Let (t_n) be a sequence in Φ such that $t_n \to t^\star \in [0, 1]$ as $n \to \infty$. Then, from the definition of Φ, there exists $x_n \in O$ such that $x_n \in T(x_n, t_n)$ for each $n \in \mathbb{N}$. Furthermore, for all $m, n \in \mathbb{N}$,

$$\begin{aligned} p_\theta(x_n, x_m) &= H_{p_\theta}(T(x_n, t_n), T(x_m, t_m)) \\ &\leq M[H_{p_\theta}(T(x_n, t_n), T(x_n, t_m)) + H_{p_\theta}(T(x_n, t_m), T(x_m, t_m))] \\ &\leq ML \mid t_n - t_m \mid + bp_\theta(x_n, x_m). \end{aligned}$$

Then, $p_\theta(x_n, x_m) \leq \frac{ML}{1-b} \mid t_n - t_m \mid$. Since every convergent sequence is a Cauchy sequence in $[0, 1]$, (t_n) is a Cauchy sequence. Therefore, we get that $\lim_{m,n\to\infty} p_\theta(x_n, x_m) = 0$ and from Lemma 1.11, (x_n) is a Cauchy sequence. Since (X, d_θ) is complete, there exists an element $x^\star$ such that $\lim_{n\to\infty} d_\theta(x_n, x^\star) = 0$. Also, T is a closed multi-valued mapping, we have that $(x_n, t_n, x_n) \in G(T)$ and $(x_n, t_n, x_n) \to (x^\star, t^\star, x^\star)$ as $n \to \infty$. Thus, $(x^\star, t^\star, x^\star) \in G(T)$ and from (i), it is clear that $x^\star \in O$ and $(t^\star, x^\star) \in \Phi$. Then, we obtain that $t^\star \in \Phi$ and Φ is closed.

Now, we show that Φ is open in $[0, 1]$. Let $t_0 \in \Phi$ and $x_0 \in O$ with $x_0 \in T(x_0, t_0)$. Since O is open in X, there exists $r_0 > 0$ such that $B_{d_\theta}(x_0, r_0) \subseteq O$. Consider an $\epsilon > 0$ with $\epsilon \leq \dfrac{r_0(1 - Mb)}{ML}$. Let $t \in (t_0 - \epsilon, t_0 + \epsilon)$. We show that $T(x, t) \subseteq \overline{B}_{d_\theta}(x_0, r_0)$, for all $x \in \overline{B}_{d_\theta}(x_0, r_0)$. Let $y \in T(x, t)$. Then, from (vi) we obtain the followings:

$$\begin{aligned} d_\theta(y, x_0) &= p_\theta(y, x_0) \\ &= p_\theta(y, T(x_0, t_0)) \\ &\leq H_{p_\theta}(T(x, t), T(x_0, t_0)) \\ &\leq M[H_{p_\theta}(T(x, t), T(x, t_0)) + H_{p_\theta}(T(x, t_0), T(x_0, t_0))] \\ &\leq ML \mid t - t_0 \mid + Mbp_\theta(x, x_0) \\ &\leq ML\epsilon + Mbr_0 \\ &\leq r_0. \end{aligned}$$

Thus, $y \in \overline{B}_{d_\theta}(x_0, r_0)$ and for each $t \in (t_0 - \epsilon, t_0 + \epsilon)$, $T(., t) : \overline{B}_{d_\theta}(x_0, r_0) \to CB(\overline{B}_{d_\theta}(x_0, r_0))$. Therefore all assumptions of Theorem 2.2 are satisfied and there exists a point $x' \in \overline{B}_{d_\theta}(x_0, r_0)$ such that $f(x') = p_\theta(x', T(x', t)) = 0$. Since $x' \in K$, from the condition (vi), we get $d_\theta(x', x') = p_\theta(x', x') = 0$. Thus, we obtain from Theorem 2.2, T has a fixed point in $\overline{B}_{d_\theta}(x_0, r_0) \subseteq K$. Due to the condition (i), the fixed point has to be in O for all $t \in [0, 1]$. Thus, $(t_0 - \epsilon, t_0 + \epsilon) \subseteq \Phi$ and Φ is open set in $[0, 1]$. Consequently, $T(., 1)$ has a fixed point. □

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the article.

References

[1] Adamo, M.S.S., Vetro, C., *Fixed point and homotopy results for mixed multi-valued mappings in 0-complete partial metric spaces*, Nonlinear Analysis: Modelling and Control, **20**(2)(2015), 159–174.

[2] Alamgir, N., Kiran, Q., Aydi, H., Mukheimer, A., *A Mizoguchi Takahashi type fixed point theorem in complete extended b-metric spaces*, Mathematics, **7**(2019), 478.

[3] Alqahtani, B., Fulga, A., Karapınar, E., *Common fixed point results on an extended b-metric space*, J. Inequal. Appl. **2018**(2018), 158.

[4] Bakhtin, I.A., *The contraction mapping principle in almost metric spaces*, Func. Anal. Gos. Ped. Inst. Unianowsk, **30**(1989), 26–37.

[5] Chifu, C., *Common fixed point results in extended b-metric spaces endowed with a directed graph*, Results Nonlinear Anal., **2**(1)(2019), 18–24.

[6] Demma, M., Saadati R., Vetro P., *Multi-valued operators with respect wt-distance on metric type spaces*, Bull. Iranian Math. Soc., **42**(6)(2016), 1571–82.

[7] Eroğlu, I., *Some fixed point results for extended P_{p_θ}-contractions via extended wt-distance and its applications*, Creat. Math. Inform., **34**(1)(2024), 11–21.

[8] Feng, Y., Liu, S., *Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings*, J. Math. Anal. Appl., **371**(1)(2006), 103–112.

[9] Huang, H., Singh, Y.M., Khan, M.S., Radenović, S., *Rational type contractions in extended b-metric spaces*, Symmetry, **13**(2021), 614.

[10] Hussain, N., Saadati, R., Agrawal, R.P., *On the topology and wt-distance on metric type spaces*, Fixed Point Theory Appl., **2014**(2014), 88.

[11] Kada, O.,Suzuki, T., Takahashi, W., *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japonica, **44**(2)(1996), 381–391.

[12] Kamran, T., Samreen, M., UL Ain, Q., *A generalization of b-metric space and some fixed point theorems*, Mathematics, **5**(2)(2017), 19.

[13] Karapınar, E., Khojasteh, F., Mitrović, ZD., *A proposal for revisiting Banach and Caristi type theorems in b-metric spaces*, Mathematics, **7**(4)(2019), 308.

[14] Mizoguchi, N., Takahashi, W., *Fixed point theorem for multivalued mappings on complete metric space*, J. Math. Anal. Appl., **141**(1989), 177–188.

[15] Öztürk, V., *Fixed point theorems for almost contractions in extended B-metric spaces*, Universal Journal Of Mathematics And Applications, **4**(3)(2021), 101–106.

[16] Mitrović, Z.D., Işık, H., Radenović, *The new results in extended b -metric spaces and applications*, International Journal of Nonlinear Analysis and Applications, **11**(1)(2020), 473–482

[17] Romaguera, S., *An application of wt-distances to characterize complete b-metric spaces*, Axioms, **12**(121)(2023).

[18] Romaguera, S., *On the correlation between Banach contraction principle and Caristi's fixed point theorem in b-metric spaces*, Mathematics, **10**(136)(2022).

[19] Samreen, M., Kamran, T., Postolache, M., *Extended b-metric space, extended b-comparison function and nonlinear contractions*, U. Politeh. Buch. Ser.A, **80**(4)(2018), 21–28.

[20] Shatanawi, W., Abodayeh, K., Mukheimer, A., *Some fixed point theorems in extended b-metric spaces*, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, **80**(4)(2018), 71–78.

[21] Subashi, L., *Some topological properties of extended b-metric space*, Proceedings of The 5th International Virtual Conference on Advanced Scientific Results (SCIECONF-2017), **5**(2017), 164–167.

[22] Lin, L.J., Du, S., *Some equivalent formulations of generalized Ekeland's variational principle and their applications*, Nonlinear Anal., **67**(2007), 187–199 .

[23] Vetro, C., Vetro, F., *A homotopy fixed point theorem in 0-complete partial metric space*, Filomat **29**(9)(2015), 2037–2048.

[24] Younis, M., Singh, D., Altun, I., Chauhan, V., *Graphical structure of extended b-metric spaces: an application to the transverse oscillations of a homogeneous bar*, International Journal of Nonlinear Sciences and Numerical Simulation, **23**(7-8)(2022), 1239–1252.