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Some Results Associated with the Hyperbolic Sine Function

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Abstract. In this paper, we examine several characteristics of analytical functions related to the hyperbolic sine function and analyze the behavior of the hyperbolic sine function inside and at the boundary of the unit disk.

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1. Introduction

Let p be an analytic function in the unit disc $D = \{z : |z| < 1\}$, $p(0) = 0$ and $p : D \to D$. In accordance with the classical Schwarz lemma, for any point z in the unit disc D, we have $|p(z)| \leq |z|$ for all $z \in D$ and $|p'(0)| \leq 1$. In addition, if the equality $|p(z)| = |z|$ holds for any $z \neq 0$, or $|p'(0)| = 1$, then p is a rotation; that is $p(z) = ze^{i\theta}$, θ real [5]. In this study, we aim to obtain the Schwarz lemma for the following class $\mathcal{T}$, which will be defined subsequently. That is, we examine the behavior of the hyperbolic sine function within the unit disk. In addition, the authors provide a class of the analytic functions depending on the function $1 + \sinh z$ and establish various conditions for the functions in the mentioned class.

We shall need the following lemma due to Jack [6].

Lemma 1.1 (Jack's lemma)**.** *Let $p(z)$ be a non-constant analytic function in D with $p(0) = 0$. If*

$$|p(z_0)| = \max\{|p(z)| : |z| \leq |z_0|\},$$

then there exists a real number $k \geq 1$ such that

$$\frac{z_0 p'(z_0)}{p(z_0)} = k.$$

Let $\mathcal{A}$ denote the class of functions $g(z) = 1 + c_1 z + c_2 z^2 + ...$ that are analytic in D. Also, let $\mathcal{T}$ be the subclass of $\mathcal{A}$ consisting of all functions $g(z)$ satisfying

$$1 + \gamma z g'(z) \prec \frac{1+z}{1-z}, \quad z \in D$$

[7] and

$$|\gamma| \cos(1) \geq 2 + |\gamma| \cosh(1), \ \gamma \in \mathbb{R}.$$

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Let $g \in \mathcal{T}$ and consider the following function

$$\vartheta(z) = \operatorname{arcsinh}\left(g(z) - 1\right),$$

where we have chosen the principle branches of the square root and logarithmic functions. Since the function arcsinh is defined as

$$\operatorname{arcsinh} z = \log\left(z + \sqrt{1 + z^2}\right),$$

so $\vartheta(z)$ is an analytic function in D and $\vartheta(0) = 0$. We show that $|\vartheta(z)| < 1$ for $z \in D$. From the definition of $\vartheta(z)$, we have

$$g(z) = \sinh \vartheta(z) + 1,$$
$$g'(z) = \vartheta'(z) \cosh \vartheta(z),$$
$$l(z) = 1 + \gamma z g'(z) = 1 + \gamma z \vartheta'(z) \cosh \vartheta(z)$$

and

$$\left|\frac{l(z) - 1}{l(z) + 1}\right| = \left|\frac{\gamma z \vartheta'(z) \cosh \vartheta(z)}{2 + \gamma z \vartheta'(z) \cosh \vartheta(z)}\right|.$$

We suppose that there exists a point $z_0 \in D$ such that

$$\max_{|z| \le |z_0|} |\vartheta(z)| = |\vartheta(z_0)| = 1.$$

From the Jack's lemma, we have

$$\vartheta(z_0) = e^{i\theta} \text{ and } \frac{z_0 \vartheta'(z_0)}{\vartheta(z_0)} = k,$$

for $\theta \in [-\pi, \pi]$. Therefore, we take

$$\begin{aligned}\left|\frac{l(z_0) - 1}{l(z_0) + 1}\right| &= \left|\frac{\gamma z_0 \vartheta'(z_0) \cosh \vartheta(z_0)}{2 + \gamma z_0 \vartheta'(z_0) \cosh \vartheta(z_0)}\right| \\ &= \left|\frac{\gamma k e^{i\theta} \cosh e^{i\theta}}{2 + \gamma k e^{i\theta} \cosh e^{i\theta}}\right| \\ &\ge \frac{|\gamma|\, k \left|\cosh e^{i\theta}\right|}{2 + |\gamma|\, k \left|\cosh e^{i\theta}\right|}.\end{aligned}$$

With the simple calculations, we take

$$\left|\cosh e^{i\theta}\right|^2 = \cosh^2(\cos\theta)\cos^2(\sin\theta) + \sinh^2(\cos\theta)\sin^2(\sin\theta) = \Psi(\theta).$$

The function $\Psi(\theta)$ has critical points $0, \pm\pi, \pm\frac{\pi}{2}$ in the interval $[-\pi, \pi]$. Also, $\Psi(\theta)$ is even function in this interval. Therefore, we obtain

$$\max\left(\Psi(\theta)\right) = \Psi(0) = \Psi(\pi) = \cosh^2(1)$$

and

$$\min\left(\Psi(\theta)\right) = \Psi\left(\frac{\pi}{2}\right) = \cos^2(1).$$

From these expressions, we take

$$\cos(1) \le \left|\cosh e^{i\theta}\right| \le \cosh(1).$$

Thus, we obtain

$$\left|\frac{l(z_0) - 1}{l(z_0) + 1}\right| \ge \frac{|\gamma|\, k \left|\cosh e^{i\theta}\right|}{2 + |\gamma|\, k \left|\cosh e^{i\theta}\right|} \ge \frac{|\gamma|\, k \cos(1)}{2 + |\gamma|\, k \cosh(1)}.$$

Let

$$\sigma(k) = \frac{|\gamma|\, k \cos(1)}{2 + |\gamma|\, k \cosh(1)}.$$

Therefore, we take

$$\sigma'(k) = \frac{2\,|\gamma| \cos(1)}{\left(2 + |\gamma|\, k \cosh(1)\right)^2} > 0.$$

Here, $\sigma(k)$ is an increasing function and hence it will have its minimum value at $k = 1$. Thus, we obtain

$$\left|\frac{l(z_0) - 1}{l(z_0) + 1}\right| \geq \frac{|\gamma| \cos(1)}{2 + |\gamma| \cosh(1)}$$

and from $|\gamma| \cos(1) \geq 2 + |\gamma| \cosh(1)$

$$\left|\frac{l(z_0) - 1}{l(z_0) + 1}\right| \geq 1.$$

This contradicts $g(z) \in \mathcal{T}$. This means that, there is no point $z_0 \in D$ such that $\max_{|z| \leq |z_0|} |\vartheta(z)| = |\vartheta(z_0)| = 1$. Hence, we take $|\vartheta(z)| < 1$ in D. From the Schwarz lemma, we take $|\vartheta'(0)| \leq 1$. Therefore, we have

$$\vartheta(z) = \operatorname{arcsinh}(g(z) - 1),$$

$$\sinh \vartheta(z) + 1 = g(z) = 1 + c_1 z + c_2 z^2 + ...,$$

$$\sinh \vartheta(z) = c_1 z + c_2 z^2 + ...$$

and

$$c_1 + c_2 z + ... = \frac{\sinh \vartheta(z)}{z}.$$

Passing to limit as z tends to 0 in the last equality, we obtain

$$c_1 = \vartheta'(0) \cosh \vartheta(0)$$

and

$$|c_1| \leq 1.$$

Let us now show that this inequality is sharp. Let

$$g(z) = 1 + \sinh z.$$

Then,

$$g'(z) = \cosh z,$$

$$g'(0) = c_1 = \cosh 0 = 1$$

and

$$|c_1| = 1.$$

We thus obtain the following lemma.

Lemma 1.2. *If $g \in \mathcal{T}$, then we have the inequality*

$$|c_1| \leq 1.$$

The result is sharp for the function

$$g(z) = 1 + \sinh z.$$

One of the important applications of Schwarz lemma involves studies at the boundary of the unit disc. Some of these studies, which are called the boundary version of Schwarz Lemma, are about estimating from below the modulus of the derivative of the function at some boundary point of the unit disc [1–4, 8–10, 12, 13]. The boundary version of the Schwarz Lemma is given as follows [11–15]:

Lemma 1.3. *Let p be an analytic function in D, $p(0) = 0$ and $p(D) \subset D$. If $p(z)$ extends continuously to boundary point $1 \in \partial D = \{z : |z| = 1\}$, and if $|p(1)| = 1$ and $p'(1)$ exists, then*

$$\left|p'(1)\right| \geq \frac{2}{1 + |p'(0)|} \tag{1.1}$$

and

$$\left|p'(1)\right| \geq 1. \tag{1.2}$$

Moreover, the equality in (1.1) holds if and only if

$$g(z) = z\frac{z - a}{1 - az}$$

for some $a \in (-1, 0]$. Also, the equality in (1.2) holds if and only if $p(z) = ze^{i\theta}$.

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel [14].

Lemma 1.4 (Julia-Wolff lemma)**.** *Let p be an analytic function in D, $p(0) = 0$ and $p(D) \subset D$. If, in addition, the function p has an angular limit $p(1)$ at $1 \in \partial D$, $|p(1)| = 1$, then the angular derivative $p'(1)$ exists and $1 \leq |p'(1)| \leq \infty$.*

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma for the class $\mathcal{T}$ and examine the behavior of the hyperbolic sine function at the boundary of the unit disk.

Theorem 2.1. *Let $g \in \mathcal{T}$. Assume that, for $1 \in \partial D$, g has an angular limit $g(1)$ at the point 1, $g(1) = 1 + \sinh 1$. Then, we have the inequality*

$$\left|g'(1)\right| \geq \cosh(1). \tag{2.1}$$

The inequality (2.1) is sharp with extremal function

$$g(z) = 1 + \sinh z.$$

Proof. Consider the function

$$\vartheta(z) = \operatorname{arcsinh}(g(z) - 1).$$

With simple edits, we have

$$g(z) = \sinh \vartheta(z) + 1$$

and $|\vartheta(1)| = 1$ for g(1) = 1 + sinh 1. Therefore, from the Schwarz lemma at the boundary, we take $|\vartheta'(1)| \geq 1$. With the simple calculations, we obtain

$$g'(1) = \vartheta'(1) \cosh \vartheta(1) = \vartheta'(1) \cosh(1),$$

$$1 \leq \left|\vartheta'(1)\right| = \frac{|g'(1)|}{\cosh(1)}$$

and

$$\left|g'(1)\right| \geq \cosh(1).$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$g(z) = 1 + \sinh z.$$

Then,

$$g'(z) = \cosh z$$

and

$$\left|g'(1)\right| = \cosh(1).$$

□

The inequality (2.1) can be strengthened from below by taking into account, $c_1 = g'(0)$, which is the first coefficient of the expansion of the function $g(z) = 1 + c_1 z + c_2 z^2 + \ldots$.

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left|g'(1)\right| \geq \frac{2\cosh(1)}{1 + |c_1|}. \tag{2.2}$$

The equality in (2.2) occurs for the function

$$g(z) = 1 + \sinh z.$$

Proof. If we apply the inequality (1.1) to the analytic function $\vartheta(z)$ given in Theorem 2.1, we obtain

$$\frac{2}{1 + |\vartheta'(0)|} \leq \left|\vartheta'(1)\right| = \frac{|g'(1)|}{\cosh(1)}.$$

Since $|\vartheta'(0)| = |c_1|$, we take

$$\left|g'(1)\right| \geq \frac{2\cosh(1)}{1 + |c_1|}.$$

Now, we shall show that the inequality (2.2) is sharp. Let

$$g(z) = 1 + \sinh z.$$

Then,

$$\left|g'(1)\right| = \cosh(1).$$

In addition, we have

$$1 + c_1 z + c_2 z^2 + ... = 1 + \sinh z,$$

$$c_1 z + c_2 z^2 + ... = \sinh z$$

and

$$c_1 + c_2 z + ... = \frac{\sinh z}{z}.$$

Passing to limit ($z \to 0$) in the last equality yields $c_1 = 1$. Therefore, we take

$$\frac{2\cosh(1)}{1 + |c_1|} = \frac{2\cosh(1)}{1+1} = \cosh(1).$$

□

An interesting special case of Theorem 2.2 is when $c_1 = 0$, in which case inequality (2.2) implies $|g'(1)| \geq 2\cosh 1$.

The inequality (2.2) can be strengthened as below by taking into account $c_2 = \frac{g''(0)}{2!}$ which is the coefficient in the expansion of the function $g(z) = 1 + c_1 z + c_2 z^2 + ...$

Theorem 2.3. *Let $g \in \mathcal{T}$. Assume that, for $1 \in \partial D$, g has an angular limit $g(1)$ at the point 1, $g(1) = 1 + \sinh 1$. Then, we have the inequality*

$$\left|g'(1)\right| \geq \cosh 1 \left(1 + \frac{2(1 - |c_1|)^2}{1 - |c_1|^2 + |c_2|}\right).$$

This result is sharp for the function

$$g(z) = 1 + \sinh z^2.$$

Proof. Let $\vartheta(z)$ be the same as in the proof of Theorem 2.1 and $r(z) = z$. By the maximum principle, for each $z \in D$, we have the inequality $|\vartheta(z)| \leq |r(z)|$. Therefore, we have

$$u(z) = \frac{\vartheta(z)}{r(z)} = \frac{\operatorname{arcsinh}(g(z) - 1)}{z} = \frac{\operatorname{arcsinh}\left(c_1 z + c_2 z^2 + ...\right)}{z}.$$

Since

$$\operatorname{arcsinh}\left(c_1 z + c_2 z^2 + ...\right) = c_1 z + c_2 z^2 + ... - \frac{1}{2}\frac{\left(c_1 z + c_2 z^2 + ...\right)^3}{3} + ...,$$

we take

$$u(z) = \frac{c_1 z + c_2 z^2 + ... - \frac{1}{2}\frac{\left(c_1 z + c_2 z^2 + ...\right)^3}{3} + ...}{z} = c_1 + c_2 z + ... - \frac{1}{2}\frac{z^2\left(c_1 + c_2 z + ...\right)^3}{3} + ...$$

Here, $u(z)$ is an analytic function in D and $|u(z)| \leq 1$ for $z \in D$. In particular, we have

$$|u(0)| = |c_1| \leq 1 \tag{2.3}$$

and

$$\left|u'(0)\right| = |c_2|.$$

The auxiliary function

$$\Lambda(z) = \frac{u(z) - u(0)}{1 - \overline{u(0)}u(z)}$$

is analytic in D, $\Lambda(0) = 0$, $|\Lambda(z)| < 1$ for $|z| < 1$ and $|\Lambda(1)| = 1$ for $1 \in \partial D$. From (1.2), we obtain

$$\frac{2}{1+|\Lambda'(0)|} \leq \left|\Lambda'(1)\right| = \frac{1-|u(0)|^2}{\left|1-\overline{u(0)}u(1)\right|^2}\left|u'(1)\right| \leq \frac{1+|u(0)|}{1-|u(0)|}\left|u'(1)\right| = \frac{1+|u(0)|}{1-|u(0)|}\left(\left|\vartheta'(1)\right| - \left|r'(1)\right|\right).$$

Also, since

$$\left|\Lambda'(0)\right| = \frac{|u'(0)|}{1-|u(0)|^2} = \frac{|c_2|}{1-|c_1|^2},$$

we take

$$\frac{2}{1+\frac{|c_2|}{1-|c_1|^2}} \leq \frac{1+|c_1|}{1-|c_1|}\left(\frac{|g'(1)|}{\cosh(1)} - 1\right),$$

$$\frac{2\left(1-|c_1|^2\right)}{1-|c_1|^2+|c_2|}\frac{1-|c_1|}{1+|c_1|} + 1 \leq \frac{|g'(1)|}{\cosh(1)}$$

and

$$\left|g'(1)\right| \geq \cosh 1\left(1 + \frac{2\left(1-|c_1|\right)^2}{1-|c_1|^2+|c_2|}\right).$$

Now, let us show that this last inequality is sharp. Let$g(z) = 1+\sinh z^2$. Then,v$g'(z) = 2z\cosh z^2 v$ and $g'(1) = 2\cosh(1)$. On the other hand, we take

$$1 + c_1 z + c_2 z^2 + ... = 1 + \sinh z^2.$$

If we take the derivative of both sides of the last expression and pass to the limit for $z \to 0$, we obtain $c_1 = 0$. Similarly, using straightforward calculations, we take $c_2 = 1$. Thus, we obtain

$$\cosh(1)\left(1 + \frac{2\left(1-|c_2|\right)^2}{1-|c_1|^2+|c_2|}\right) = 2\cosh(1).$$

□

The following theorem shows the relationship between the coefficients c_1 and c_2 in the Maclaurin expansion of the function $g(z) = 1 + c_1 z + c_2 z^2 + ...$

Theorem 2.4. *Let $g \in \mathcal{T}$, $g(z) - 1$ have no zeros in D except $z = 0$ and $c_1 > 0$. Then, we have the inequality*

$$|c_2| \leq 2\left|c_1 \ln(c_1)\right|. \tag{2.4}$$

This result is sharp with equality for the function

$$g(z) = 1 + \sinh\left(z e^{\frac{1+z}{1-z}\ln c_1}\right).$$

Proof. Let $c_1 > 0$ be in the expression of the function $g(z)$. Having in mind equality (2.3) and the function $g(z) - 1$ has no zeros in D except $z = 0$, we use $\ln u(z)$ to denote the analytic branch of the logarithm normed by the condition

$$\ln u(0) = \ln(c_1) < 0.$$

The auxiliary function

$$b(z) = \frac{\ln u(z) - \ln u(0)}{\ln u(z) + \ln u(0)}$$

is analytic in the unit disc D, $|b(z)| < 1$, $b(0) = 0$. The function $b(z)$ we have expressed in Theorem 2.4 satisfies the conditions of the Schwarz lemma. Thus, if we apply the Schwarz lemma to the function $b(z)$, we obtain

$$\left|b'(0)\right| = -\frac{|c_2|}{2c_1\ln(c_1)}$$

and

$$|c_2| \leq 2\left|c_1 \ln(c_1)\right|.$$

To show that inequality (2.4) is sharp, we take an analytic function

$$g(z) = 1 + \sinh\left(ze^{\frac{1+z}{1-z}\ln c_1}\right).$$

If we rearrange the function $g(z)$, we get

$$g(z) - 1 = \sinh\left(ze^{\frac{1+z}{1-z}\ln c_1}\right)$$

and

$$\operatorname{arcsinh}(g(z) - 1) = zw(z),$$

where

$$w(z) = e^{\frac{1+z}{1-z}\ln c_1}. \tag{2.5}$$

Therefore, we take

$$w(z) = \frac{\operatorname{arcsinh}(g(z) - 1)}{z} = \frac{c_1 z + c_2 z^2 + ... - \frac{1}{2}\frac{(c_1 z + c_2 z^2 + ...)^3}{3} + ...}{z}$$
$$= c_1 + c_2 z + ... - \frac{1}{2}\frac{z^2 (c_1 + c_2 z + ...)^3}{3} + ...$$

and

$$\left|w'(0)\right| = |c_2|.$$

If we take the derivative of the function $w(z)$ defined in (2.5), we get

$$\left|w'(0)\right| = 2\left|c_1 \ln c_1\right|.$$

Thus, we obtain

$$|c_2| = 2\left|c_1 \ln c_1\right|.$$

□

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

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