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Research article

Quasi Yamabe Solitons on CR Submanifolds of Maximal CR Dimension in Kähler Manifolds

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

Abstract

In this paper we give necessary and sufficient conditions for a CR submanifold of maximal CR dimension in arbitrary Kähler manifold to admit (quasi-)Yamabe structure, with naturally chosen soliton vector field. When the ambient manifold is a non-flat complex space form, we give a complete classification of such solitons, under certain conditions.

Keywords: *Yamabe soliton, quasi-Yamabe soliton, complex space form, CR submanifold of maximal CR dimension, structure vector field, constant scalar curvature.*

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1. Introduction

Riemannian differential geometry originated in attempts to generalize the highly successful theory of compact surfaces. One consequence of the famous uniformization theorem of complex analysis is the fact that every compact surface has a conformally equivalent metric of the constant (Gaussian) curvature. However, in a higher dimension ($n \geqslant 3$) this can not be true since the Riemannian curvature tensor has the independent components of the order n^4, while a conformal change of metric allows us to choose only one unknown function. From this point of view it seems natural instead to seek a conformal change of metric that makes only the scalar curvature (the complete contraction of the curvature tensor) constant. This famous problem is known as the **Yamabe problem** and it was completely solved (see [13]) using techniques of elliptic partial differential equations and calculus of variations, without using the notions of Yamabe flow and Yamabe soliton. Therefore, for every compact Riemannian manifold (M, g) of dimension $n \geqslant 3$, it is true that there is a conformally equivalent metric $\bar{g}$ such that $(M, \bar{g})$ has the constant scalar curvature. Such metric $\bar{g}$ is then called a **Yamabe metric**.

R. Hamilton introduced the notion of **Yamabe flow** (see [9]), in which the metric on a Riemannian manifold is deformed by evolving according to the flow $\frac{\partial}{\partial t}g(t) = -\rho(t)g(t)$, where $\rho(t)$ is the scalar curvature of the metric $g(t)$. However, there are certain metrics which, instead of evolving with the flow, remain invariant up to the scaling and diffeomorphisms - they are called self-similar solutions of the flow. One-parameter family of metrics $g(t) = \sigma(t)\psi_t^* g(0)$, for some smooth function $\sigma(t)$ and some one-parameter family of diffeomorphisms ψ_t, which is a self-similar solution of the Yamabe flow, corresponds to the **Yamabe solitons**. In dimension $n = 2$, the Yamabe flow is equivalent to the Ricci flow, which under suitable conditions evolves an initial metric to an Einstein metric. However, in dimension $n > 2$ the Yamabe and Ricci flows do not agree, since the former one evolves an initial metric to a new one with constant scalar curvature within the same conformal class, while the latter one does not in general. Yamabe flows and Yamabe solitons have been studied quite extensively recently (see [10], [12]), as well as Ricci solitons (see [4], [5]). For any geometric soliton, there exists a vector field V (the

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soliton vector field) which relates the Lie derivative of the metric $\mathcal{L}_V g$ with the geometric object defining the flow under consideration.

The aim of submanifold geometry is to understand geometric invariants of submanifolds and to classify submanifolds according to given geometric data. In light of this, the purpose of this paper is to study an important class of submanifolds of Kähler manifolds - CR submanifolds with maximal CR dimension, which are also (quasi-) Yamabe solitons, with a specially, naturally chosen soliton vector field. In Theorem 3.1 we prove necessary and sufficient conditions which have to be satisfied on such a submanifold, when the ambient space is a Kähler manifold. Moreover, when the ambient space is a non-flat complex space form, we are able to give a complete classification of such solitons, under certain conditions.

2. Preliminaries

A Riemannian manifold (M, g) is a **Yamabe soliton** if it admits a vector field V such that

$$\frac{1}{2}\mathcal{L}_V g = (\rho - \lambda)g, \tag{2.1}$$

where $\mathcal{L}_V$ denotes the Lie derivative in the direction of the vector field V, ρ is the scalar curvature of the metric g and λ is a real constant. A vector field V is then called a soliton vector field for (M, g).

A Riemannian manifold (M, g) is a **quasi-Yamabe soliton** if it admits a vector field V such that

$$\frac{1}{2}\mathcal{L}_V g = (\rho - \lambda)g + \mu\eta \otimes \eta, \tag{2.2}$$

where μ is some function, λ is a real constant and η is a dual 1-form of V, given by $\eta(X) = g(X, V)$.

2.1. CR submanifolds of maximal CR dimension of a Kähler manifold.

Let $(\overline{M}^{n+p}, J, \bar{g})$ be a real $(n + p)$-dimensional almost Hermitian manifold such that natural almost complex structure J is the endomorphism of the tangent bundle $T\overline{M}$ satisfying $J^2 = -I$ and $\bar{g}$ is the Riemannian metric of $\overline{M}$ satisfying the Hermitian condition $\bar{g}(J\overline{X}, J\overline{Y}) = \bar{g}(\overline{X}, \overline{Y})$ for any $\overline{X}, \overline{Y} \in T\overline{M}$. Furthermore, let M^n be a real n-dimensional submanifold of $\overline{M}$ defined via the isometric immersion $\imath : M \to \overline{M}$, with Riemannian metric g on M isometrically induced from $\bar{g}$ in such a way that $g(X, Y) = \bar{g}(\imath X, \imath Y)$ for all X, Y tangent to M. We denote by TM and $T^{\perp}M$ the tangent and normal bundle of M, respectively.

It is known that, for any $x \in M$, the holomorphic subspace $H_xM = JT_xM \cap T_xM$ of T_xM is the maximal J-invariant subspace of the tangent space T_xM at x. In general, the dimension of H_xM varies with x, but if the subspace H_xM has constant dimension for any $x \in M$, the submanifold M is called the Cauchy-Riemann submanifold or briefly **CR submanifold** and the constant complex dimension of H_xM is called the CR dimension of M (see [8] for more details). This important class of submanifolds includes both almost complex submanifolds (when $H_xM = T_xM$ is as maximal as possible) and totally real submanifolds (when $H_xM = \{0\}$ is as minimal as possible), which are widely investigated in the literature. In this paper we consider the case when M^n is a CR submanifold of complex $\frac{n+p}{2}$-dimensional almost Hermitian manifold $\overline{M}^{\frac{n+p}{2}}$ whose CR dimension is $\frac{n-1}{2}$, called **maximal CR dimension**. In this case, the above definition of CR submanifolds coincide with Bejancu's definition given in [1]. Besides the typical example of a real hypersurface M^n of $\overline{M}^{\frac{n+1}{2}}$, other important examples are real hypersurfaces of complex submanifolds of $\overline{M}$ and odd-dimensional ϕ'-invariant submanifolds of real hypersurfaces M' of $\overline{M}$, where ϕ' is the almost contact structure of M'. Consequently, M is necessarily odd–dimensional and there exists a unit vector field ξ normal to M such that $JTM \subset TM \oplus \mathrm{span}\{\xi\}$, that is, for any $X \in TM$ we have the following decomposition in tangential and normal components:

$$J\imath X = \imath\,\phi X + \eta(X)\xi, \tag{2.3}$$

where η is one-form on M and ϕ is a skew–symmetric endomorphism acting on TM. Moreover, using (2.3), the Hermitian property of J implies

$$J\xi = -\imath U, \quad g(U, X) = \eta(X), \quad g(U, U) = 1. \tag{2.4}$$

Applying J to (2.3) and (2.4) and comparing the tangential and normal part to M, we obtain

$$\begin{aligned} \phi^2 X &= -X + \eta(X)U, & \eta(\phi X) &= 0, \\ g(\phi X, \phi Y) &= g(X,Y) - \eta(X)\eta(Y), & \phi U &= 0. \end{aligned} \tag{2.5}$$

These relations imply that (ϕ, η, U, g) is an **almost contact metric structure** on M.

Since it is well-known that the subbundle $T_1^\perp M = \{\eta \in T^\perp M | \,\overline{g}(\eta, \xi) = 0\}$ is J-invariant, we can choose a local J-adapted orthonormal basis $\xi, \xi_1, \ldots, \xi_q, \xi_{1^*}, \ldots, \xi_{q^*}$ of $T^\perp M$ so that $\xi_{a^*} = J\xi_a$, $a = 1, \ldots, q$, $q = \frac{p-1}{2}$.

Next, let $\overline{\nabla}$ and ∇ denote the Levi-Civita connection of $\overline{M}$ and M, respectively, and D the induced normal connection from $\overline{\nabla}$ to $T^\perp M$. As $D_X\xi = \sum_{a=1}^{q}\{s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*}\}$, the distinguished vector field ξ is parallel with respect to the normal connection D, namely

$$D_X\xi = 0, \tag{2.6}$$

if and only if $s_a = s_{a^*} = 0$, for all $a = 1, \ldots, q$. Consequently, the Gauss and Weingarten formulae (both for the distinguished normal ξ and for the normals ξ_a, ξ_{a^*}) hold in the following manner:

$$\overline{\nabla}_{\imath X}\imath Y = \imath\nabla_X Y + g(AX,Y)\xi + \sum_{a=1}^{q}\{g(A_aX,Y)\xi_a + g(A_{a^*}X,Y)\xi_{a^*}\}, \tag{2.7}$$

$$\overline{\nabla}_{\imath X}\xi = -\imath AX + D_X\xi = -\imath AX, \tag{2.8}$$

$$\overline{\nabla}_{\imath X}\xi_a = -\imath A_aX + D_X\xi_a = -\imath A_aX + \sum_{b=1}^{q}\{s_{ab}(X)\xi_b + s_{ab^*}(X)\xi_{b^*}\}, \tag{2.9}$$

$$\overline{\nabla}_{\imath X}\xi_{a^*} = -\imath A_{a^*}X + D_X\xi_{a^*} = -\imath A_{a^*}X + \sum_{b=1}^{q}\{s_{a^*b}(X)\xi_b + s_{a^*b^*}(X)\xi_{b^*}\}, \tag{2.10}$$

for all $X, Y \in TM$, where A, A_a, A_{a^*} are the shape operators for the normals ξ, ξ_a, ξ_{a^*}, $a = 1, \ldots, q$, $q = \frac{p-1}{2}$, respectively and s's are the coefficients of the normal connection D. Recall that real hypersurfaces of totally geodesic complex submanifolds of $\overline{M}$ and odd-dimensional ϕ'-invariant submanifolds of real hypersurfaces M' of $\overline{M}$, where ϕ' is the almost contact structure of M', such that $A'X' = \lambda' X' + \mu u'(X')U'$, as well as real hypersurfaces of $\overline{M}$, are examples of CR submanifolds of maximal CR dimension with $D\xi = 0$. For more details we refer to [8].

When the ambient almost Hermitian manifold $(\overline{M}, J)$ is a Kähler manifold ($\overline{\nabla}J = 0$), taking the covariant derivative of (2.3) and comparing the tangential and normal part, we conclude that

$$\nabla_X U = \phi AX, \tag{2.11}$$

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)U, \tag{2.12}$$

$$(\nabla_X\eta)Y = g(\phi AX, Y). \tag{2.13}$$

Further, if $D\xi = 0$, taking the covariant derivative of $\xi_{a^*} = J\xi_a$ and using (2.3), (2.4), (2.6), (2.9) and (2.10), it follows that

$$\begin{aligned} &A_aU = 0, && A_a = -\phi A_{a^*}, && A_a\phi = -\phi A_a, \\ &A_{a^*}U = 0, && A_{a^*} = \phi A_a, && A_{a^*}\phi = -\phi A_{a^*}, \\ &\mathrm{Tr}A_a = \mathrm{Tr}A_{a^*} = 0 && \text{for all } a = 1, \ldots, q. \end{aligned} \tag{2.14}$$

2.2. Complex space forms $\overline{M}^m(c)$.

A Kähler manifold $(\overline{M}, \bar{g}, J)$ is called a *complex space form* if it has constant holomorphic sectional curvature c. The only complete, simply connected complex space forms, of a complex dimension m, are: complex projective space $\mathbb{CP}^m$ ($c > 0$), complex Euclidean space $\mathbb{C}^m$ ($c = 0$) and complex hyperbolic space $\mathbb{CH}^m$ ($c < 0$). It is well-known (see [8], [15]) that the Riemannian curvature tensor $\overline{R}$ of $\overline{M}^m(c)$ is given by

$$\overline{R}(\bar{X},\bar{Y})\bar{Z} = \frac{c}{4}\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y} + \bar{g}(J\bar{Y},\bar{Z})J\bar{X} - \bar{g}(J\bar{X},\bar{Z})J\bar{Y} + 2\bar{g}(\bar{X},J\bar{Y})J\bar{Z}\}. \tag{2.15}$$

Consequently, for a CR submanifold of maximal CR dimension M^n of a complex space form $\overline{M}^{\frac{n+p}{2}}$, the Gauss equation, as well as the Codazzi equation for the shape operator with respect to parallel distinguished vector field ξ ($D\xi = 0$), become

$$\begin{aligned} R(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ &- 2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY, \\ &+ \sum_{a=1}^{q}(g(A_aY,Z)A_aX - g(A_aX,Z)A_aY) \\ &+ \sum_{a=1}^{q}(g(A_{a^*}Y,Z)A_{a^*}X - g(A_{a^*}X,Z)A_{a^*}Y), \end{aligned} \tag{2.16}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)U\}, \tag{2.17}$$

for all X, Y, Z tangent to M. Therefore, using (2.14) and (2.16), we obtain that the Ricci tensor Ric, the Ricci operator S and the scalar curvature ρ of M are given by:

$$\mathrm{Ric}(X,Y) = \frac{c}{4}((n+2)g(X,Y) - 3\eta(X)\eta(Y)) + (\mathrm{Tr}A)g(AX,Y) - g(AX,AY), \tag{2.18}$$

$$SX = \frac{c}{4}((n+2)X - 3\eta(X)U) + (\mathrm{Tr}A)AX - A^2X, \tag{2.19}$$

$$\rho = (n+3)(n-1)\frac{c}{4} + (\mathrm{Tr}A)^2 - \mathrm{Tr}(A^2). \tag{2.20}$$

For the case of a complex projective and complex hyperbolic space, we will use the well-known complete classification of real hypersurfaces whose almost contact structure and shape operator commute, which will prove extremely useful in the proofs of Theorem 3.3 and Theorem 3.4.

Theorem 2.1. *[14][16] The only complete real hypersurfaces in the non-flat complex space forms $\overline{M}^n(c)$, $n \geqslant 2$, whose almost contact structure and shape operator commute are the following hypersurfaces (these hypersurfaces are known in the literature as hypersurfaces of type A):*

- *geodesic hyperspheres in $\mathbb{CP}^n$;*
- *tubes over totally geodesic complex projective space $\mathbb{CP}^k$ in $\mathbb{CP}^n$, $1 \leqslant k \leqslant n-2$, $n \geqslant 3$;*
- *horospheres in $\mathbb{CH}^n$;*
- *geodesic hyperspheres in $\mathbb{CH}^n$;*
- *tubes over totally geodesic complex hyperbolic hyperplanes $\mathbb{CH}^{n-1}$ in $\mathbb{CH}^n$;*
- *tubes over totally geodesic complex hyperbolic space $\mathbb{CH}^k$ in $\mathbb{CH}^n$, $1 \leqslant k \leqslant n-2$, $n \geqslant 3$.*

3. Main results

Let us now investigate real n-dimension CR submanifolds M of maximal CR dimension in a Kähler manifold $(\overline{M}^{\frac{n+p}{2}}, J, \bar{g})$, which are the (quasi)-Yamabe solitons with a soliton vector field specially chosen to be a structure vector field $U = -J\xi$, where ξ is a distinguished normal vector field on M, determined by (2.3). We will call such submanifolds **maximal CR (quasi)-Yamabe solitons**.

Recall that a vector field X in a Riemannian manifold is a **Killing vector field** when its flow is locally an isometry, namely if X satisfies the so called Killing equation $\mathcal{L}_X g = 0$. Observe that the following result holds not only for the hypersurfaces, but for all CR submanifolds.

Lemma 3.1. *The structure vector field U of a CR submanifold of maximal CR dimension in a Kähler manifold is Killing if and only if the shape operator A and the almost contact metric structure ϕ commute.*

Proof. From the definition of the Lie derivative of the metric tensor g in direction of U

$$(\mathcal{L}_U g)(X,Y) = g(\nabla_X U, Y) + g(\nabla_Y U, X), \tag{3.1}$$

and (2.11), it follows that

$$(\mathcal{L}_U g)(X,Y) = g((\phi A - A\phi)X, Y). \tag{3.2}$$

Therefore, U is a Killing vector field if and only if $g((\phi A - A\phi)X, Y) = 0$ holds for all $X, Y \in TM$, which is equivalent to the condition $A\phi = \phi A$. □

Remark 3.1. Using (2.5), the condition $A\phi = \phi A$ implies $AU = \alpha U$, where the function α is given by $\alpha = g(AU, U)$.

Furthermore, X is a **geodesic vector field** if the acceleration of its integral curve is proportional to velocity. Moreover, a unit vector field on a Riemannian manifold is a geodesic vector field when its integral curves are geodesics, i.e. have zero acceleration. Next, we will give a sufficient condition for the structure vector field U of a CR submanifold of maximal CR dimension to be a geodesic vector field.

Proposition 3.1. *Let M be a CR submanifold of maximal CR dimension, with the structure vector field U, in a Kähler manifold $\overline{M}$. If M is a quasi-Yamabe soliton with U as a soliton vector field, then U is a geodesic vector field.*

Proof. Using (2.2) and (3.2) it follows that M is a quasi-Yamabe soliton with the soliton vector field $U = -J\xi$ if and only if the following relation holds for every $X, Y \in TM$

$$\frac{1}{2}g((\phi A - A\phi)X, Y) = (\rho - \lambda)g(X, Y) + \mu g(X, U)g(Y, U). \tag{3.3}$$

Therefore, as U is a unit vector field, the endomorphism ϕ is skew-symmetric and $\phi U = 0$, substituting $X = U$ and $Y = \nabla_U U$ in (3.3), it follows $g(\phi AU, \nabla_U U) = 0$. Consequently, using (2.11), we conclude $\nabla_U U = 0$, namely U is a geodesic vector field. □

Remark 3.2. Since (2.11) implies $\nabla_U U = \phi AU$, from the definition it follows that the vector field U is geodesic if and only if U is in the kernel of the endomorphism ϕ, i.e. U is a principal vector field for the shape operator A. This means that $AU = \alpha U$ holds, where the function α is given by $\alpha = g(AU, U)$. When M is a real hypersurface, this is one way to define a Hopf hypersurface and it is known that, for a Hopf hypersurface of a complex space form, the function α is constant. In Proposition 3.2 we will present a new sufficient condition for α to be constant, in the case when M is a CR submanifold of maximal CR dimension in a non-flat complex space form.

Now, let us prove that a CR submanifold of maximal CR dimension M of a Kähler manifold is a maximal CR quasi-Yamabe soliton if and only if it is a maximal CR Yamabe soliton and we find the necessary and sufficient conditions for M to be a maximal CR Yamabe soliton.

Theorem 3.1. *For a CR submanifold M of maximal CR dimension of an arbitrary Kähler manifold $\overline{M}$, the following conditions are equivalent:*

1. *M is a maximal CR quasi-Yamabe soliton;*
2. *M is a maximal CR Yamabe soliton;*
3. *the structure vector field of M is a Killing vector field and the scalar curvature of M is constant.*

Proof. Implications $(2) \Rightarrow (1)$ and $(3) \Rightarrow (2)$ follow directly from the definitions (2.1) and (2.2) of maximal CR (quasi)-Yamabe solitons.

Let us proceed to the proof of the implications $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$. Supposing that M is a maximal CR quasi-Yamabe soliton, i.e. a CR submanifold of maximal CR dimension which is a quasi-Yamabe soliton with the soliton vector field $U = -J\xi$ and using Proposition 3.1, we conclude that U is a geodesic vector field and $AU = \alpha U = g(AU, U)U$. Therefore, substituting $X = Y = U$ into (3.3), we get $\rho = \lambda - \mu$. If α is the only eigenvalue of A, choosing arbitrary $Z \perp U$ such that $AZ = \alpha Z$ and replacing X and Y in (3.3) by Z, leads to

$$\rho = \lambda. \tag{3.4}$$

If A has more than one eigenvalue, let $Z \perp U$ be a vector field on M such that $AZ = \nu Z$, i.e. let Z be the eigenvector of A with the principal curvature function $\nu \neq \alpha$. The same arguments asE those above again lead to relation (3.4).

Therefore, using (3.4), we conclude that $\mu = 0$, i.e. every maximal CR quasi-Yamabe soliton is also a maximal CR Yamabe soliton. Moreover, its scalar curvature ρ is constant, since λ is constant. We conclude the proof by showing that if M is a maximal CR Yamabe soliton, then U is a Killing vector field. Since $\rho = \lambda$ and $\mu = 0$, relation (3.3) implies that $g((\phi A - A\phi)X, Y) = 0$ holds for all tangent vector fields X, Y on M, which, according to Lemma 3.1, implies that U is a Killing vector field. □

Example 3.1. Real hypersurfaces of complex manifolds are very important in our study since it is well-known that they are CR submanifolds of maximal CR dimension. It is proved in [3] that the only complete real hypersurfaces in the non-flat complex space forms, which are (quasi)-Yamabe solitons with the Reeb vector field as a soliton vector field, are those known in the literature as the hypersurfaces of type A: geodesic hyperspheres, tubes over totally geodesic complex projective space (in a complex projective space), horospheres, geodesic hyperspheres, tubes over totally geodesic complex hyperbolic hyperplanes and tubes over totally geodesic complex hyperbolic space (in a complex hyperbolic space). However, it is well-known (see [8]) that these manifolds M^n can be considered as CR submanifolds with maximal CR dimension isometrically immersed in complex space forms $\overline{M}(c)$ ($\overline{M} = \mathbb{CP}^{\frac{n+p}{2}}$ or $\overline{M} = \mathbb{CH}^{\frac{n+p}{2}}$), with the real codimension $p > 1$. Moreover, it is established that they are real $n-$dimensional hypersurfaces of the non-flat complex space forms M' ($M' = \mathbb{CP}^{\frac{n+1}{2}}$ or $M' = \mathbb{CH}^{\frac{n+1}{2}}$) which are totally geodesic complex submanifolds of a complex space form $\overline{M}$ ($\overline{M} = \mathbb{CP}^{\frac{n+p}{2}}$ or $\overline{M} = \mathbb{CH}^{\frac{n+p}{2}}$). We also recall that in this case ξ is parallel with respect to the normal connection.

More precisely, let M' be a complex submanifold of a non-flat complex space form $(\overline{M}^{\frac{n+p}{2}}, J)$ with totally geodesic immersion $j : M' \to \overline{M}$ and let M^n be a real hypersurface of a complex manifold (M', J') with immersion $\imath' : M \to M'$, where J' is the induced complex structure of M'. Moreover, let M be a quasi Yamabe soliton in M' with the Reeb vector field as a soliton vector field. We will prove that $\imath = j \circ \imath'$ is the immersion of M in $\overline{M}$ which endows it with a maximal CR quasi Yamabe soliton structure. We denote the Levi-Civita connections of $M, M', \overline{M}$ by $\nabla, \nabla', \overline{\nabla}$, respectively. Furthermore, let ξ' be the unit normal vector field of M in M' and ξ_a, $a = 1, \dots, p-1$ be the unit normal vector fields of M' in $\overline{M}$. Let us denote the shape operators corresponding to unit normal vector fields $\xi = j\xi', \xi_a$ of M in $\overline{M}$ by A, A_a, $a = 1, \dots, p-1$.

Consequently, the Gauss formula implies that for all $X, Y \in TM$ the following relations hold

$$\begin{aligned} \overline{\nabla}_X(\imath Y) &= \overline{\nabla}_X((j \circ \imath')Y) = j\nabla'_X(\imath' Y) = j(\imath'\nabla_X Y + h'(X,Y)) \\ &= j(\imath'\nabla_X Y + g(A'X,Y)\xi') = \imath\nabla_X Y + g(A'X,Y)j\xi', \end{aligned} \tag{3.5}$$

$$\overline{\nabla}_X(\imath Y) = \imath\nabla_X Y + h(X,Y) = \imath\nabla_X Y + g(AX,Y)\xi + \sum_{a=1}^{p-1} g(A_a X, Y)\xi_a. \tag{3.6}$$

Comparing (3.5) and (3.6), we derive that $A = A'$ and $A_a = 0$. Consequently, since from [3] it follows that the scalar curvature ρ' of M in M' is constant, using (2.20), we conclude that the scalar curvature ρ of M in $\overline{M}$ is also constant.

Moreover, since M' is a complex submanifold of $\overline{M}$, relation $JjX' = jJ'X'$ holds for any $X' \in TM'$. Thus, we compute

$$\begin{aligned} J\imath X &= J(j \circ \imath')X = jJ'\imath' X = j(\imath'\phi' X + \eta'(X)\xi') \\ &= \imath\phi' X + \eta'(X)j\xi' = \imath\phi' X + \eta'(X)\xi, \end{aligned} \tag{3.7}$$

where we denoted by ϕ' and η' the almost contact structure and $1-$form for the immersion of M in M', defined by the relation for hypersurfaces, analogous to (2.3). Comparing (2.3) and (3.7), we obtain $\phi' = \phi$, $\eta' = \eta$. Consequently, we deduce that $A\phi = A'\phi' = \phi' A' = \phi A$, so the induced almost contact structure ϕ and the shape operator A commute. From [3] it follows that the structure vector field $U' = -J'\xi'$ of M in M' is a Killing vector field. Since $A\phi = A'\phi' = \phi' A' = \phi A$, using Lemma 3.1 we conclude that the structure vector field $U = -J\xi$ of M in $\overline{M}$ is also Killing and that the conditions of Theorem 3.1 are satisfied. Therefore, M^n is a maximal CR quasi-Yamabe soliton in $\overline{M}^{\frac{n+p}{2}}$.

Remark 3.3. Let M be a CR submanifold of maximal CR dimension of a complex space form $\overline{M}$. It is well-known (see [8]) that if the shape operator A associated to ξ has only one eigenvalue, then $\overline{M}$ is a complex Euclidean space.

We will now restrict our further investigation of the maximal CR quasi Yamabe solitons when the ambient space is a non-flat complex space form and the distinguished normal vector field ξ is parallel with respect to the normal connection D. It turns out, with the assumptions above, that all the eigenvalues of the shape operator associated to ξ are constant, namely the following proposition holds.

Proposition 3.2. *Let M be a maximal CR quasi Yamabe soliton of a non-flat complex space form with distinguished normal vector field ξ parallel with respect to the normal connection. Then the shape operator A, associated to ξ, has at most three distinct eigenvalues and they are constant.*

Proof. Under the hypotheses of Proposition 3.2, using Remark 3.1, Remark 3.2 and Theorem 3.1, we have concluded that $AU = \alpha U$ and $A\phi = \phi A$. Let us first prove that α is constant. Differentiating $AU = \alpha U$ covariantly, with respect to arbitrary tangent vector field X, we obtain $(\nabla_X A)U + A\nabla_X U = (X\alpha)U + \alpha\nabla_X U$. Since ∇A is symmetric, after taking the scalar product with arbitrary tangent vector field Y and using (2.11), we derive

$$g((\nabla_X A)Y, U) = (X\alpha)\eta(Y) + \alpha g(\phi AX, Y) - g(A\phi AX, Y). \tag{3.8}$$

Switching the roles of X and Y, and using $A\phi = \phi A$, relation (3.8) now implies

$$g((\nabla_X A)Y - (\nabla_Y A)X, U) = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + 2\alpha g(\phi AX, Y) - 2g(A\phi AX, Y). \tag{3.9}$$

Since the Codazzi equation (2.17) implies $g((\nabla_X A)Y - (\nabla_Y A)X, U) = -\frac{c}{2}g(\phi X, Y)$, relation (3.9) leads to

$$(X\alpha)\eta(Y) - (Y\alpha)\eta(X) + 2\alpha g(\phi AX, Y) + \frac{c}{2}g(\phi X, Y) - 2g(A\phi AX, Y) = 0. \tag{3.10}$$

Replacing Y by U in (3.10), we conclude

$$X\alpha = (U\alpha)\eta(X). \tag{3.11}$$

Differentiating covariantly (3.11), then interchanging X and Y and subtracting, gives

$$Y(U\alpha)\eta(X) - X(U\alpha)\eta(Y) = (U\alpha)(g(\phi AX, Y) - g(\phi AY, X)). \tag{3.12}$$

As $A\phi = \phi A$, A is symmetric and ϕ is skew-symmetric, substituting Y for U in (3.12), we deduce

$$X(U\alpha) = U(U\alpha)\eta(X). \tag{3.13}$$

Substituting (3.13) in (3.12), yields

$$(U\alpha)\, g(\phi AX, Y) = 0 \tag{3.14}$$

for all tangent vector fields X, Y. If there is a point $x \in M$ such that $(U\alpha)(x) \neq 0$, then there is a neighborhood $\mathcal{U}(x)$ such that $\phi A = 0 = A\phi$ on $\mathcal{U}$. Let us note that putting (3.11) back in (3.10) gives that relation

$$\phi A^2 X - \alpha\phi AX - \frac{c}{4}\phi X = 0 \tag{3.15}$$

holds for all tangent vector fields X. Consequently, $\phi A = 0$ and (3.15) would imply $c = 0$, which is a contradiction, since $\overline{M}$ is a non-flat complex space form. Thus we have proved that $U\alpha = 0$. Now, (3.11) directly implies that α is a constant function.

Finally, since A is a symmetric operator, let $X \perp U$ be another eigenvector with the corresponding eigenvalue λ. Then, according to (3.15), it follows

$$\lambda^2 - \alpha\lambda - \frac{c}{4} = 0. \tag{3.16}$$

Therefore, the shape operator A has at most three distinct eigenvalues and they are constant. Moreover, since $\overline{M}$ is a non-flat complex space form, having in mind Remark 3.3, we conclude that the shape operator A has two or three distinct, constant eigenvalues. □

Remark 3.4. Let us note that we have confirmed one of the necessary conditions for a CR submanifold of maximal CR dimension of a non-flat complex space form with $D\xi = 0$ to be a quasi Yamabe soliton. Namely, using relation (2.20) and Proposition 3.2, we deduce that the scalar curvature of M^n must be constant.

Further, we continue our investigation of maximal CR quasi Yamabe solitons M^n in a non-flat complex space form $\overline{M}$ under the additional assumption $A_a = 0 = A_{a^*}$, $a = 1, \ldots, q$, $q = \frac{p-1}{2}$. Using the codimension reduction technique, we will prove in Proposition 3.3 that there exists a a real $(n+1)$-dimensional totally geodesic non-flat complex space form M' of $\overline{M}$ such that M is a real hypersurface of M'.

First, let us define $N_0 = \{\eta \in T_x^{\perp}M : A_\eta = 0\}$ and note that, under the conditions stated above,

$$N_0(x) = \text{span}\,\{\xi_1(x), \ldots, \xi_q(x), \xi_{1^*}(x), \ldots, \xi_{q^*}(x)\}.$$

More precisely, the assumption $A_a = 0 = A_{a^*}$, $a = 1, \ldots, q$, implies

$$\text{span}\,\{\xi_1(x), \ldots, \xi_q(x), \xi_{1^*}(x), \ldots, \xi_{q^*}(x)\} \subset N_0(x).$$

On the other hand, for any $\nu \in N_0(x)$, $\nu = l\xi + \sum_{a=1}^{q}\{l^a\xi_a + l^{a^*}\xi_{a^*}\}$, we obtain $0 = A_\nu = lA + \sum_{a=1}^{q}\{l^a A_a + l^{a^*}A_{a*}\} = lA$. As $A \not\equiv 0$, it follows that $l = 0$ and

$$\begin{aligned} \nu &= \sum_{a=1}^{q}\{p^a\xi_a + p^{a^*}\xi_{a^*}\} \in \text{span}\,\{\xi_1(x),\ldots,\xi_q(x),\xi_{1^*}(x),\ldots,\xi_{q^*}(x)\}, \text{ i.e.} \\ N_0(x) &\subset \text{span}\,\{\xi_1(x),\ldots,\xi_q(x),\xi_{1^*}(x),\ldots,\xi_{q^*}(x)\}. \end{aligned}$$

Furthermore, since $J\xi_a = \xi_{a^*}$, we have $JN_0(x) = N_0(x)$ and consequently we conclude that

$$H_0(x) = JN_0(x) \cap N_0(x) = \text{span}\,\{\xi_1(x),\ldots,\xi_q(x),\xi_{1^*}(x),\ldots,\xi_{q^*}(x)\}$$

is the maximal J-invariant subspace of N_0 and $JH_0(x) = H_0(x)$, since J is an isomorphism. Hence the orthogonal complement $H_1(x)$ of $H_0(x)$ in $T_x^\perp M$ is spanned by $\{\xi\}$ and its dimension is one. Since by assumption ξ is parallel with respect to the normal connection, we can apply the following (codimension reduction) theorems (for a complex projective space and for a complex hyperbolic space), for $H(x) = H_0(x)$ and therefore $H_2(x) = H_1(x) = \text{span}\{\xi\}$ and $r = 1$.

Theorem 3.2. *[17], [11] Let M be a real n-dimensional submanifold of a real $(n+p)$-dimensional complex projective space $\mathbb{CP}^{\frac{n+p}{2}}$ (complex hyperbolic space $\mathbb{CH}^{\frac{n+p}{2}}$) and $H(x)$ be a J-invariant subspace of $H_0(x)$. If the orthogonal complement $H_2(x)$ of $H(x)$ in $T_x^\perp M$ is invariant under parallel translation with respect to the normal connection and r is the constant dimension of H_2, then there exists a real $(n+r)$-dimensional totally geodesic complex projective subspace $\mathbb{CP}^{\frac{n+r}{2}}$ (complex hyperbolic subspace $\mathbb{CH}^{\frac{n+r}{2}}$) such that $M \subset \mathbb{CP}^{\frac{n+r}{2}}$ ($M \subset \mathbb{CH}^{\frac{n+r}{2}}$).*

Summarizing, we conclude that maximal CR quasi Yamabe soliton M^n may be regarded as a real hypersurface of $\mathbb{CP}^{\frac{n+1}{2}}$ ($\mathbb{CH}^{\frac{n+1}{2}}$), which are totally geodesic submanifolds in $\mathbb{CP}^{\frac{n+p}{2}}$ ($\mathbb{CH}^{\frac{n+p}{2}}$) and the following proposition holds.

Proposition 3.3. *Let M^n be a maximal CR quasi Yamabe soliton of a complex projective space $\mathbb{CP}^{\frac{n+p}{2}}$ (respectively a complex hyperbolic space $\mathbb{CH}^{\frac{n+p}{2}}$) with distinguished normal vector field ξ parallel with respect to the normal connection. If the shape operators for the normals ξ_a, ξ_{a^*}, $a = 1,\ldots,q$, $q = \frac{p-1}{2}$ vanish identically, then there exists a totally geodesic complex projective subspace $\mathbb{CP}^{\frac{n+1}{2}}$ (respectively complex hyperbolic subspace $\mathbb{CH}^{\frac{n+1}{2}}$) such that M^n is a real hypersurface of $\mathbb{CP}^{\frac{n+1}{2}}$ (respectively $\mathbb{CH}^{\frac{n+1}{2}}$).*

In what follows we denote $\mathbb{CP}^{\frac{n+1}{2}}$ ($\mathbb{CH}^{\frac{n+1}{2}}$) by M' and $\mathbb{CP}^{\frac{n+p}{2}}$ ($\mathbb{CH}^{\frac{n+p}{2}}$) by $\overline{M}$. Let $\imath_1$ be the immersion of M into M' and let $\imath_2$ be the totally geodesic immersion of M' into $\overline{M}$. Then, using the Gauss formula (2.7) for hypersurfaces, it follows that

$$\nabla'_X \imath_1 Y = \imath_1 \nabla_X Y + g'(A'X,Y)\xi', \tag{3.17}$$

where ξ' is the unit vector field to M in M', A' is its shape operator. Since M' is totally geodesic in $\overline{M}$, using (3.17), we compute for the immersion $\imath = \imath_2 \circ \imath_1$ of M in $\overline{M}$

$$\overline{\nabla}_X \imath Y = \imath_2 \nabla'_X \imath_1 Y = \imath_2(\imath_1 \nabla_X Y + g'(A'X,Y)\xi'), \tag{3.18}$$

since M' is totally geodesic in $\overline{M}$. Comparing (3.18) and (2.7), we conclude $\xi = \imath_2\xi'$ and $A' = A$.

Further, as M' is a complex submanifold of $\overline{M}$, relation $J\imath_2 X' = \imath_2 J'X'$ holds for any $X' \in TM'$, where J' is the induced complex structure of M'. Thus, we compute

$$\begin{aligned} J\imath X &= J\imath_2 \circ \imath_1 X = \imath_2 J'\imath_1 X = \imath_2(\imath_1\phi'X + \eta'(X)\xi') \\ &= \imath\phi'X + \eta'(X)\imath_2\xi' = \imath\phi'X + \eta'(X)\xi. \end{aligned} \tag{3.19}$$

Comparing (2.3) and (3.19), we conclude that $\phi' = \phi$, $\eta' = \eta$. Consequently, as M is a maximal CR quasi Yamabe soliton of a non-flat complex space form, Theorem 3.1 implies that $A\phi = \phi A$. Thus, we deduce that M is a real hypersurface of a non-flat complex space form M' which satisfies $A'\phi' = \phi' A'$, so we can apply known results from the hypersurface theory, specifically Theorem 2.1. This finishes the proofs of the following theorems.

Theorem 3.3. *Let M^n, $n \geq 3$, be a maximal CR quasi Yamabe soliton in a complex projective space $\mathbb{CP}^{\frac{n+p}{2}}$. If vector field ξ is parallel with respect to the normal connection and all the shape operators with respect to other normal vector fields vanish, then M^n is locally congruent to:*

- *geodesic hypersphere;*
- *tube over totally geodesic complex projective space* $\mathbb{CP}^k$ *in* $\mathbb{CP}^{\frac{n+1}{2}}$, $1 \leqslant k \leqslant \frac{n-1}{2}$.

Theorem 3.4. *Let* M^n, $n \geq 3$, *be a maximal CR quasi Yamabe soliton in a complex hyperbolic space* $\mathbb{CH}^{\frac{n+p}{2}}$. *If vector field* ξ *is parallel with respect to the normal connection and all the shape operators with respect to other normal vector fields vanish, then* M^n *is locally congruent to:*

- *horosphere;*
- *geodesic hypersphere;*
- *tube over totally geodesic complex hyperbolic hyperplane;*
- *tube over totally geodesic complex hyperbolic space* $\mathbb{CH}^k$ *in* $\mathbb{CH}^{\frac{n+1}{2}}$, $1 \leqslant k \leqslant \frac{n-1}{2}$.

Remark 3.5. Taking into account the proofs of Theorem 3.3 and Theorem 3.4, and using the results from [3], we may conclude that if M^n $(n \geq 3)$ is a maximal CR quasi Yamabe soliton in $\mathbb{CP}^{\frac{n+p}{2}}$ ($\mathbb{CH}^{\frac{n+p}{2}}$, respectively), with $D\xi = 0$ and $A_a = 0 = A_{a^*}$, $a = 1, \dots, q$, $q = \frac{p-1}{2}$, then M^n is a quasi-Yamabe soliton in $\mathbb{CP}^{\frac{n+1}{2}}$ ($\mathbb{CH}^{\frac{n+1}{2}}$, respectively). More precisely, under the above conditions, it follows that $A\phi = \phi A$, $A = A'$, $\phi = \phi'$, which implies $A'\phi' = \phi' A'$. Moreover, since Proposition 3.2 claims that the eigenvalues of $A(= A')$ are constant, it follows that $\rho' = (n^2 - 1)c + (\mathrm{Tr}A')^2 - \mathrm{Tr}(A'^2)$ is constant. Thus, Theorem 3.1. from [3] asserts that M^n is a quasi-Yamabe soliton in $M' = \mathbb{CP}^{\frac{n+1}{2}}$ ($\mathbb{CH}^{\frac{n+1}{2}}$, respectively).

In Riemannian geometry, the structure of a submanifold is encoded in the second fundamental form. Let us recall that in [6] the authors studied a certain algebraic relation between the naturaly induced almost contact structure tensor ϕ and the second fundamental form h of CR submanifolds of maximal CR dimension M^n in a complex space form $\overline{M}$. They proved that if the condition

$$h(\phi X, Y) + h(X, \phi Y) = 0 \tag{3.20}$$

is satisfied for all $X, Y \in TM$, then $A\phi = \phi A$ and $AU = \alpha U$. Moreover, the ambient manifold $\overline{M}$ has to be a complex Euclidean space or the distinguished vector field ξ is parallel with respect to a normal connection D. Consequently, for a non-flat space form, they proved that $A_a = 0 = A_{a^*}$. Therefore, in Proposition 3.3, Theorem 3.3 and Theorem 3.4, we may replace the conditions: the vector field ξ is parallel with respect to the normal connection and all the shape operators with respect to other normal vector fields vanish, with the condition (3.20). Proceeding analogously to the proofs of Theorem 3.3 and Theorem 3.4 the following proposition holds.

Theorem 3.5. *Let* M^n, $n \geq 3$, *be a maximal CR quasi Yamabe soliton in a complex non-flat space form* $\overline{M}^{\frac{n+p}{2}}$. *If the condition* (3.20) *is satisfied, then* M^n *is locally congruent to:*

- *geodesic hypersphere;*
- *tube over totally geodesic complex projective space* $\mathbb{CP}^k$ *in* $\mathbb{CP}^{\frac{n+1}{2}}$, $1 \leqslant k \leqslant \frac{n-1}{2}$.
- *horosphere;*
- *geodesic hypersphere;*
- *tube over totally geodesic complex hyperbolic hyperplane;*
- *tube over totally geodesic complex hyperbolic space* $\mathbb{CH}^k$ *in* $\mathbb{CH}^{\frac{n+1}{2}}$, $1 \leqslant k \leqslant \frac{n-1}{2}$.

In [7] the authors proved that if for n-dimensional compact, minimal CR submanifold of maximal CR dimension of a complex projective space $\mathbb{CP}^{\frac{n+p}{2}}$ the scalar curvature ρ of M satisfies

$$\rho \geq (n+2)(n-1),$$

then the naturaly induced almost contact structure tensor ϕ and the shape operator A commute, $A_a = A_{a^*} = 0$, $a = 1, \dots, q$ and $\rho = (n+2)(n-1)$. This, using Theorem 3.1, proves the following theorem.

Theorem 3.6. *Let* M *be an* n*-dimensional compact, minimal CR submanifold of maximal CR dimension of* $\mathbb{CP}^{\frac{n+p}{2}}$. *If the scalar curvature* ρ *of* M *satisfies*

$$\rho \geq (n+2)(n-1), \tag{3.21}$$

then M *is a maximal CR quasi-Yamabe soliton.*

Remark 3.6. Note that in [7] the authors also proved if M is an n-dimensional compact, minimal CR submanifold of maximal CR dimension of $\mathbb{CP}^{\frac{n+p}{2}}$ and if the scalar curvature ρ of M satisfies (3.21), then M is congruent to a geodesic hypersphere or a tube over totally geodesic complex projective space $\mathbb{CP}^k$ in $\mathbb{CP}^{\frac{n+1}{2}}$, $1 \leqslant k \leqslant \frac{n-1}{2}$.

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Author's contributions

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