CONSTRUCTIVE MATHEMATICAL ANALYSIS
7 (2024), No. 2, pp. 77-89
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

Fractional proportional linear control systems: A geometric perspective on controllability and observability

KHIZRA BUKHSH, AWAIS YOUNUS, AIMAN MUKHEIMER, AND THABET ABDELJAWAD*

ABSTRACT. The paper presents a detailed analysis of control and observation of generalized Caputo proportional fractional time-invariant linear systems. The focus is on identifying controllable states and observable systems within the controllable subspace, null space, and unobservable subspace of the proposed system. The necessary conditions for the controllable subspace and the necessary and sufficient conditions for observability criteria are firmly established. The controllable subspace is treated geometrically as the set of controllable states, while the observable system is characterized by a zero unobservable subspace. The results are reinforced by examples and will immensely benefit future studies on fractional-order control systems.

Keywords: Controllable subspace, unobservable subspace, controllability, observability, fractional proportional control system.

2020 Mathematics Subject Classification: 34A08, 93B05, 93B07, 93B27.

1. INTRODUCTION

Control theory is a crucial field that directs the behavior of engineered processes and machines towards a desired state, all while guaranteeing stability and reducing errors. Its ultimate goal is to identify the optimal solution to control problems. When appraising a solution, two factors must be taken into account: the capability to transition from any starting state to any desired state by using the appropriate control inputs, and the capacity to establish the initial state of the system when the output is known, with knowledge of the input. In 1960, Kalman [15] proposed controllability and observability concepts that are now fundamental in control theory.

Fractional derivatives are crucial in various fields like control theory, finance, and nanotechnology. For further interest, we refer to [4]. Li et al. [18] discussed the use of a proportional derivative controller for controlling the output, denoted as u, at a given time t. The algorithm is defined with two shape control parameters is given by

$$u(t) = k_p E(t) + k_d \frac{d}{dt} E(t).$$

In this context, E, k_p, and k_d represent the error, proportional gain, and derivative gain, respectively. Anderson et al. [1] introduced the proportional derivative of order θ as:

$$D^{\theta}\phi(\vartheta) = k_1(\theta,\vartheta)\,\phi(\vartheta) + k_0(\theta,\vartheta)\,\phi^{'}(\vartheta).$$

Received: 17.03.2024; Accepted: 04.06.2024; Published Online: 10.06.2024
*Corresponding author: Thabet Abdeljawad; tabdeljawad@psu.edu.sa
DOI: 10.33205/cma.1454113

In the given equation, the variable ϕ represents a differentiable function, while k_0 and k_1 are continuous functions defined on the interval $[0,1]\times\mathbb{R}$ with values in the interval $[0,\infty)$. The parameter θ belongs to the interval $[0,1]$ and satisfies the following conditions $\forall\,\vartheta\in\mathbb{R}$:

$$\lim_{\theta\to 0^+} k_0(\theta,\vartheta)=0,\ \lim_{\theta\to 1^-} k_0(\theta,\vartheta)=1,\ k_0(\theta,\vartheta)\neq 0,\ \theta\in(0,1],$$
$$\lim_{\theta\to 0^+} k_1(\theta,\vartheta)=1,\ \lim_{\theta\to 1^-} k_1(\theta,\vartheta)=0,\ k_1(\theta,\vartheta)\neq 0,\ \theta\in[0,1).$$

As the order θ approaches 0, this local derivative converges to the original function. This property enhances the effectiveness of conformable derivatives. The findings presented in above result have enabled Dawei et al. [7] to demonstrate the control of complex network models. Jarad et al. [12] introduced a novel result concerning fractional operators derived from enhanced conformable derivatives. In a subsequent work, Jarad et al. [11] further improved and modified the aforementioned result.

Various studies have explored the controllability and observability properties of mathematical models in different fields. Several researchers [3, 6, 9, 10, 19, 24, 25, 26, 28] have studied various aspects of controllability and observability in different types of dynamic systems, including time-fractional, heat equation, conformable fractional, robotic arms, fractional-order differential, and stochastic singular systems.

This paper outlines critical geometric criteria that are essential for determining the controllability and observability of Caputo proportional fractional linear control systems:

$$\begin{aligned} {}^{c}D^{\theta,\varrho,\phi}x(\vartheta) &= Ax(\vartheta)+Bu(\vartheta),\\ y(\vartheta) &= Cx(\vartheta)+Du(\vartheta),\ \vartheta\in[0,T], \end{aligned} \tag{1.1}$$

with the initial condition $x(b)=x_b$. Geometric properties provide valuable insights into linear fractional control systems for engineers and researchers. These insights can guide the analysis, design, and optimization processes of the system. Geometric methods are also employed in the design of feedback control systems. Techniques such as pole placement and linear quadratic regulator (LQR) control involve manipulating the system's poles in the complex plane to achieve desired performance and stability objectives.

The paper is structured in the following manner: Section 2 presents crucial definitions and lemmas. Section 3 establishes the property of the matrix Mittag-Leffler function in the context of the generalized Caputo proportional fractional derivative. Subsection 3.1 derives geometric criteria for controllability using the Gramian controllability matrix and discusses the necessary controllability condition for Caputo proportional fractional linear time-invariant system (1.1). Subsection 3.2 discusses the necessary and sufficient observability conditions for the system (1.1). Section 4 provides pertinent examples that support the presented results. Lastly, Section 5 concludes the paper.

2. Basic Notions

Definition 2.1 ([11]). *For $\varrho\in(0,1]$ & $\theta\in\mathbb{C}$ with $Re(\theta)\geq 0$, Caputo type's left derivative, defined as:*

$$\begin{aligned} \left({}^{c}D^{\theta,\varrho,\phi}h\right)(\vartheta) &= {}_{a}I^{m-\theta,\varrho,\phi}\left(D^{m,\varrho,\phi}h\right)(\vartheta)\\ &= \frac{1}{\varrho^{m-\theta}\Gamma(m-\theta)}\int_a^{\vartheta} e^{\frac{\varrho-1}{\varrho}(\phi(\vartheta)-\phi(\tau))}\left(\phi(\vartheta)-\phi(\tau)\right)^{m-\theta-1}\\ &\quad\times\left(D^{m,\varrho,\phi}h\right)(\tau)\,\phi^{'}(\tau)\,d\tau. \end{aligned} \tag{2.2}$$

Remark 2.1 ([11]). *Consider $\varrho=1$ in Definition 2.1,* *(1) If $\phi(\vartheta) = \vartheta$ in (2.2), we get the Riemann-Liouville fractional operators.*

(2) If $\phi(\vartheta) = \frac{\vartheta^{\mu}}{\mu}$ in (2.2), we get the Katugampola fractional operators.

(3) If $\phi(\vartheta) = \ln\vartheta$ in (2.2), we get the Hadamard fractional operators.

(4) If $\phi(\vartheta) = \frac{(\vartheta - a)^{\mu}}{\mu}$ in (2.2), we get the fractional operators mentioned in [12].

The Mittag-Leffler functions have significant importance in the field of fractional calculus [17, 21, 29].

Definition 2.2 ([17, 21, 29]). *The Mittag-Leffler function is given by*

$$E_{\theta}(z) = \sum_{j=0}^{\infty} \frac{z^{j}}{\Gamma(j\theta + 1)}, z \in \mathbb{C}, Re(\theta) > 0.$$

The Mittag-Leffler function is defined by two parameters, θ and β [17, 21, 29]

$$E_{\theta,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^{j}}{\Gamma(j\theta + \beta)}, z \in \mathbb{C}, Re(\theta) > 0, Re(\beta) > 0.$$

Theorem 2.1 ([5]). *Consider a linear system of generalized Caputo proportional fractional derivative with parameters ϱ and θ, where ϱ and θ are in the interval $(0, 1)$. Let ϕ be a continuous, strictly increasing function. The system is represented as follows:*

$$\begin{cases} \left({}^{c}D^{\theta,\varrho,\phi}x\right)(\vartheta) = Ax(\vartheta) + Bu(\vartheta), \\ x(b) = x_{b}. \end{cases} \tag{2.3}$$

Here, $x : [b, T] \to \mathbb{R}^{n}$, $u : [b, T] \to \mathbb{R}^{m}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are matrices, and A satisfies the condition that $\det(\lambda I - A) \neq 0$. Then the solution of equation (2.3) for the time-invariant case is given by:

$$\begin{aligned} x(\vartheta) &= e^{\frac{\varrho-1}{\varrho}(\phi(\vartheta)-\phi(b))} E_{\theta}\left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(b))^{\theta}\right) x_{b} \\ &+ \varrho^{-\theta} \int_{b}^{\vartheta} e^{\frac{\varrho-1}{\varrho}(\phi(\vartheta)-\phi(\tau))} (\phi(\vartheta) - \phi(\tau))^{\theta-1} \\ &\times E_{\theta,\theta}\left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(\tau))^{\theta}\right) Bu(\tau)\phi'(\tau)\, d\tau. \end{aligned} \tag{2.4}$$

Definition 2.3 ([23]). *The controllable subspace for the linear state equation (1.1) is defined as the subspace of X, denoted by $\langle A|\mathfrak{B}\rangle$, where $\mathfrak{B} = Im(B)$, as follows:*

$$\langle A|\mathfrak{B}\rangle = \mathfrak{B} + A\mathfrak{B} + \cdots + A^{n-1}\mathfrak{B}.$$

Definition 2.4 ([23]). *System (1.1) is called state controllable on $[b, t_{f}]$, $t_{f} > 0$; $\exists$ an input signal $u(\cdot) : [b, t_{f}] \to \mathbb{R}^{m}$ proposed solution of (2.3) fulfills $x(t_{f}) = 0$.*

Let us consider the controllability Gramian matrix from [5]:

$$\begin{aligned} W_{c}[b, t_{f}] := \varrho^{-\theta} \int_{b}^{t_{f}} & e^{\frac{\varrho-1}{\varrho}(\phi(t_{f})-\phi(\tau))} (\phi(t_{f}) - \phi(\tau))^{\theta-1} \\ &\times E_{\theta,\theta}\left(\varrho^{-\theta} A (\phi(t_{f}) - \phi(\tau))^{\theta}\right)(B) \\ &\times (B)^{*} E_{\theta,\theta}^{*}\left(\varrho^{-\theta} A (\phi(t_{f}) - \phi(\tau))^{\theta}\right) \phi'(\tau)\, d\tau, \end{aligned} \tag{2.5}$$

where the matrix transpose is represented as $*$.

The geometric approach to analyzing observability for the linear state equation (1.1) initiates from a reversed concept as:

Definition 2.5 ([23]). *The unobservable subspace N for the linear state equation (1.1) is defined as the subspace of X*

$$N = \cap_{i=0}^{\infty} \ker \left[CA^i \right].$$

Remark 2.2 ([23]). *N is an invariant subspace for A.*

Definition 2.6 ([23]). *System (1.1) is called state observable on $[b, t_f]$ for any initial condition $x(b) = x_b \in \mathbb{R}^n$ the system's uniqueness is found by its corresponding input $u(\vartheta)$ and output $y(\vartheta)$, $\vartheta \in [b, t_f]$; $t_f \in [b, T]$.*

Let us consider the observability Gramian matrix from [5]:

$$\begin{aligned} W_o[b, t_f] := \int_b^{t_f} e^{\frac{\varrho-1}{\varrho}(\phi(\vartheta)-\phi(b))} E_\theta^* \left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(b))^\theta \right) C^* \\ \times C E_\theta \left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(b))^\theta \right) d\vartheta, \end{aligned} \tag{2.6}$$

where the matrix transpose is represented as $*$.

Theorem 2.2 ([5]). *System (1.1) is observable on $[b, t_f]$ iff $|W_o[b, t_f]| \neq 0$ for some $t_f > 0$.*

Let us recall the Cayley-Hamilton theorem for fractional continuous-time linear systems.

Theorem 2.3 ([13]). *Let $\Psi(\lambda) = \det[I_n\lambda - f(A)] = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ be the characteristic polynomial of the matrix $f(A)$. Then the matrix $f(A)$ satisfies its characteristic equation, i.e.*

$$[f(A)]^n + a_{n-1}[f(A)]^{n-1} + \cdots + a_1[f(A)] + a_0 I_n = 0.$$

3. Main Results

We first establish a preliminary result.

Proposition 3.1. *There exist analytic functions $\theta_o(t), \theta_1(t), \ldots, \theta_{n-1}(t)$ such that*

$$E_\theta \left(A \left(\frac{\phi(t) - \phi(0)}{\varrho} \right)^\theta \right) = \sum_{k=0}^{n-1} \theta_k(t) [f(A)]^k. \tag{3.7}$$

Proof. The $n \times n$ matrix generalized Caputo proportional fractional differential equation

$$\left({}^c D^{\theta,\varrho,\phi} x \right)(t) = Ax(t), \; x(0) = I,$$

has the unique solution

$$x(t) = e^{\frac{\varrho-1}{\varrho}(\phi(t)-\phi(0))} E_\theta \left(A \left(\frac{\phi(t) - \phi(0)}{\varrho} \right)^\theta \right).$$

The matrix generalized Caputo proportional fractional differential equation characterizing the Mittag-Leffler function, we can establish (3.7) by showing that there exist scalar analytic functions $\theta_o(t), \theta_1(t), \ldots, \theta_{n-1}(t)$ such that

$$\begin{aligned}\sum_{k=0}^{n-1} {}^{c}D^{\theta,\varrho,\phi}\theta_k(t)[f(A)]^k &= \sum_{k=0}^{n-1}\theta_k(t)[f(A)]^{k+1}, \\ \sum_{k=0}^{n-1}\theta_k(0)[f(A)]^k &= I.\end{aligned} \tag{3.8}$$

The Cayley-Hamilton Theorem 2.3 implies

$$[f(A)]^n = -a_0 I - a_1[f(A)] - \cdots - a_{n-1}[f(A)]^{n-1}.$$

Then (3.8) can be completely formulated using $I, A, \ldots, A^{n-1}$ as

$$\begin{aligned}\sum_{k=0}^{n-1} {}^{c}D^{\theta,\varrho,\phi}\theta_k(t)[f(A)]^k &= \sum_{k=0}^{n-2}\theta_k(t)[f(A)]^{k+1} - \theta_{n-1}(t)[f(A)]^n \\ &= \sum_{k=0}^{n-2}\theta_k(t)[f(A)]^{k+1} - \sum_{k=0}^{n-1}a_k\theta_{n-1}(t)[f(A)]^k \\ &= \sum_{k=1}^{n-1}\theta_{k-1}(t)[f(A)]^k - a_0\theta_{n-1}(t)I \\ &\quad -\sum_{k=1}^{n-1}a_k\theta_{n-1}(t)[f(A)]^k.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{k=0}^{n-1} {}^{c}D^{\theta,\varrho,\phi}\theta_k(t)[f(A)]^k = -a_0\theta_{n-1}(t)I + \sum_{k=1}^{n-1}[\theta_{k-1}(t) - a_k\theta_{n-1}(t)][f(A)]^k, \\ &\sum_{k=0}^{n-1}\theta_k(0)[f(A)]^k = I.\end{aligned} \tag{3.9}$$

An insightful point to recognize is that addressing (3.9) involves approaching it through the consideration of coefficient equations for individual powers of A

$$\begin{bmatrix} {}^{c}D^{\theta,\varrho,\phi}\theta_o(t) \\ {}^{c}D^{\theta,\varrho,\phi}\theta_1(t) \\ \vdots \\ {}^{c}D^{\theta,\varrho,\phi}\theta_{n-1}(t)\end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & \cdots & 0 & -a_1 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & & 1 & -a_{n-1}\end{bmatrix}\begin{bmatrix}\theta_o(t) \\ \theta_1(t) \\ \vdots \\ \theta_{n-1}(t)\end{bmatrix}, \quad \begin{bmatrix}\theta_o(0) \\ \theta_1(0) \\ \vdots \\ \theta_{n-1}(0)\end{bmatrix} = \begin{bmatrix}1 \\ 0 \\ \vdots \\ 0\end{bmatrix}.$$

This show existence of analytic functions implies an exact solution to this linear state equation. $\theta_o(t), \theta_1(t), \ldots, \theta_{n-1}(t)$ that satisfy (3.9), and hence (3.8). □

3.1. **Controllability.** The subsequent Proposition furnishes the necessary instrument to demonstrate that $\langle A|\mathfrak{B}\rangle$ precisely constitutes the collection of states that can be controlled.

Proposition 3.2. *For any* $t_a > 0$, $\langle A|\mathfrak{B}\rangle = Im[W_c(0, t_a)]$.

Proof. For any $n \times 1$ vector x_o, setting $t_a > 0$

$$\begin{aligned} W_c[b,t_a]x_o = \varrho^{-\theta} \int_b^{t_a} & e^{\frac{\varrho-1}{\varrho}(\phi(t_a)-\phi(\tau))} (\phi(t_a)-\phi(\tau))^{\theta-1} \\ & \times (B) E_{\theta,\theta}\left(\varrho^{-\theta} A (\phi(t_a)-\phi(\tau))^{\theta}\right) \\ & \times (B)^* E^*_{\theta,\theta}\left(\varrho^{-\theta} A (\phi(t_a)-\phi(\tau))^{\theta}\right) \phi'(\tau) x_o d\tau. \end{aligned}$$

Since $E_{\theta,\theta}\left(\varrho^{-\theta} A (\phi(t_a)-\phi(\tau))^{\theta}\right) = \sum_{k=0}^{n-1} {}_k\tilde{p}_k(t)(A)^k, \theta > 0$ [20]. Therefore,

$$\begin{aligned} W_c[b,t_a]x_o = \sum_{k=0}^{n-1} (A)^k B \varrho^{-\theta} \int_b^{t_a} & \tilde{p}_k(t) e^{\frac{\varrho-1}{\varrho}(\phi(t_a)-\phi(\tau))} (\phi(t_a)-\phi(\tau))^{\theta-1} \\ \times (B)^* E^*_{\theta,\theta} & \left(A \left(\frac{\phi(t_a)-\phi(\tau)}{\varrho} \right)^{\theta} \right) \phi'(\tau) x_o d\tau. \end{aligned}$$

Because every column of $(A)^k B$ is in $(A)^k \mathfrak{B}$, and the kth-summand mentioned above represents linear combination of the columns of $(A)^k B$. This implies that,

$$\begin{aligned} W_c[b,t_a]x_o &\in \mathfrak{B} + A\mathfrak{B} + \cdots + (A)^{n-1}\mathfrak{B} \\ &\in \langle A|\mathfrak{B}\rangle. \end{aligned}$$

Hence,

$$Im[W_c(b,t_a)] \subset \langle A|\mathfrak{B}\rangle.$$

It is obvious that, $\langle A|\mathfrak{B}\rangle$ corresponds to the range space of the controllability Gramian matrix

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

associated with the linear state equation (1.1). Construct an invertible $n \times n$ matrix P by selecting a set of column vectors that form a basis for $\langle A|\mathfrak{B}\rangle$ and extend this basis to the entire space X. Subsequently, altering the state variables in accordance with the transformation given by $z(t) = P^{-1}x(t)$ results in a novel linear state equation expressed in terms of the transformed state variable $z(t)$, along with corresponding coefficient matrices

$$P^{-1}AP = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}, P^{-1}B = \begin{bmatrix} \hat{B}_{11} \\ 0 \end{bmatrix}.$$

The given expressions can be utilized to represent $W_c[b,t_a]$ in (2.5) as

$$\begin{aligned} W_c[b,t_a] = \varrho^{-\theta} P \int_b^{t_a} & e^{\frac{\varrho-1}{\varrho}(\phi(t_a)-\phi(\tau))} (\phi(t_a)-\phi(\tau))^{\theta-1} \\ \times E_{\theta,\theta} & \left(\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \left(\frac{\phi(t_a)-\phi(\tau)}{\varrho} \right)^{\theta} \right) \begin{bmatrix} \hat{B}_{11} \\ 0 \end{bmatrix} \\ \times (B)^* E^*_{\theta,\theta} & \left(\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \left(\frac{\phi(t_a)-\phi(\tau)}{\varrho} \right)^{\theta} \right) \phi'(\tau) d\tau P^T. \end{aligned}$$

This implies that

$$W_c[b,t_a] = P \begin{bmatrix} \hat{W}_1[b,t_a] & 0 \\ 0 & 0 \end{bmatrix} P^T,$$

where

$$\hat{W}_1[b,t_a] = \varrho^{-\theta}\int_b^{t_a} e^{\frac{\varrho-1}{\varrho}(\phi(t_a)-\phi(\tau))}(\phi(t_a)-\phi(\tau))^{\theta-1} \times E_{\theta,\theta}\left(\hat{A}_{11}\left(\frac{\phi(t_a)-\phi(\tau)}{\varrho}\right)^{\theta}\right)\hat{B}_{11} \times \left(\hat{B}_{11}\right)^{*} E^{*}_{\theta,\theta}\left(\hat{A}_{11}\left(\frac{\phi(t_a)-\phi(\tau)}{\varrho}\right)^{\theta}\right) g^{'}(\tau)\,d\tau$$

is a non-singular matrix. This illustration demonstrates that any vector of the form

$$P\begin{bmatrix} z \\ 0 \end{bmatrix} \tag{3.10}$$

is contained in $Im\,[W\,(b,t_a)]$. For setting

$$x = \left[P^T\right]^{-1}\begin{bmatrix} \hat{W}_1[b,t_a]\,z \\ 0 \end{bmatrix}$$

we obtain

$$W_1[b,t_1]\,x = P\begin{bmatrix} z \\ 0 \end{bmatrix}.$$

The structure of $A^k B = P\begin{bmatrix} \hat{A}^k_{11}\hat{B}_{11} \\ 0 \end{bmatrix}$, $k = 0,1,\ldots$ is represented as (3.10), it implies that

$$\langle A|\mathfrak{B}\rangle \subset Im\,[W_c\,(b,t_a)]\,.$$

Hence, we conclude that $\langle A|\mathfrak{B}\rangle = Im\,[W_c\,(b,t_a)]\,.$ □

Theorem 3.4. *If a vector x_b belongs to the set of controllable states for the linear state equation (1.1), then $x_b \in \langle A|\mathfrak{B}\rangle$.*

Proof. If state x_b can be controlled, then $\exists$ a positive finite time t_a such that

$$0 = x\,(t_a) = e^{\frac{\varrho-1}{\varrho}(\phi(t_a)-\phi(b))}E_{\theta}\left(\varrho^{-\theta}A\,(\phi(t_a)-\phi(b))^{\theta}\right)x_b + \varrho^{-\theta}\int_b^{t_a} e^{\frac{\varrho-1}{\varrho}(\phi(t_a)-\phi(\tau))}(\phi(t_a)-\phi(\tau))^{\theta-1} \times E_{\theta,\theta}\left(\varrho^{-\theta}A\,(\phi(t_a)-\phi(\tau))^{\theta}\right)Bu\,(\tau)\,\phi^{'}(\tau)\,d\tau.$$

$$e^{\frac{\varrho-1}{\varrho}(\phi(t_a)-\phi(b))}E_\theta\left(\varrho^{-\theta}A\left(\phi\left(t_a\right)-\phi\left(b\right)\right)^\theta\right)x_b$$
$$=-\,\varrho^{-\theta}\int\limits_b^{t_a}e^{\frac{\varrho-1}{\varrho}(\phi(t_a)-\phi(\tau))}\left(\phi\left(t_a\right)-\phi\left(\tau\right)\right)^{\theta-1}E_{\theta,\theta}\left(\varrho^{-\theta}A\left(\phi\left(t_a\right)-\phi\left(\tau\right)\right)^\theta\right)$$
$$\times Bu\left(\tau\right)\phi^{'}\left(\tau\right)d\tau.$$
$$E_\theta\left(\varrho^{-\theta}A\left(\phi\left(t_a\right)-\phi\left(b\right)\right)^\theta\right)x_b=-\varrho^{-\theta}\int\limits_b^{t_a}e^{\frac{\varrho-1}{\varrho}(\phi(b)-\phi(\tau))}\left(\phi\left(t_a\right)-\phi\left(\tau\right)\right)^{\theta-1}$$
$$\times E_{\theta,\theta}\left(\varrho^{-\theta}A\left(\phi\left(t_a\right)-\phi\left(\tau\right)\right)^\theta\right)Bu\left(\tau\right)\phi^{'}\left(\tau\right)d\tau.$$
$$x_bE_\theta\left(\varrho^{-\theta}A\left(\phi\left(t_a\right)-\phi\left(b\right)\right)^\theta\right)=-\varrho^{-\theta}\int\limits_b^{t_a}e^{\frac{\varrho-1}{\varrho}(\phi(b)-\phi(\tau))}\left(\phi\left(t_a\right)-\phi\left(\tau\right)\right)^{\theta-1}$$
$$\times E_{\theta,\theta}\left(\varrho^{-\theta}A\left(\phi\left(t_a\right)-\phi\left(\tau\right)\right)^\theta\right)Bu\left(\tau\right)g^{'}\left(\tau\right)d\tau.$$
$$x_b=-\varrho^{-\theta}\int\limits_b^{t_a}e^{\frac{\varrho-1}{\varrho}(\phi(b)-\phi(\tau))}\left(\phi\left(t_a\right)-\phi\left(\tau\right)\right)^{\theta-1}$$
$$\times E_{\theta,\theta}\left(\varrho^{-\theta}A\left(\phi\left(t_a\right)-\phi\left(\tau\right)\right)^\theta\right)BE_\theta^{-1}\left(\varrho^{-\theta}A\left(\phi\left(t_a\right)-\phi\left(b\right)\right)^\theta\right)u\left(\tau\right)g^{'}\left(\tau\right)d\tau.$$

Since $E_{\theta,\theta}\left(\varrho^{-\theta}A\left(\phi\left(t_a\right)-\phi\left(\tau\right)\right)^\theta\right)=\sum\limits_{k=0}^{n-1}{}_k\tilde{p}_k\left(t\right)\left(A\right)^k,\theta>0$ [20]. Therefore,

$$x_b=-\sum_{k=0}^{n-1}\left(A\right)^kB\varrho^{-\theta}\int_b^{t_a}\tilde{p}_k\left(t\right)e^{\frac{\varrho-1}{\varrho}(\phi(b)-\phi(\tau))}\left(\phi\left(t_a\right)-\phi\left(\tau\right)\right)^{\theta-1}$$
$$\times E_\theta^{-1}\left(\varrho^{-\theta}A\left(\phi\left(t_a\right)-\phi\left(b\right)\right)^\theta\right)u\left(\tau\right)\phi^{'}\left(\tau\right)d\tau.$$
$$x_b=\sum_{k=0}^{n-1}\left(A\right)^kB\varrho^{-\theta}\int_{t_a}^{b}\tilde{p}_k\left(t\right)e^{\frac{\varrho-1}{\varrho}(\phi(b)-\phi(\tau))}\left(\phi\left(t_a\right)-\phi\left(\tau\right)\right)^{\theta-1}$$
$$\times E_\theta^{-1}\left(\varrho^{-\theta}A\left(\phi\left(t_a\right)-\phi\left(b\right)\right)^\theta\right)u\left(\tau\right)\phi^{'}\left(\tau\right)d\tau.$$

Because each column of $(A)^k B$ is in $(A)^k \mathfrak{B}$, and the kth-summand mentioned above represents linear combination of the columns of $(A)^k B$. This implies that,

$$\begin{aligned}x_b &\in \mathfrak{B}+A\mathfrak{B}+\cdots+\left(A\right)^{n-1}\mathfrak{B}\\ &\in \left\langle A|\mathfrak{B}\right\rangle.\end{aligned}$$

□

Theorem 3.5. *If X is the set of controllable states for the linear state equation (1.1), then it implies that X is contained in controllable subspace $\langle A|\mathfrak{B}\rangle$.*

3.2. **Observability.** The subsequent proposition furnishes the requisite technique for demonstrating the observability of a given system.

Proposition 3.3. *For any $t_f>0$, $N=\ker\left(W_o\left[b,t_f\right]\right)$.*

Proof. Suppose that $v \in \ker(W_o)$, which means that $W_o v = 0$. Then, we have:

$$\begin{aligned} v^* W_o v = & \int_b^{t_f} v^* e^{\frac{\varrho-1}{\varrho}(\phi(\vartheta)-\phi(b))} E_\theta^* \left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(b))^\theta\right) C^* \\ & \times C E_\theta \left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(b))^\theta\right) v d\vartheta. \\ 0 = & \int_b^{t_f} e^{\frac{\varrho-1}{\varrho}(\phi(\vartheta)-\phi(b))} \left(E_\theta \left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(b))^\theta\right) v\right)^* C^* \\ & \times C \left(E_\theta \left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(b))^\theta\right) v\right) dv. \\ 0 = & \int_b^{t_f} e^{\frac{\varrho-1}{\varrho}(\phi(\vartheta)-\phi(b))} \left\| C \left(E_\theta \left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(b))^\theta\right) v\right) \right\|^2 dv. \end{aligned}$$

Since $\left\| C \left(E_\theta \left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(b))^\theta\right) v\right) \right\|^2 \geq 0$ for all $t \geq 0$, we must have

$$C \left(E_\theta \left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(b))^\theta\right) v\right) = 0$$

for all $t \geq 0$. This implies that

$$E_\theta \left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(b))^\theta\right) v \in \ker(C), \forall t \geq 0.$$

Which means that v belongs to the unobservable subspace N.

Further, suppose that $w \in N$, which means that there exists no input $u(t)$ such that $x(0) = w$ and $y(t) = Cx(t) + Du(t) = 0$ for all $t \geq 0$. This implies that the output of the system cannot distinguish between the initial state w and the zero state $x = 0$. Therefore, we have:

$$0 = \int_b^{t_f} \|y(t)\|^2 dt = \int_b^{t_f} x^*(t) C^* C x(t) dt.$$

Now,

$$\begin{aligned} W_o w := & \int_b^{t_f} e^{\frac{\varrho-1}{\varrho}(\phi(\vartheta)-\phi(b))} E_\theta^* \left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(b))^\theta\right) C^* \\ & \times C E_\theta \left(\varrho^{-\theta} A (\phi(\vartheta) - \phi(b))^\theta\right) w d\vartheta = 0, \end{aligned}$$

where the last step follows from the fact that C^*C is a positive semi-definite matrix. Therefore, we have $x^*(t) C^* C x(t) = 0$ for all $t \geq 0$. This implies that $w \in \ker(W_o)$. Hence, $N = \ker(W_o)$. □

The following Theorem gives the geometric type criterion for a system to be observable.

Theorem 3.6. *The linear state equation (1.1) is observable if and only if $N = \{0\}$.*

Proof. Consider the system (1.1) is observable on $[b, t_f]$. We have to show that $N = \{0\}$. It follows that observability Gramian matrix is invertible as system is observable,

$$\ker(W_o[b, t_f]) = \{0\}.$$

By using proposition 3.3, we have

$$N = \{0\}.$$

Conversely suppose that $N = \{0\}$. By using proposition 3.3, we have

$$\ker\left(W_o\left[b, t_f\right]\right) = \{0\}.$$

It follows that observability Gramian matrix is invertible. Then by Theorem 2.2, linear state equation (1.1) is observable. □

4. Numerical Examples

Let's provide two examples to demonstrate the application of our findings.

Example 4.1. *Suppose the following 3-dimensional linear time invariant system on* $[0,5]$*:*

$$\left({}^{c}D^{\frac{1}{2},\frac{1}{2},\phi}x\right)(\vartheta) = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 4 & 1 \\ 1 & 5 & 1 \end{pmatrix} x(\vartheta) + \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} u(\vartheta), \quad x(0) = 0. \tag{4.11}$$

Let us denote

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 4 & 1 \\ 1 & 5 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix},$$

then, one can obtain

$$\mathfrak{B} = Im(B) = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

The process of computing a basis for a subspace entails choosing columns from a set of matrices in such a way that they are not linearly dependent.

$$\left[B\ AB\ A^2B\right] = \begin{bmatrix} 1 & 2 & 4 & 5 & 10 & 29 \\ 0 & 1 & 3 & 9 & 22 & 54 \\ 1 & 1 & 2 & 8 & 21 & 58 \end{bmatrix}.$$

And, we observe that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ *columns are linearly independent. Therefore, the controllable subspace of* $\mathbb{R}^3$ *is given by*

$$\langle A|\mathfrak{B}\rangle = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \right\} = \mathbb{R}^3.$$

Hence by using Theorem 3.5, system (4.11) is controllable.

Example 4.2. *Suppose the following 3-dimensional linear time invariant system on* $[0,5]$*:*

$$\left({}^{c}D^{\frac{1}{2},\frac{1}{2},\phi}x\right)(\vartheta) = \begin{pmatrix} 1 & 6 & 5 \\ 7 & 2 & 4 \\ 8 & 9 & 3 \end{pmatrix} x(\vartheta)\, y(\vartheta) = \begin{pmatrix} 0 & 5 & 1 \\ 4 & 2 & 1 \end{pmatrix} x(\vartheta)\, x(0) = 0. \tag{4.12}$$

Let us denote

$$A = \begin{pmatrix} 1 & 6 & 5 \\ 7 & 2 & 4 \\ 8 & 9 & 3 \end{pmatrix}, C = \begin{pmatrix} 0 & 5 & 1 \\ 4 & 2 & 1 \end{pmatrix};$$

then, one can obtain

$$\ker(C) = span\left\{\begin{bmatrix}-3/20\\-1/5\\1\end{bmatrix}\right\};$$

$$\ker(CA) = span\left\{\begin{bmatrix}-262/1097\\-735/1097\\1\end{bmatrix}\right\};$$

$$\ker\left(CA^2\right) = span\left\{\begin{bmatrix}-3373/84859\\-58320/84859\\1\end{bmatrix}\right\}.$$

Creating a basis for a subspace entails the process of choosing linearly independent columns from a set of matrices

$$\left[\ker(C)\ \ker(CA)\ \ker\left(CA^2\right)\right] = \begin{bmatrix}-3/20 & -262/1097 & -3373/84859\\-1/5 & -735/1097 & -58320/84859\\1 & 1 & 1\end{bmatrix}.$$

And we observe that, all columns are linearly independent. Therefore, the unobservable subspace $N \subseteq \mathbb{R}^3$ *is*

$$N = \ker(C) \cap \ker(CA) \cap \ker\left(CA^2\right) = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}.$$

Hence by using Theorem 3.6, system (4.12) is observable.

5. Conclusion

This paper focuses on the controllability and observability analysis of generalized Caputo proportional fractional linear time-invariant control systems using geometric analysis. The authors establish the geometric characterization of the controllable subspace and unobservable subspace of such systems. They also discuss the connections with the controllability and observability Gramian matrices of the considered systems. The paper also presents a necessary criterion for controllability based on the controllable subspace, as well as a necessary and sufficient criterion for observability based on the unobservable subspace. The authors validate their findings through examples. By expanding the scope of the systems studied, the paper generalizes some known results and demonstrates the potential for exploring the combination of control theory with generalized Caputo proportional fractional operators, as indicated by recent research.

Acknowledgment

The authors A. Mukheimer and T. Abdeljawad would like to thank Prince sultan University for the support through TAS research lab.

References

[1] D. R. Anderson, D. J. Ulness: *Newly defined conformable derivatives*, Adv. Dyn. Syst. Appl, **10** (2015), 109–137.

[2] Z. Al-Zhour: *Controllability and observability behaviors of a non-homogeneous conformable fractional dynamical system compatible with some electrical applications*, Alexandria Engineering Journal, **61** (2022), 1055–1067.

[3] K. Balachandran, M. Matar and J. J. Trujillo: *Note on controllability of linear fractional dynamical systems*, J. Control Decis., **3** (2016), 267–279.

[4] M. Bohner, K. S. Vidhyaa, E. Thandapani: *Oscillation of noncanonical second- order advanced differential equations via canonical transform*, Constr. Math. Anal., **5** (1) (2022), 7–13.

[5] K. Bukhsh, A. Younus: *On the controllability and observability of fractional proportional linear systems*, Internat. J. Systems Sci., **54** (2023), 1410–1422.
[6] A. Da Silva: *Controllability of linear systems on solvable Lie groups*, SIAM J. Control Optim., **54** (2016), 372–390.
[7] D. Ding, X. Zhang, J. Cao, N. Wang and D. Liang: *Bifurcation control of complex networks model via PD controller*, Neurocomputing, **175** (2016), 1–9.
[8] Z. Ge: *Controllability and observability of stochastic singular systems in Banach spaces*, J. Syst. Sci. Complex., **35** (2022), 194–204.
[9] K. A. Grasse, N. Ho: *Simulation relations and controllability properties of linear and nonlinear control systems*, SIAM J. Control Optim., **53** (2015), 1346–1374.
[10] T. L. Guo: *Controllability and observability of impulsive fractional linear time-invariant system*, Comput. Math. Appl., **64** (2012), 3171–3182.
[11] F. Jarad, M. A. Alqudah and T. Abdeljawad: *On more general forms of proportional fractional operators*, Open Math., **18** (2020), 167–176.
[12] F. Jarad, T. Abdeljawad and J. Alzabut: *Generalized fractional derivatives generated by a class of local proportional derivatives*, E. P. J. Special Topics, **226** (2017), 3457–3471.
[13] T. Kaczorek: Cayley–Hamilton theorem for fractional linear systems, In Theory and Applications of Non-integer Order Systems: 8th Conference on Non-integer Order Calculus and Its Applications, Zakopane, Poland, Springer International Publishing, (2017), 45–55.
[14] R. E. Kalman: *Mathematical description of linear dynamical systems*, Journal of the Society for Industrial and Applied Mathematics, Series A: Control, **1** (1963), 152–192.
[15] R. E. Kalman: *Contributions to the theory of optimal control*, Bol. Soc. Mat. Mexicana, **5** (1960), 102–119.
[16] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 204 (2006).
[17] A. A. Kilbas, O. I. Marichev and S. G. Samko: *Fractional integrals and derivatives (theory and applications)*, CRC Press, India (1993).
[18] Y. Li, K. H. Ang and G. C. Chong: *PID control system analysis and design*, IEEE Control Systems Magazine, **26** (2006), 32–41.
[19] G. Mazanti,: *Relative controllability of linear difference equations*, SIAM J. Control Optim., **55** (2017), 3132–3153.
[20] M. Popolizio: *On the matrix Mittag–Leffler function: theoretical properties and numerical computation*, Mathematics, **7** (2019), 1140.
[21] I. Podlubny: *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier (1998).
[22] J. D. J. Rubio, E. Orozco, D. A. Cordova, M. A. Islas, J. Pacheco, G. J. Gutierrez, ... and D. Mujica-Vargas: *Modified linear technique for the controllability and observability of robotic arms*, IEEE Access, **10** (2022), 3366-3377.
[23] W. J. Rugh: *Linear system theory*, Prentice-Hall, Inc (1996).
[24] J. L. Šajić, S. Langthaler, J. Schröttner and C. Baumgartner: *System identification and mathematical modeling of the pandemic spread COVID-19 in Serbia*, IFAC-PapersOnLine, **55** (2022), 19–24.
[25] V. Singh, D. N. Pandey: *Controllability of multi-term time-fractional differential systems*, J. Control Decis., **7** (2020), 109–125.
[26] J. Wang, Y. Zheng, K. Li and Q. Xu: *DeeP-LCC: Data-enabled predictive leading cruise control in mixed traffic flow*, IEEE Transactions on Control Systems Technology, **31** (6) (2023), 2760–2776.
[27] L. Wang, Q. Yan and H. Yu: *Constrained approximate null controllability of the coupled heat equation with impulse controls*, SIAM J. Control Optim., **59** (2021), 3418–3446.
[28] J. Wei: *The controllability of fractional control systems with control delay*, Comput. Math. Appl., **64** (2012), 3153–3159.
[29] H. Zhang, I. Ahmad, G. Rahman and S. Ahmad: *Investigation for Existence, Controllability & Observability of a Fractional order Delay Dynamical System*, Authorea Preprints, (2022).

KHIZRA BUKHSH
BAHAUDDIN ZAKARIYA UNIVERSITY
CENTRE FOR ADVANCED STUDIES IN PURE AND APPLIED MATHEMATICS
MULTAN, 60800, PAKISTAN
GHAZI UNIVERSITY
DEPARTMENT OF MATHEMATICS
DERA GHAZI KHAN, 32000, PAKISTAN
ORCID: 0009-0002-9809-9047
Email address: khizrabukhsh716@gmail.com

AWAIS YOUNUS
BAHAUDDIN ZAKARIYA UNIVERSITY
CENTRE FOR ADVANCED STUDIES IN PURE AND APPLIED MATHEMATICS
MULTAN, 60800, PAKISTAN
ORCID: 0000-0002-0590-6691
Email address: awais@bzu.edu.pk

AIMAN MUKHEIMER
PRINCE SULTAN UNIVERSITY
DEPARTMENT OF MATHEMATICS AND SCIENCES
RIYADH, 11586, SAUDI ARABIA
ORCID: 0000-0001-8798-3297
Email address: Mukheimer@psu.edu.sa

THABET ABDELJAWAD
PRINCE SULTAN UNIVERSITY
DEPARTMENT OF MATHEMATICS AND SCIENCES
RIYADH, 11586, SAUDI ARABIA
ORCID: 0000-0002-8889-3768
Email address: tabdeljawad@psu.edu.sa