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On existence of coincidence solutions for a generalized system of functional equations

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Abstract

In this paper, we consider a class of generalized system of functional equations which arise in multistage decision process. We show that the coincidence solutions for this system of functional equations exist. The results presented here unify the results due to several authors. A numerical example is illustrated to justify the results.

1 Introduction

In multistage decision process, in a natural way, functional equations arise. Bellman [4] first presented the existence of solutions for some classes of functional equations arising in dynamic programming. In optimization, dynamic programming, because of its wide applicability, induces much interest among people of various disciplines. The origin of the theory of dynamic programming lies in the domain of multistage decision process. The approximate dynamic programming approach can solve the optimization control problem of nonlinear system. Methods of dynamic programming and Bellman principle are used to accomplish the multistage optimization. The link between the optimal control problem and the Bellman equation is provided by dynamic programming. For details see [17, 16].

The initial form of a functional equations in multistage decision processes was proposed by Bellman and Lee [6] which is as follows $f(x) = \text{opt}_{y\in D} H(x, y, f(T(x, y))), \forall x \in S$. Here x and y represent the state and decision vectors respectively, S and D denotes the state space and decision space respectively and T represents the transformation of the process. $f(x)$ represents the optimal return function with initial state x and opt denotes $\sup$ or $\inf$.

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This type of functional equation was first applied in engineering, control theory and economic theory (see, [4, 5, 7, 18, 19, 20]). Some applications of these ideas to the calculus of variations and theory of integral equations are found in [1, 2, 3]. In [13], Liu and Ume studied functional equations arising in multistage decision process for their existence, uniqueness and iterative approximation of solutions. Liu and Kang [12], studied the properties of functional equations.

The system of functional equations of dynamic programming, initiated by Chang [9] as

$$\begin{aligned} f(x) &= \sup_{y\in D}\{u(x,y)+G(x,y,g(a(x,y)))\} \\ g(x) &= \sup_{y\in D}\{u(x,y)+F(x,y,f(a(x,y)))\}. \end{aligned} \tag{1}$$

Chang and Ma [10] discussed the following type of system of functional equations:

$$\begin{aligned} f(x) &= \sup_{y\in D}\{v(x,y)+G(x,y,g(a(x,y)))\} \\ g(x) &= \sup_{y\in D}\{u(x,y)+F(x,y,f(b(x,y)))\}. \end{aligned} \tag{2}$$

We propose a class of generalized system of functional equations in the field of discrete optimization in line with Bellman's equation arising in dynamic programming of multistage decision process.

$$\begin{aligned} f(x) = \operatorname*{opt}_{y\in D}\{&p(x,y)+\operatorname{opt}\{r(x,y),C(x,y,g(c(x,y))),\\ &u_i(x,y)+A_i(x,y,g(a_i(x,y))): i=1,2,\cdots,m\}\},\\ g(x) = \operatorname*{opt}_{y\in D}\{&q(x,y)+\operatorname{opt}\{s(x,y),D(x,y,f(d(x,y))),\\ &v_i(x,y)+B_i(x,y,f(b_i(x,y))): i=1,2,\cdots,m\}\}. \end{aligned} \tag{3}$$

Here x and y indicate the state and decision vectors respectively. c, d, a_i and b_i for $i=1,2,\cdots,m$ refer the transformation of the processes. $f(x)$ and $g(x)$ signify the optimal return functions with initial state x.

In this paper, we establish the existence of coincidence solutions of system of functional equations (3). Here, the functions $\overline{f}$ and $\overline{g}$ are said to be coincidence solutions of the system of functional equations (3) if the following condition holds

$$\left.\begin{aligned} \overline{f}(x) = \operatorname*{opt}_{y\in D}\{&p(x,y)+\operatorname{opt}\{r(x,y),C(x,y,\overline{g}(c(x,y))),\\ &u_i(x,y)+A_i(x,y,\overline{g}(a_i(x,y))): i=1,2,\cdots,m\}\},\\ \overline{g}(x) = \operatorname*{opt}_{y\in D}\{&q(x,y)+\operatorname{opt}\{s(x,y),D(x,y,\overline{f}(d(x,y))),\\ &v_i(x,y)+B_i(x,y,\overline{f}(b_i(x,y))): i=1,2,\cdots,m\}\}. \end{aligned}\right\} \tag{4}$$

The organization of the paper is as follows. Section 2 presents some basic notations and results. In section 3, we establish some sufficient conditions ensuring the existence of coincidence solutions for the class of system of functional equations (3). The results presented in this paper unify the results of Chang [9], Chang and Maa [10], Liu [11], Liu et al.[14], Liu et al. [15], Bhakta and Mitra [8]. Finally, a numerical example is presented to demonstrate the results in section 4.

2 Preliminaries

Suppose $\Re = (-\infty,+\infty)$, $\Re^+ = [0,+\infty)$ and $\Re^- = (-\infty,0]$. $[t]$ denotes the largest integer not exceeding t, for any $t \in \Re$ and $(X,\|.\|)$ and $(Y,\|.\|')$ be real Banach spaces. $S \subseteq X$ be the state space and $D \subseteq Y$ be the decision space.

Consider $\Phi_1 = \{(\varphi,\psi) \mid \varphi,\psi : \Re^+ \to \Re^+$ are nondecreasing and $\sum_{n=0}^{\infty}\psi(\varphi^n(t)) < \infty$ for $t > 0\}$ and $BB(S) = \{ f \mid f : S \to \Re \textit{ is bounded on bounded subsets of } S\}$.

Let k be a positive integer and $f, g \in BB(S)$, let,

$$d_k(f,g) = \sup\{| f(x) - g(x)| : x \in \overline{B}(0,k)\},$$

$$d(f,g) = \sum_{k=1}^{\infty}\frac{1}{2^k}.\frac{d_k(f,g)}{1+d_k(f,g)},$$

where $\overline{B}(0,k) = \{x : x \in S \textit{ and } \| x \| \le k\}$. Here, $(BB(S),d)$ is a complete metric space.

We need following lemmas to prove our main results.

Lemma 2.1 *Let* $a_i, b_i \in \Re$ *for* $i = 1,2,\cdots,m$, *then* $\max\{| a_i + b_i | : i = 1,2,\cdots,m\} \le \max\{| a_i | : i = 1,2,\cdots,m\} + \max\{| b_i | : i = 1,2,\cdots,m\}$.

Lemma 2.2 *If* $a,b,c \in \Re$, *then* $|\operatorname{opt}\{a,b,c\}| \le \max\{| a |,| b |,| c |\}$.

3 Existence of coincidence solutions

First, we show the existence of coincidence solutions for the system of functional equations (3).

Theorem 3.1 *Let* $c,d,a_i,b_i : S \times D \to S$, $p,q,r,s,u_i,v_i : S \times D \to \Re$ *and* $C,D,A_i,B_i : S \times D \times \Re \to \Re$ *for* $i = 1,2,\cdots,m$, *and* (φ,ψ) *be in* Φ_1 *satisfying the following assumptions:*

(D_1) $\max\{| p(x,y)|,| q(x,y)|,| r(x,y)|,| s(x,y)|,| u_i(x,y)|,| v_i(x,y)| : i = 1,2,\cdots,m\} \le \psi(\| x \|)$, for all $(x,y) \in S \times D$,

(D_2) $\max\{\| c(x,y) \|,\| d(x,y) \|,\| a_i(x,y) \|,\| b_i(x,y) \| : i = 1,2,\cdots,m\} \le \varphi(\| x \|)$, for all $(x,y) \in S \times D$,

(D_3) $\operatorname{opt}\{s(x,y), D(x,y,z), v_i(x,y) + B_i(x,y,z) : i = 1,2,\cdots,m\} \ge 0$ and $\max\{| C(x,y,z)|,$ $| D(x,y,z)|,| A_i(x,y,z)|,| B_i(x,y,z)| : i = 1,2,\cdots,m\} \le | z |$, for all $(x,y,z) \in S \times D \times \Re$

Consider the following cases of system of functional equations:

Case I: Suppose $\text{opt}_{y\in D} = \sup_{y\in D}$ and $\text{opt} = \text{opt}$ in $f(x)$ and $g(x)$ of (3) and (φ,ψ) satisfying (D_1), (D_2) and (D_3) along with the following additional assumption:

$(D_4)$for any fixed $(x,y)\in S\times D, C(x,y,.), D(x,y,.), A_i(x,y,.)$ and $B_i(x,y,.)$ are both left continuous and nondecreasing with respect to the third argument on $\Re$.

Case II: Suppose $\text{opt}_{y\in D} = \inf_{y\in D}$ in $f(x)$, $\text{opt}_{y\in D} = \sup_{y\in D}$ in $g(x)$ and $\text{opt} = \text{opt}$ in $f(x)$ and $g(x)$ of (3) and (φ,ψ) satisfying (D_1), (D_2) and (D_3) along with the following additional assumption:

(D_5) $C(x,y,.)$ and $A_i(x,y,.)$ are both left continuous and nondecreasing with respect to the third argument on $\Re$ and $D(x,y,.), B_i(x,y,.)$ are both right continuous and nondecreasing with respect to the third argument on $\Re$, for $(x,y)\in S\times D$.

Case III: Suppose $\text{opt}_{y\in D} = \inf_{y\in D}$ and $\text{opt} = \text{opt}$ in $f(x)$ and $g(x)$ of (3) and (φ,ψ) satisfying (D_1) and (D_2) along with the following additional assumptions:

(D_6) $\text{opt}\{s(x,y), D(x,y,z), v_i(x,y)+B_i(x,y,z): i=1,2,\cdots,m\}\le 0$ and $\max\{|C(x,y,z)|,$

$|D(x,y,z)|, |A_i(x,y,z)|, |B_i(x,y,z)|: i=1,2,\cdots,m\}\le |z|$,

$(D_7)$for $(x,y)\in S\times D, C(x,y,.), D(x,y,.), A_i(x,y,.)$ and $B_i(x,y,.)$ are both right continuous and nondecreasing with respect to the third argument on $\Re$.

Then the system of functional equations (3) has a coincidence solution.

Proof. The proof is carried out by the following steps.

Step 1. First we show the monotonicity of $\{g_{2n}(x)\}_{n\ge 0}$ and $\{f_{2n+1}(x)\}_{n\ge 0}$. We need to consider following three cases.

Case I: We consider the followings.

$$g_0(x) = \sup_{y\in D} q(x,y), \quad \forall x\in S, \tag{5}$$

$$\begin{aligned} g_{2n}(x) = \sup_{y\in D}\{q(x,y) + \mathrm{opt}\{s(x,y), D(x,y,f_{2n-1}(d(x,y)), \\ v_i(x,y) + B_i(x,y,f_{2n-1}(b_i(x,y)) : i = 1,2,\cdots,m)\}\}, \quad n \geq 1, \end{aligned} \tag{6}$$

$$\begin{aligned} f_{2n+1}(x) = \sup_{y\in D}\{p(x,y) + opt\{r(x,y), C(x,y,g_{2n}(c(x,y))), \\ u_i(x,y) + A_i(x,y,g_{2n}(a_i(x,y)) : i = 1,2,\cdots,m)\}\}, \quad n \geq 0. \end{aligned} \tag{7}$$

With the assumption (D_4) and following a similar argument given in [14], we obtain

$$g_0(x) \leq g_2(x) \leq \cdots \leq g_{2n}(x) \leq g_{2n+2}(x) \leq \cdots, \tag{8}$$

$$f_1(x) \leq f_3(x) \leq \cdots \leq f_{2n-1}(x) \leq f_{2n+1}(x) \leq \cdots. \tag{9}$$

This implies the sequences $\{g_{2n}(x)\}_{n\geq 0}$ and $\{f_{2n+1}(x)\}_{n\geq 0}$ are monotone.

Case II : We consider the followings.

$$g_0(x) = \sup_{y\in D} q(x,y), \quad \forall x \in S, \tag{10}$$

$$\begin{aligned} g_{2n}(x) = \sup_{y\in D}\{q(x,y) + \mathrm{opt}\{s(x,y), D(x,y,f_{2n-1}(d(x,y)), \\ v_i(x,y) + B_i(x,y,f_{2n-1}(b_i(x,y)) : i = 1,2,\cdots,m)\}\}, \quad n \geq 1, \end{aligned} \tag{11}$$

$$\begin{aligned} f_{2n+1}(x) = \inf_{y\in D}\{p(x,y) + opt\{r(x,y), C(x,y,g_{2n}(c(x,y))), \\ u_i(x,y) + A_i(x,y,g_{2n}(a_i(x,y)) : i = 1,2,\cdots,m)\}\}, \quad n \geq 0, \end{aligned} \tag{12}$$

With the assumption (D_5) and following a similar argument given in [14], we obtain

$$g_0(x) \leq g_2(x) \leq \cdots \leq g_{2n}(x) \leq g_{2n+2}(x) \leq \cdots, \tag{13}$$

$$f_1(x) \geq f_3(x) \geq \cdots \geq f_{2n-1}(x) \geq f_{2n+1}(x) \geq \cdots. \tag{14}$$

Applying the similar approach of the proof of Case 1, we conclude the sequences $\{g_{2n}(x)\}_{n\geq 0}$ and $\{f_{2n+1}(x)\}_{n\geq 0}$ are monotone.

Case III : We consider the followings.

$$g_0(x) = \inf_{y\in D} q(x,y), \quad \forall x \in S, \tag{15}$$

$$\begin{aligned} g_{2n}(x) = \inf_{y\in D} \{ & q(x,y) + \mathrm{opt}\{ s(x,y), D(x,y,f_{2n-1}(d(x,y)), \\ & v_i(x,y) + B_i(x,y,f_{2n-1}(b_i(x,y)) : i = 1,2,\cdots,m)\}\}, \quad n \ge 1, \end{aligned} \tag{16}$$

$$\begin{aligned} f_{2n+1}(x) = \inf_{y\in D} \{ & p(x,y) + opt\{ r(x,y), C(x,y,g_{2n}(c(x,y))), \\ & u_i(x,y) + A_i(x,y,g_{2n}(a_i(x,y)) : i = 1,2,\cdots,m)\}\}, \quad n \ge 0, \end{aligned} \tag{17}$$

With the assumption (D_6), (D_7) and following a similar argument given in [14], we obtain

$$g_0(x) \ge g_2(x) \ge \cdots \ge g_{2n}(x) \ge g_{2n+2}(x) \ge \cdots, \tag{18}$$

$$f_1(x) \ge f_3(x) \ge \cdots \ge f_{2n-1}(x) \ge f_{2n+1}(x) \ge \cdots. \tag{19}$$

Applying the similar approach of the proof of Case I, we conclude the sequences $\{g_{2n}(x)\}_{n\ge0}$ and $\{f_{2n+1}(x)\}_{n\ge0}$ are monotone.

Step 2. Now we show that these two sequences $\{g_{2n}(x)\}_{n\ge0}$ and $\{f_{2n+1}(x)\}_{n\ge0}$ are bounded for each of the three cases.

Case I: For $k = [\|x\|] + 1$ by (D_1) and (5), we obtain

$$|g_0(x)| = |\sup_{y\in D} q(x,y)| \le \psi(\|x\|). \tag{20}$$

In view of Lemma 2.1, Lemma 2.2 and $(D_1)-(D_3)$ and following a similar argument of the proof given in [14], we obtain

$$|g_{2n}(x)| \le 2\sum_{i=0}^{2n} \psi(\phi^i(\|x\|)) \le 2\sum_{i=0}^{\infty} \psi(\phi^i(k)), \quad \forall n \ge 0, \tag{21}$$

$$|f_{2n+1}(x)| \le 2\sum_{i=0}^{2n+1} \psi(\phi^i(\|x\|)) \le 2\sum_{i=0}^{\infty} \psi(\phi^i(k)), \quad \forall n \ge 0. \tag{22}$$

This implies that $\{g_{2n}(x)\}_{n\ge0}$ and $\{f_{2n+1}(x)\}_{n\ge0}$ are bounded.

Case II: Proof follows from the similar argument of Case I.

Case III: Proof follows from the similar argument of Case I.

Step 3. Finally, we show the existence of coincidence solutions of the system of functional equation (3). We consider the three cases separately.

Case I: Suppose,

$$\left.\begin{aligned} P(x) = \sup_{y\in D} \{ & p(x,y) + \mathrm{opt}\{ r(x,y), C(x,y,g(c(x,y))), \\ & u_i(x,y) + A_i(x,y,g(a_i(x,y))) : i = 1,2,\cdots,m\}\}, \\ Q(x) = \sup_{y\in D} \{ & q(x,y) + \mathrm{opt}\{ s(x,y), D(x,y,f(d(x,y))), \\ & v_i(x,y) + B_i(x,y,f(b_i(x,y))) : i = 1,2,\cdots,m\}\}. \end{aligned}\right\} \tag{23}$$

Inequalities (8), (9) and (23) imply for any $(x,y) \in S \times D$ and $n \ge 0$,

$$\left.\begin{array}{c} p(x,y)+\mathrm{opt}\{r(x,y),C(x,y,g(c(x,y))),u_i(x,y)+A_i(x,y,g_{2n}(a_i(x,y))):i=1,2,\cdots,m\} \\ \le f_{2n+1}(x)\le P(x), \\ q(x,y)+\mathrm{opt}\{s(x,y),D(x,y,f(d(x,y))),v_i(x,y)+B_i(x,y,f_{2n-1}(b_i(x,y))):i=1,2,\cdots,m\} \\ \le g_{2n}(x)\le Q(x), \end{array}\right\} \tag{24}$$

Following the Case I of Step 1 and Step 2 these two sequences are monotone and bounded. This implies $f,g\in BB(S)$. Considering (8) and (9) with the assumption (D_4) and letting $n\to\infty$ in (24) with $f,g\in BB(S)$, we obtain

$$\left.\begin{array}{c} p(x,y)+\mathrm{opt}\{r(x,y),C(x,y,g(c(x,y))),u_i(x,y)+A_i(x,y,g(a_i(x,y))):i=1,2,\cdots,m\} \\ \le f(x)\le P(x), \\ q(x,y)+\mathrm{opt}\{s(x,y),D(x,y,f(d(x,y))),v_i(x,y)+B_i(x,y,f(b_i(x,y))):i=1,2,\cdots,m\} \\ \le g(x)\le Q(x), \end{array}\right\} \tag{25}$$

which implies that

$$\left.\begin{array}{c} P(x)\le f(x)\le P(x), \\ Q(x)\le g(x)\le Q(x). \end{array}\right\} \tag{26}$$

Hence, $P(x)=f(x)$ and $Q(x)=g(x)$, for $x\in S$.

Case II: Applying the similar approach of Case I, we obtain

$$\left.\begin{array}{c} p(x,y)+\mathrm{opt}\{r(x,y),C(x,y,g(c(x,y))),u_i(x,y)+A_i(x,y,g(a_i(x,y))):i=1,2,\cdots,m\} \\ \ge f(x)\ge P(x), \\ q(x,y)+\mathrm{opt}\{s(x,y),D(x,y,f(d(x,y))),v_i(x,y)+B_i(x,y,f(b_i(x,y))):i=1,2,\cdots,m\} \\ \le g(x)\le Q(x), \end{array}\right\} \tag{27}$$

which implies that

$$\left.\begin{array}{c} P(x)\ge f(x)\ge P(x), \\ Q(x)\le g(x)\le Q(x). \end{array}\right\} \tag{28}$$

Hence, $P(x)=f(x)$ and $Q(x)=g(x)$, for $x\in S$.

Case III: Applying the similar approach of Case I, we obtain

$$\left.\begin{array}{c} p(x,y)+\mathrm{opt}\{r(x,y),C(x,y,g(c(x,y))),u_i(x,y)+A_i(x,y,g(a_i(x,y))):i=1,2,\cdots,m\} \\ \ge f(x)\ge P(x), \\ q(x,y)+\mathrm{opt}\{s(x,y),D(x,y,f(d(x,y))),v_i(x,y)+B_i(x,y,f(b_i(x,y))):i=1,2,\cdots,m\} \\ \ge g(x)\ge Q(x), \end{array}\right\} \tag{29}$$

which implies that

$$\left.\begin{array}{l} P(x) \geq f(x) \geq P(x), \\ Q(x) \geq g(x) \geq Q(x). \end{array}\right\} \tag{30}$$

Hence, $P(x) = f(x)$ and $Q(x) = g(x)$, for $x \in S$.

Hence, system of functional equation (3) has a coincidence solution.

Remark 3.2

1. In case $r = s = C = D = A_3 = A_4 = \cdots = A_m = B_3 = B_4 = \cdots = B_m = u_i = v_i = 0$ for $i = 1,2,\cdots,m$, then [Theorem 3.1, Case I] reduces to Theorem 3.1 of Liu et al.[14].

2. If $opt_{y\in D} = \sup_{y\in D}$, $\mathrm{opt} = \max$ and $p = q = u_2 = u_3 = \cdots = u_m = v_2 = v_3 = \cdots = v_m = A_2 = A_3 = \cdots = A_m = B_2 = B_3 = \cdots = B_m = 0$, then [Theorem 3.1, Case I] reduces to Theorem 3.1 of Liu et al. [15].

3. In case $\psi = I$, $r = s = C = D = u_i = v_i = 0$ for $i = 1,2,\cdots,m$ and $A_1 = A_2 = \cdots = A_m, B_1 = B_2 = \cdots = B_m, a_1 = a_2 = \cdots = a_m, b_1 = b_2 = \cdots = b_m$, then [Theorem 3.1, Case II] reduces to Theorem 4.1 of Liu [11].

4. In case $r = s = C = D = u_i = v_i = A_3 = A_4 = \cdots = A_m = B_3 = B_4 = \cdots = B_m = 0$ for $i = 1,2,\cdots,m$, then [Theorem 3.1, Case II], reduces to Theorem 3.2 of Liu et al. [14].

5. If $\psi = I$, $r = s = C = D = u_i = v_i = 0$ for $i = 1,2,\cdots,m$ and $A_1 = A_2 = \cdots = A_m, B_1 = B_2 = \cdots = B_m, a_1 = a_2 = \cdots = a_m, b_1 = b_2 = \cdots = b_m$, then [Theorem 3.1, Case I] reduces to Theorem 4.2 of Liu [11], which, in turn, generalizes Theorem 2.3 of Bhakta and Mitra [8].

6. In case $r = s = C = D = u_i = v_i = A_3 = A_4 = \cdots = A_m = B_3 = B_4 = \cdots = B_m = 0$ for $i = 1,2,\cdots,m$, then [Theorem 3.1, Case III], reduces to Theorem 3.3 of Liu et al. [14].

7. Theorem 3.2 and Theorem 3.3 of Liu et al. [15] are special cases of [Theorem 3.1, Case I].

8. Theorem 4.1 of Chang [9] is a special case of [Theorem 3.1, Case I].

4 Numerical example

Consider the following system of functional equation.

Example 4.1 *Let* $X = Y = \Re, S = [1,\infty), D = \Re^+$. *Define* $c, d, a_i, b_i : S \times D \to S$, $p, q, r, s, u_i, v_i : S \times D \to \Re$ *and* $A_i, B_i : S \times D \times \Re \to \Re$ *for* $i = 1,2,3$, *and* $\varphi, \psi : \Re^+ \to \Re^+$ *by*

$\psi(t) = t^2, \varphi(t) = \dfrac{t}{3},$

$$\begin{aligned} f(x) = \sup_{y\in D}\{ & \frac{x^2}{2+\sin(x+2y^2)} + opt\{\frac{x^2}{1+\cos(2y+3x^2)}, C(x,y,g(\frac{x}{4}\cos(1+3xy^3))), \\ & x^2\sin(3x+4y^2) + A_1(x,y,g(\frac{\sin x}{3+x+y^2})), \frac{x^3}{1+xy} + A_2(x,y,g(\frac{x\cos(x+2y)}{5+y^2})), \\ & x^2\cos(1+4xy^3) + A_3(x,y,g(\frac{x^2}{3+x\sin(2x+3y^2)}))\}\} \\ g(x) = \sup_{y\in D}\{ & \frac{x^2}{1+y^2} + opt\{\frac{x^3}{1+xy^2}, D(x,y,f(\frac{x}{3+y^2})), \frac{x^4}{1+x^2y^5} \\ & + B_1(x,y,f(\frac{x}{3+|\cos(3x^3+y^2)|})), \frac{x^2|\sin(3x^3y)|}{1+\cos(2y^2)} \\ & + B_2(x,y,f(\frac{x}{5+|\cos(y^2+5)|})), \frac{x^2|\cos(2x^3y)|}{1+y^2} + B_3(x,y,f(\frac{x^2}{7+xy}))\}\} \end{aligned} \tag{31}$$

where,

$$C(x,y,z) = \begin{cases} 0, & if\ z<0 \\ \dfrac{z}{1+x+y}, & if\ z\ge 0, \end{cases} \quad D(x,y,z) = \begin{cases} 0, & if\ z<0 \\ \dfrac{z|\sin(x+2y)|}{(1+x^2y)}, & if\ z\ge 0, \end{cases}$$

$$A_1(x,y,z) = \begin{cases} \dfrac{z}{1+x+y^2}, & if\ z<0 \\ \dfrac{z|\sin(3x+2y)|}{2+x+y^2}, & if\ z\ge 0, \end{cases} \quad B_1(x,y,z) = \begin{cases} 0, & if\ z<0 \\ \dfrac{z}{3+|\sin(x+2y)|}, & if\ z\ge 0, \end{cases}$$

$$A_2(x,y,z) = \begin{cases} 0, & if\ z<0 \\ \dfrac{z|\cos(x-y^2)|}{2+x+y^2}, & if\ z\ge 0, \end{cases} \quad B_2(x,y,z) = \begin{cases} 0, & if\ z<0 \\ \dfrac{z}{1+x+y^2|\sin(2x+3y)|}, & if\ z\ge 0, \end{cases}$$

$$A_3(x,y,z) = \begin{cases} \dfrac{z}{1+x+y}, & if\ z<0 \\ \dfrac{z|\cos(2xy^3)|}{(2+x^2+y)}, & if\ z\ge 0, \end{cases} \quad B_3(x,y,z) = \begin{cases} 0, & if\ z<0 \\ \dfrac{z}{1+xy+x^2}, & if\ z\ge 0. \end{cases}$$

Note that the results of [8, 9, 11, 14, 15] are not applicable due to positive values of $|p|$ and $|q|$ for verifying the existence of coincidence solutions, since

$$|p(x,y)| = \frac{x^2}{2+|\sin(x+2y^2)|} > 0, \quad \forall (x,y)\in S\times D,$$

$$|r(x,y)| = \frac{x^2}{1+|\cos(2y+3x^2)|} > 0, \quad \forall (x,y) \in S \times D.$$

One can verify the existence of coincidence solutions by applying the result proposed in this paper for all possible values of $|p|$ and $|q|$. The example considered is basically the case of

Case I of Theorem 3.1. Now we compute

$$|p(x,y)| = \left|\frac{x^2}{2+\sin(x+2y^2)}\right| \le x^2 = \psi(x),$$

$$|c(x,y)| = \left|\frac{x\cos(1+3xy^3)}{4}\right| \le \frac{x}{3} = \phi(x),$$

$$|A_1(x,y,z)| = \begin{cases} \left|\dfrac{z}{1+x+y^2}\right| \le |z|, & if \;\; z<0 \\ \left|\dfrac{z\,|\sin(3x+2y)|}{2+x+y^2}\right| \le |z|, & if \;\; z>0 \\ 0 & if \;\; z=0. \end{cases}$$

It is easy to show that all the assumptions $(D_1)-(D_4)$ for Theorem 3.1 are satisfied for this example. Based on the Theorem 3.1 we conclude that system of functional equations (31) has a coincidence solution.

5 Conclusion

Functional equations are well studied in the literature of dynamic programming and arise in a number of applications in engineering, control theory and economic theory. Bellman first presented the existence of solutions for some classes of functional equations arising in dynamic programming. We consider a generalized system of functional equations and prove the existence of coincidence solution. Theorem 3.1 unifies the results due to [8, 9, 11, 14, 15] which is illustrated with the help of an example. To find a coincidence solution for a system of functional equation one has to develop a suitable algorithm which remains as an interesting work for future research.

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