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A Sturm comparison criterion for impulsive hyperbolic equations on a rectangular prism

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Abstract. In this paper, new Sturmian comparison results and oscillatory properties of linear impulsive hyperbolic equations are obtained on a rectangular prism under fixed moment of impulse effects. Besides the Kreith's results [9, 10], this paper represents an extension of earlier findings obtained on the rectangular domain in the plane to the results obtained in rectangular prism in space.

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1. Introduction

In 1969, Kreith [9] obtained a remarkable analogue of the Sturm comparison theorem between the pair of hyperbolic boundary value problems of the form

$$\begin{aligned} &u_{tt} - u_{xx} + p(x,t)u = 0 \\ &u_x(x_j,t) + (-1)^j r_j(t)u(x_j,t) = 0; \qquad (j = 1,2) \end{aligned} \tag{1}$$

and

$$\begin{aligned} &v_{tt} - v_{xx} + q(x,t)v = 0, \\ &v_x(x_j,t) + (-1)^j s_j(t)v(x_j,t) = 0; \qquad (j = 1,2) \end{aligned} \tag{2}$$

on the rectangular domain:

$$D = \{(x,t) : x_1 < x < x_2,\ t_1 < t < t_2)\}.$$

Theorem 1. *Let z_1 be a solution of problem (1) satisfying*

$$z_1(x,t_1) = z_1(x,t_2) = 0; \quad x_1 \le x \le x_2,$$

which is positive for $(x,t) \in [x_1,x_2] \times (t_1,t_2)$. If $q \ge p$ on D and $s_j \ge r_j$ $(j = 1,2)$ on $[t_1,t_2]$, then every solution z_2 of problem (2) has a zero in

$$\bar{D} = \{(x,t) : x_1 \le x \le x_2,\ t_1 \le t \le t_2)\}.$$

For the proof of Theorem 1, we address the readers [9, Theorem 1]. See also the monograph by Kreith [10, pp. 24–26].

Impulsive differential equations have been an interesting area for mathematics, physics, biology, chemistry, engineering, medicine etc. As far as impulsive ordinary differential equations are considered, there are many studies in terms of the existence of periodic solutions, asymptotic behavior, stability, Sturmian theory and oscillatory behavior of their solutions, see for example the book by Lakshmikantham, Bainov and Simeonov [11]. When partial differential equations under the impulse effect is considered, there are fewer publications compared to ordinary impulsive differential equations. Some of the noteworthy contributions have been made by Bainov et al. [1–4] for the first order impulsive partial differential inequalities,

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by Fu et al. [7] for the oscillation of impulsive hyperbolic systems, by Bainov and Simeonov [5] for the oscillatory behavior of impulsive differential equations, by Minchev [14] for the oscillation criteria of nonlinear hyperbolic differential and difference equations under impulse effect, by Cui et al. [6] for some problems on oscillation of impulsive hyperbolic differential systems with several retarded arguments, by Luo et al. [12] for oscillatory behavior of nonlinear impulsive partial functional differential equations, by Zhu et al. [18] for oscillation criteria of impulsive neutral hyperbolic equations, by Hernández et al. [8] for the existence of solutions of impulsive partial functional differential equations, by Ning et al. [15] for the oscillation of system of impulsive hyperbolic equations and by Luo et al. [13] for oscillatory solutions of impulsive quasilinear hyperbolic systems with delay. Oscillation theory for impulsive partial differential equations has received great attention and has been developing quite rapidly in recent years. As far as the Sturm theory is concerned, it seems there is only a single work [16] for impulsive hyperbolic equations in the literature. Recently, present authors [16] give some Sturm-type comparison criteria for impulsive hyperbolic equations on a rectangular domain. They attempted to give analogical comparison results for the couple of impulsive hyperbolic problems

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + f(x,t)u(x,t) = 0; & (x,t) \in \Gamma \setminus \Gamma_{\text{imp}}, \\ \Delta u_t(x,t) + f_k(x,t)u(x,t) = 0; & (x,t) \in \Gamma_{\text{imp}} \end{cases} \tag{3}$$

satisfying the boundary conditions

$$u_x(x_j,t) + (-1)^j r_j(t)u(x_j,t) = 0; \qquad (j = 1,2) \tag{4}$$

and

$$\begin{cases} v_{tt}(x,t) - v_{xx}(x,t) + g(x,t)v(x,t) = 0; & (x,t) \in \Gamma \setminus \Gamma_{\text{imp}}, \\ \Delta v_t(x,t) + g_k(x,t)v(x,t) = 0; & (x,t) \in \Gamma_{\text{imp}} \end{cases} \tag{5}$$

satisfying the boundary conditions

$$v_x(x_j,t) + (-1)^j s_j(t)v(x_j,t) = 0; \quad (j = 1,2), \tag{6}$$

where

$$\begin{aligned} \Gamma :=& \{(x,t) : x \in (x_1,x_2),\ t \in (t_1,t_2)\} \quad \text{and} \\ \Gamma_{\text{imp}} :=& \{(x,t) \in \Gamma : t = \tau_k, k \in \mathbb{N}\}, \end{aligned}$$

$r_j, s_j \in \mathrm{C}([t_1,t_2],\mathbb{R})$ for $j = 1,2$, and $f, g, f_k, g_k \in \mathrm{C}(\Gamma,\mathbb{R})$ for $k \in \mathbb{N}$. Here $\{\tau_k\}$ is real-valued sequence such that

$$\tau_1 < \tau_2 < \cdots < \tau_k < \tau_{k+1} < \cdots \qquad (k \in \mathbb{N})$$

with $\lim_{n\to\infty} \tau_n = \infty$, and the operator Δ is the impulse operator defined as $\Delta\nu(x,\tau) = \nu(x,\tau^+) - \nu(x,\tau^-)$, where

$$\nu(x,\tau^{\pm}) = \lim_{(x,t)\to(x,\tau^{\pm})} \nu(x,\tau).$$

Theorem 2 ([16]). *Let u be a solution of problem* (3)–(4) *which is positive on Γ and satisfies $u(x,t_1) = u(x,t_2) = 0$ for all $x \in [x_1,x_2]$. If $g > f$ on Γ, $s_j > r_j$ $(j = 1,2)$ in $[t_1,t_2]$, and $g_k > f_k$ $(k \in \mathbb{N})$ on Γ_{imp}, then every solution v of problem* (5)–(6) *has a zero in closure $\bar{\Gamma}$ of Γ.*

Fix $x_0, y_0, t_0 \in \mathbb{R}$. Let $\mathcal{I} = (x_1,x_2) \subset [x_0,\infty)$, $\mathcal{J} = (y_1,y_2) \subset [y_0,\infty)$ and $\mathcal{K} = (t_1,t_2) \subset [t_0,\infty)$ be non-degenerate intervals.

Define the rectangular prism

$$\Omega = \mathcal{I} \times \mathcal{J} \times \mathcal{K},$$

and the domains

$$\begin{aligned} \mathcal{K}_{\text{imp}} &:= \{t \in \mathcal{K} : t = \tau_k, k \in \mathbb{N}\} \quad \text{and} \\ \Omega_{\text{imp}} &:= \mathcal{I} \times \mathcal{J} \times \mathcal{K}_{\text{imp}}, \end{aligned}$$

where $\{\tau_k\}$ is as defined previously.

Denote by $\mathrm{C}_{\text{imp}}\left(\bar{\Omega},\mathbb{R}\right)$ the set of functions $w : \bar{\Omega} \to \mathbb{R}$ satisfying the following properties:

(i) $w(x,y,t)$ is a continuous function for $(x,y,t) \in \bar{\Omega} \setminus \bar{\Omega}_{\text{imp}}$

(ii) There exist limits

$$\lim_{\substack{(x,y,t)\to(x,y,\tau_k)\\ t>\tau_k}} w(x,y,t) = w(x,y,\tau_k^+) \qquad (k\in\mathbb{N})$$

and

$$\lim_{\substack{(x,y,t)\to(x,y,\tau_k)\\ t<\tau_k}} w(x,y,t) = w(x,y,\tau_k^-) \qquad (k\in\mathbb{N})$$

for all $(x,y)\in\bar{\mathcal{I}}\times\bar{\mathcal{J}}$.

(iii) $\nu(x,y,t)$ is piecewise left continuous function at each τ_k, $k\in\mathbb{N}$, i.e.

$$\lim_{\substack{(x,y,t)\to(x,y,\tau_k)\\ t<\tau_k}} \nu(x,y,t) = \nu(x,y,\tau_k)$$

for each $k\in\mathbb{N}$ and $(x,y)\in\bar{\mathcal{I}}\times\bar{\mathcal{J}}$.

In this work, we give some Sturm-type comparison results for solutions of the couple of impulsive hyperbolic problems of the form

$$\begin{cases} u_{tt}-\triangle u+f(x,y,t)u=0; & (x,y,t)\in\Omega\setminus\Omega_{\text{imp}}\\ \Delta u_t+f_k(x,y,t)u=0; & (x,y,t)\in\Omega_{\text{imp}}\end{cases} \tag{7}$$

satisfying the boundary conditions

$$\begin{aligned} u_x(x_j,y,t)+(-1)^j r_j(t)u(x_j,y,t)=0; &\qquad (y,t)\in\bar{\mathcal{J}}\times\bar{\mathcal{K}},\\ u_y(x,y_j,t)+(-1)^j r_{j+2}(t)u(x,y_j,t)=0; &\qquad (x,t)\in\bar{\mathcal{I}}\times\bar{\mathcal{K}},\end{aligned} \tag{8}$$

and

$$\begin{cases} v_{tt}-\triangle v+g(x,y,t)v=0; & (x,y,t)\in\Omega\setminus\Omega_{\text{imp}}\\ \Delta v_t+g_k(x,y,t)v=0; & (x,y,t)\in\Omega_{\text{imp}}\end{cases} \tag{9}$$

satisfying the boundary conditions

$$\begin{aligned} v_x(x_j,y,t)+(-1)^j s_j(t)v(x_j,y,t)=0; &\qquad (y,t)\in\bar{\mathcal{J}}\times\bar{\mathcal{K}},\\ v_y(x,y_j,t)+(-1)^j s_{j+2}(t)v(x,y_j,t)=0; &\qquad (x,t)\in\bar{\mathcal{I}}\times\bar{\mathcal{K}}\end{aligned} \tag{10}$$

for $j=1,2$, where $f,g:\bar{\Omega}\to\mathbb{R}$, $r_\ell,s_\ell:\bar{\mathcal{K}}\to\mathbb{R}$ are continuous functions for $\ell=1,2,3,4$,

$$\Delta w(x,y,t)=w(x,y,t^+)-w(x,y,t^-),$$

and $\triangle$ is the usual *Laplace operator*:

$$\triangle=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

A function $z\neq 0$ is defined to be a solution of (7)–(8) (respectively (9)–(10)) if

- $z\in \mathrm{C}\left(\bar{\Omega},\mathbb{R}\right)$ (i.e., $\Delta z(x,y,\tau_k)=0$ for all $k\in\mathbb{N}$) and $z_t\in \mathrm{C}_{\text{imp}}\left(\bar{\Omega},\mathbb{R}\right)$;
- there exist second-order partial derivatives z_{tt}, z_{xx} and z_{yy} satisfying the first equation in (7) for each $(x,y,t)\in\Omega\setminus\Omega_{\text{imp}}$;
- z satisfies the second equation in (7) in Ω_{imp} and the boundary conditions given in (8).

Recently, present authors [17] considered the pair of Problems (7)–(8) and (9)–(10) without impulse effect, i.e. $f_k(x,y,t)\equiv 0\equiv g_k(x,y,t)$, and they obtained some Sturm-type comparison results between them.

Motivated by Theorems 1 and 2, and the results given in [17], we consider impulsive hyperbolic equations on a rectangular prism and their oscillatory properties. The results obtained in this work are conceivable as impulsive extension of those given in [17].

2. Linear Comparison Results

Based on the Kreith's comparison result obtained on the rectangular domain in the plane, we interfere to obtain an analogic result for the solutions of the couple of impulsive problems (7)–(8) and (9)–(10) on a *rectangular prism* in three-space.

Main result of the paper is the following.

Theorem 3 (Sturm comparison theorem). *Let u be a solution of problem* (7)–(8) *satisfying the initial conditions*

$$u(x,y,t_1) = u(x,y,t_2) = 0; \qquad (x,y) \in \bar{\mathcal{I}} \times \bar{\mathcal{J}}, \tag{11}$$

which is positive on Ω. If the inequalities

$$g(x,y,t) \geq f(x,y,t); \qquad (x,y,t) \in \Omega, \tag{12}$$

$$s_j(t) \geq r_j(t); \qquad t \in \bar{\mathcal{K}} \qquad (j = 1,2,3,4), \tag{13}$$

and

$$g_k(x,y,t) \geq f_k(x,y,t); \qquad (x,y,t) \in \Omega_{\text{imp}} \qquad (k \in \mathbb{N}) \tag{14}$$

hold, then every solution v of problem (9)–(10) *has a zero in $\bar{\Omega}$.*

Proof. Suppose to contrary that v has no zero in $\bar{\Omega}$. Without loss of generality we may assume that $v > 0$ in $\bar{\Omega}$. The proof of the case that $v < 0$ in $\bar{\Omega}$ is similar.

Multiplying the first equations in (7) and (9) by v and u respectively, and subtracting, we see that the identity

$$\big[uv_x - vu_x\big]_x + \big[uv_y - vu_y\big]_y + \big[vu_t - uv_t\big]_t = \big[g(x,y,t) - f(x,y,t)\big]uv \tag{15}$$

holds for all $(x,y,t) \in \bar{\Omega}$. Integrating both sides of (15) over Ω, we obtain

$$\begin{aligned} &\iiint\limits_{\Omega} \big[g(x,y,t) - f(x,y,t)\big]uv\mathrm{d}V \\ &\quad = \iiint\limits_{\Omega} \Big\{\big[uv_x - vu_x\big]_x + \big[uv_y - vu_y\big]_y + \big[vu_t - uv_t\big]_t\Big\}\,\mathrm{d}V, \end{aligned} \tag{16}$$

where $\mathrm{d}V$ is the volume element. The functions under integral signs have discontinuities of first kind at the jump points τ_k, so we divide the domain Ω into $(n+1)$ sub-domains in the following way:

$$\begin{aligned} \Omega_0 &:= \{(x,y,t) : (x,y) \in \mathcal{I} \times \mathcal{J}, t \in (t_1, \tau_1]\}, \\ \Omega_k &:= \{(x,y,t) : (x,y) \in \mathcal{I} \times \mathcal{J}, t \in (\tau_k, \tau_{k+1}]\}; \qquad k = 1,2,\ldots,n-1, \\ \Omega_n &:= \{(x,y,t) : (x,y) \in \mathcal{I} \times \mathcal{J}, t \in (\tau_n, t_2)\}. \end{aligned}$$

This allows us to apply the divergence theorem to each triple integral

$$\iiint\limits_{\Omega_m} \Big\{\big[uv_x - vu_x\big]_x + \big[uv_y - vu_y\big]_y + \big[vu_t - uv_t\big]_t\Big\}\,\mathrm{d}V \tag{17}$$

for $m = 0,1,\ldots,n$. We also note that each partition defined above satisfy

(i) $\displaystyle\bigcap_{\ell=0}^{n} \Omega_\ell = \emptyset$;

(ii) $\Omega = \displaystyle\bigcup_{\ell=0}^{n} \Omega_\ell$.

Clearly, we have from (i) and (ii) that

$$\begin{aligned} &\iiint\limits_{\Omega} \big[g(x,y,t) - f(x,y,t)\big]uv\mathrm{d}V \\ &\quad = \iiint\limits_{\Omega_0} \big[g(x,y,t) - f(x,y,t)\big]uv\mathrm{d}V + \sum_{k=1}^{n-1} \iiint\limits_{\Omega_k} \big[g(x,y,t) - f(x,y,t)\big]uv\mathrm{d}V \\ &\qquad + \iiint\limits_{\Omega_n} \big[g(x,y,t) - f(x,y,t)\big]uv\mathrm{d}V. \end{aligned} \tag{18}$$

We note that each Ω_m, $m = 0,1,\ldots,n$, is a simple solid region with the piecewise smooth boundary $\mathcal{S}_m$. Applying divergence theorem to the smooth vector field

$$\mathbf{F}(x,y,t) := (uv_x - vu_x)\mathbf{i} + (uv_y - vu_y)\mathbf{j} + (vu_t - uv_t)\mathbf{k}, \tag{19}$$

on Ω_m, $m = 0, 1, \ldots, n$, the integral given in (17) turns out to be

$$\begin{aligned}
&\iiint\limits_{\Omega_m} \Big\{ \big[uv_x - vu_x\big]_x + \big[uv_y - vu_y\big]_y + \big[vu_t - uv_t\big]_t \Big\} \mathrm{d}V \\
&= \iiint\limits_{\Omega_m} \operatorname{div} \mathbf{F} \mathrm{d}V \qquad \Bigg(= \iiint\limits_{\Omega_m} \nabla \bullet \mathbf{F} \mathrm{d}V \Bigg) \\
&= \oiint\limits_{\mathcal{S}_m} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S
\end{aligned} \tag{20}$$

for $m = 0, 1, \ldots, n$, where $\hat{\mathbf{N}}$ is the unit outward normal to the surface $\mathcal{S}_m$ and the ∇ is the usual nabla (gradient) operator defined by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial t}\mathbf{k}.$$

Since $\mathcal{S}_m$ $(= \partial\Omega_m)$, $m = 0, 1, \ldots, n$, is the union of six regions, it can be expressed as

$$\mathcal{S}_m = \bigcup_{\mu=1}^{6} \mathcal{S}_{m\mu}, \tag{21}$$

where each $\mathcal{S}_{m\mu}$, $\mu = 1, \ldots, 6$, are disjoint, rectangular, oriented, closed surfaces. It follows from the fact (21) that, the integral on the right-hand side of (20) can be expressed as

$$\begin{aligned}
&\iiint\limits_{\Omega_m} \Big\{ \big[uv_x - vu_x\big]_x + \big[uv_y - vu_y\big]_y + \big[vu_t - uv_t\big]_t \Big\} \mathrm{d}V \\
&= \sum_{\mu=1}^{6} \iint\limits_{\mathcal{S}_{m\mu}} \mathbf{F} \bullet \hat{\mathbf{N}}_{m\mu} \mathrm{d}S, \qquad (m = 0, 1, \ldots, n),
\end{aligned} \tag{22}$$

where each $\hat{\mathbf{N}}_{m\mu}$ are the unit outward normal vectors to each surface $\mathcal{S}_{m\mu}$ and defined by

$$\begin{array}{lll}
\hat{\mathbf{N}}_{m1} = -\mathbf{i}, & \hat{\mathbf{N}}_{m2} = \mathbf{i}, & \hat{\mathbf{N}}_{m3} = -\mathbf{j} \\
\hat{\mathbf{N}}_{m4} = \mathbf{j}, & \hat{\mathbf{N}}_{m5} = -\mathbf{k}, & \hat{\mathbf{N}}_{m6} = \mathbf{k}
\end{array} \tag{23}$$

for $m = 0, 1, \ldots, n$, and $\mathbf{F}$ is defined in (19).

Now, we start with the first integral in the right-hand side on (18). Taking $m = 0$ in (22) and using (15) and (23), it can be expressed as

$$\iiint\limits_{\Omega_0} \big[g(x, y, t) - f(x, y, t)\big] uv \mathrm{d}V = \oiint\limits_{\mathcal{S}_0} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S = \sum_{\mu=1}^{6} \iint\limits_{\mathcal{S}_{0\mu}} \mathbf{F} \bullet \hat{\mathbf{N}}_{0\mu} \mathrm{d}S, \tag{24}$$

where each surfaces $\mathcal{S}_{0\mu}$ are defined by

$$\begin{aligned}
\mathcal{S}_{01} &= \{(x, y, t) : x = x_1, y \in \mathcal{J}, t \in (t_1, \tau_1]\}, \\
\mathcal{S}_{02} &= \{(x, y, t) : x = x_2, y \in \mathcal{J}, t \in (t_1, \tau_1]\}, \\
\mathcal{S}_{03} &= \{(x, y, t) : y = y_1, x \in \mathcal{I}, t \in (t_1, \tau_1]\}, \\
\mathcal{S}_{04} &= \{(x, y, t) : y = y_2, x \in \mathcal{I}, t \in (t_1, \tau_1]\}, \\
\mathcal{S}_{05} &= \{(x, y, t) : t = t_1, (x, y) \in \mathcal{I} \times \mathcal{J}\}
\end{aligned}$$

and

$$\mathcal{S}_{06} = \{(x, y, t) : t = \tau_1, (x, y) \in \mathcal{I} \times \mathcal{J}\}.$$

Then by using the initial conditions (8) and (10), each integral on the right-hand side of (24) turn out to be

$$\begin{aligned}
\iint\limits_{\mathcal{S}_{01}} \mathbf{F} \bullet \hat{\mathbf{N}}_{01} \mathrm{d}S &= -\iint\limits_{\mathcal{S}_{01}} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S \\
&= -\int_{t_1}^{\tau_1} \int_{y_1}^{y_2} [uv_x - vu_x](x_1, y, t) \mathrm{d}y \mathrm{d}t
\end{aligned}$$

$$= \int_{t_1}^{\tau_1} \int_{y_1}^{y_2} [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t) \mathrm{d}y \mathrm{d}t, \tag{25}$$

$$\begin{aligned}\iint\limits_{\mathcal{S}_{02}} \mathbf{F} \bullet \hat{\mathbf{N}}_{02} \mathrm{d}S &= \iint\limits_{\mathcal{S}_{02}} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S \\ &= \int_{t_1}^{\tau_1} \int_{y_1}^{y_2} [uv_x - vu_x](x_2, y, t) \mathrm{d}y \mathrm{d}t \\ &= \int_{t_1}^{\tau_1} \int_{y_1}^{y_2} [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \mathrm{d}y \mathrm{d}t,\end{aligned} \tag{26}$$

$$\begin{aligned}\iint\limits_{\mathcal{S}_{03}} \mathbf{F} \bullet \hat{\mathbf{N}}_{03} \mathrm{d}S &= -\iint\limits_{\mathcal{S}_{03}} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S \\ &= -\int_{t_1}^{\tau_1} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_1, t) \mathrm{d}x \mathrm{d}t \\ &= \int_{t_1}^{\tau_1} \int_{x_1}^{x_2} [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t) \mathrm{d}x \mathrm{d}t,\end{aligned} \tag{27}$$

$$\begin{aligned}\iint\limits_{\mathcal{S}_{04}} \mathbf{F} \bullet \hat{\mathbf{N}}_{04} \mathrm{d}S &= \iint\limits_{\mathcal{S}_{04}} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S \\ &= \int_{t_1}^{\tau_1} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_2, t) \mathrm{d}x \mathrm{d}t \\ &= \int_{t_1}^{\tau_1} \int_{x_1}^{x_2} [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \mathrm{d}x \mathrm{d}t,\end{aligned} \tag{28}$$

$$\iint\limits_{\mathcal{S}_{05}} \mathbf{F} \bullet \hat{\mathbf{N}}_{05} \mathrm{d}S = -\iint\limits_{\mathcal{S}_{05}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = -\int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, t_1) \mathrm{d}x \mathrm{d}y \tag{29}$$

and

$$\iint\limits_{\mathcal{S}_{06}} \mathbf{F} \bullet \hat{\mathbf{N}}_{06} \mathrm{d}S = \iint\limits_{\mathcal{S}_{06}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = \int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, \tau_1) \mathrm{d}x \mathrm{d}y. \tag{30}$$

Summing up equations (25)–(30), equation (24) turns out to be

$$\begin{aligned}&\iiint\limits_{\Omega_0} \big[g(x, y, t) - f(x, y, t)\big] uv \mathrm{d}V \\ &= \int_{t_1}^{\tau_1} \int_{y_1}^{y_2} \Big\{ [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t) \\ &\quad + [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \Big\} \mathrm{d}y \mathrm{d}t \\ &\quad + \int_{t_1}^{\tau_1} \int_{x_1}^{x_2} \Big\{ [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t) \\ &\quad + [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \Big\} \mathrm{d}x \mathrm{d}t \\ &\quad + \int_{y_1}^{y_2} \int_{x_1}^{x_2} \Big\{ [vu_t - uv_t](x, y, \tau_1) - [vu_t - uv_t](x, y, t_1) \Big\} \mathrm{d}x \mathrm{d}y.\end{aligned} \tag{31}$$

Similarly, by taking $m = n$ in (22) and using (15) and (23), the last integral in the right-hand side on (18) can be expressed as

$$\iiint\limits_{\Omega_n} \big[g(x, y, t) - f(x, y, t)\big] uv \mathrm{d}V = \oiint\limits_{\mathcal{S}_n} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S = \sum_{\mu=1}^{6} \iint\limits_{\mathcal{S}_{n\mu}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n\mu} \mathrm{d}S, \tag{32}$$

where

$$\begin{aligned}
\mathcal{S}_{n1} &= \{(x,y,t) : x = x_1, y \in \mathcal{J}, t \in (\tau_n, t_2]\},\\
\mathcal{S}_{n2} &= \{(x,y,t) : x = x_2, y \in \mathcal{J}, t \in (\tau_n, t_2]\},\\
\mathcal{S}_{n3} &= \{(x,y,t) : y = y_1, x \in \mathcal{I}, t \in (\tau_n, t_2]\},\\
\mathcal{S}_{n4} &= \{(x,y,t) : y = y_2, x \in \mathcal{I}, t \in (\tau_n, t_2]\},\\
\mathcal{S}_{n5} &= \{(x,y,t) : t = \tau_n, (x,y) \in \mathcal{I} \times \mathcal{J}\}
\end{aligned}$$

and

$$\mathcal{S}_{n6} = \{(x,y,t) : t = t_2, (x,y) \in \mathcal{I} \times \mathcal{J}\}.$$

Boundary conditions (8) and (10) imply that each integral on the right-hand side of (32) turn out to be

$$\begin{aligned}
\iint\limits_{\mathcal{S}_{n1}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n1} \mathrm{d}S &= -\iint\limits_{\mathcal{S}_{n1}} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S\\
&= -\int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} [uv_x - vu_x](x_1, y, t) \mathrm{d}y \mathrm{d}t\\
&= \int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t) \mathrm{d}y \mathrm{d}t, \qquad (33)
\end{aligned}$$

$$\begin{aligned}
\iint\limits_{\mathcal{S}_{n2}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n2} \mathrm{d}S &= \iint\limits_{\mathcal{S}_{n2}} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S\\
&= \int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} [uv_x - vu_x](x_2, y, t) \mathrm{d}y \mathrm{d}t\\
&= \int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \mathrm{d}y \mathrm{d}t, \qquad (34)
\end{aligned}$$

$$\begin{aligned}
\iint\limits_{\mathcal{S}_{n3}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n3} \mathrm{d}S &= -\iint\limits_{\mathcal{S}_{n3}} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S\\
&= -\int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_1, t) \mathrm{d}x \mathrm{d}t\\
&= \int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t) \mathrm{d}x \mathrm{d}t, \qquad (35)
\end{aligned}$$

$$\begin{aligned}
\iint\limits_{\mathcal{S}_{n4}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n4} \mathrm{d}S &= \iint\limits_{\mathcal{S}_{n4}} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S\\
&= \int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_2, t) \mathrm{d}x \mathrm{d}t\\
&= \int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \mathrm{d}x \mathrm{d}t, \qquad (36)
\end{aligned}$$

$$\iint\limits_{\mathcal{S}_{n5}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n5} \mathrm{d}S = -\iint\limits_{\mathcal{S}_{n5}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = -\int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, \tau_n^+) \mathrm{d}x \mathrm{d}y \qquad (37)$$

and

$$\iint\limits_{\mathcal{S}_{n6}} \mathbf{F} \bullet \hat{\mathbf{N}}_{n6} \mathrm{d}S = \iint\limits_{\mathcal{S}_{n6}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = \int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, t_2) \mathrm{d}x \mathrm{d}y. \qquad (38)$$

By addition of integrals (33)–(38), equation (32) can be expressed as

$$\iiint\limits_{\Omega_n} \big[g(x,y,t) - f(x,y,t)\big] uv \mathrm{d}V$$

$$
\begin{aligned}
&= \int_{\tau_n^+}^{t_2} \int_{y_1}^{y_2} \Big\{ [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t) \\
&+ [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \Big\} \mathrm{d}y \mathrm{d}t \\
&+ \int_{\tau_n^+}^{t_2} \int_{x_1}^{x_2} \Big\{ [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t) \\
&+ [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \Big\} \mathrm{d}x \mathrm{d}t \\
&+ \int_{y_1}^{y_2} \int_{x_1}^{x_2} \Big\{ [vu_t - uv_t](x, y, t_2) - [vu_t - uv_t](x, y, \tau_n^+) \Big\} \mathrm{d}x \mathrm{d}y. \qquad (39)
\end{aligned}
$$

Finally, we will examine the integrals in the mid part of (18), i.e.

$$
\iiint_{\Omega_k} [g(x,y,t) - f(x,y,t)] uv \mathrm{d}V = \oiint_{\mathcal{S}_k} \mathbf{F} \bullet \hat{\mathbf{N}} \mathrm{d}S = \sum_{\mu=1}^{6} \iint_{\mathcal{S}_{k\mu}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k\mu} \mathrm{d}S, \qquad (40)
$$

where

$$
\begin{aligned}
\mathcal{S}_{k1} &= \{(x,y,t) : x = x_1, y \in \mathcal{J}, t \in (\tau_k, \tau_{k+1}]\}, \\
\mathcal{S}_{k2} &= \{(x,y,t) : x = x_2, y \in \mathcal{J}, t \in (\tau_k, \tau_{k+1}]\}, \\
\mathcal{S}_{k3} &= \{(x,y,t) : y = y_1, x \in \mathcal{I}, t \in (\tau_k, \tau_{k+1}]\}, \\
\mathcal{S}_{k4} &= \{(x,y,t) : y = y_2, x \in \mathcal{I}, t \in (\tau_k, \tau_{k+1}]\}, \\
\mathcal{S}_{k5} &= \{(x,y,t) : t = \tau_k, (x,y) \in \mathcal{I} \times \mathcal{J}\}
\end{aligned}
$$

and

$$
\mathcal{S}_{k6} = \{(x,y,t) : t = \tau_{k+1}, (x,y) \in \mathcal{I} \times \mathcal{J}\}
$$

for $k = 1, 2, \ldots, n-1$.

Then integrals on the right-hand side of (40) become

$$
\begin{aligned}
\iint_{\mathcal{S}_{k1}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k1} \mathrm{d}S &= - \iint_{\mathcal{S}_{k1}} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S \\
&= - \int_{\tau_k^+}^{\tau_{k+1}} \int_{y_1}^{y_2} [uv_x - vu_x](x_1, y, t) \mathrm{d}y \mathrm{d}t \\
&= \int_{\tau_k^+}^{\tau_{k+1}} \int_{y_1}^{y_2} [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t) \mathrm{d}y \mathrm{d}t, \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
\iint_{\mathcal{S}_{k2}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k2} \mathrm{d}S &= \iint_{\mathcal{S}_{k2}} \mathbf{F} \bullet \mathbf{i} \mathrm{d}S \\
&= \int_{\tau_k^+}^{\tau_{k+1}} \int_{y_1}^{y_2} [uv_x - vu_x](x_2, y, t) \mathrm{d}y \mathrm{d}t \\
&= \int_{\tau_k^+}^{\tau_{k+1}} \int_{y_1}^{y_2} [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \mathrm{d}y \mathrm{d}t, \qquad (42)
\end{aligned}
$$

$$
\begin{aligned}
\iint_{\mathcal{S}_{k3}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k3} \mathrm{d}S &= - \iint_{\mathcal{S}_{k3}} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S \\
&= - \int_{\tau_k^+}^{\tau_{k+1}} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_1, t) \mathrm{d}x \mathrm{d}t \\
&= \int_{\tau_k^+}^{\tau_{k+1}} \int_{x_1}^{x_2} [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t) \mathrm{d}x \mathrm{d}t, \qquad (43)
\end{aligned}
$$

$$\iint_{\mathcal{S}_{k4}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k4} \mathrm{d}S = \iint_{\mathcal{S}_{k4}} \mathbf{F} \bullet \mathbf{j} \mathrm{d}S$$
$$= \int_{\tau_k^+}^{\tau_{k+1}} \int_{x_1}^{x_2} [uv_y - vu_y](x, y_2, t) \mathrm{d}x \mathrm{d}t$$
$$= \int_{\tau_k^+}^{\tau_{k+1}} \int_{x_1}^{x_2} [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \mathrm{d}x \mathrm{d}t \tag{44}$$

$$\iint_{\mathcal{S}_{k5}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k5} \mathrm{d}S = -\iint_{\mathcal{S}_{k5}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = -\int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, \tau_k^+) \mathrm{d}x \mathrm{d}y \tag{45}$$

and

$$\iint_{\mathcal{S}_{k6}} \mathbf{F} \bullet \hat{\mathbf{N}}_{k6} \mathrm{d}S = \iint_{\mathcal{S}_{k6}} \mathbf{F} \bullet \mathbf{k} \mathrm{d}S = \int_{y_1}^{y_2} \int_{x_1}^{x_2} [vu_t - uv_t](x, y, \tau_{k+1}) \mathrm{d}x \mathrm{d}y \tag{46}$$

for $k = 1, 2, \ldots, n-1$. Integrals (41)–(46) yield

$$\iiint_{\Omega_k} \big[g(x, y, t) - f(x, y, t)\big] uv \mathrm{d}V$$
$$= \int_{\tau_k^+}^{\tau_{k+1}} \int_{y_1}^{y_2} \Big\{ [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t)$$
$$+ [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \Big\} \mathrm{d}y \mathrm{d}t$$
$$+ \int_{\tau_k^+}^{\tau_{k+1}} \int_{x_1}^{x_2} \Big\{ [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t)$$
$$+ [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \Big\} \mathrm{d}x \mathrm{d}t$$
$$+ \int_{y_1}^{y_2} \int_{x_1}^{x_2} \Big\{ -[vu_t - uv_t](x, y, \tau_k^+) + [vu_t - uv_t](x, y, \tau_{k+1}) \Big\} \mathrm{d}x \mathrm{d}y \tag{47}$$

for $k = 1, 2, \ldots, n-1$.

Finally we add the integrals (31), (39) and (47) to obtain the main integral (18) as

$$\iiint_{\Omega} \big[g(x, y, t) - f(x, y, t)\big] uv \mathrm{d}V$$
$$= \left\{ \int_{t_1}^{\tau_1} + \int_{\tau_1^+}^{\tau_2} + \cdots + \int_{\tau_{n-1}^+}^{\tau_n} + \int_{\tau_n^+}^{t_2} \right\} \int_{y_1}^{y_2} \Big\{ [r_1(t) - s_1(t)] u(x_1, y, t) v(x_1, y, t)$$
$$+ [r_2(t) - s_2(t)] u(x_2, y, t) v(x_2, y, t) \Big\} \mathrm{d}y \mathrm{d}t$$
$$+ \left\{ \int_{t_1}^{\tau_1} + \int_{\tau_1^+}^{\tau_2} + \cdots + \int_{\tau_{n-1}^+}^{\tau_n} + \int_{\tau_n^+}^{t_2} \right\} \int_{x_1}^{x_2} \Big\{ [r_3(t) - s_3(t)] u(x, y_1, t) v(x, y_1, t)$$
$$+ [r_4(t) - s_4(t)] u(x, y_2, t) v(x, y_2, t) \Big\} \mathrm{d}x \mathrm{d}t$$
$$+ \int_{y_1}^{y_2} \int_{x_1}^{x_2} \Big\{ [vu_t - uv_t](x, y, t_2) - [vu_t - uv_t](x, y, t_1) + [vu_t - uv_t](x, y, \tau_1)$$
$$+ \sum_{k=1}^{n-1} \Big\{ -[vu_t - uv_t](x, y, \tau_k^+) + [vu_t - uv_t](x, y, \tau_{k+1}) \Big\}$$
$$- [vu_t - uv_t](x, y, \tau_n^+) \Big\} \mathrm{d}x \mathrm{d}y. \tag{48}$$

Noting that $\Delta u(x,y,\tau_k)=\Delta v(x,y,\tau_k)=0$, $k\in\mathbb{N}$, the impulse conditions for the functions u_t and v_t in the second lines of (7) and (9), respectively, imply that the related impulse terms in (48) can be picked up as

$$\begin{aligned}
&-[vu_t-uv_t](x,y,\tau_1)+\sum_{k=1}^{n-1}\Big\{[vu_t-uv_t](x,y,\tau_k^+)-[vu_t-uv_t](x,y,\tau_{k+1})\Big\}\\
&\quad+[vu_t-uv_t](x,y,\tau_n^+)\\
&=\sum_{k=1}^{n}\Delta[vu_t-uv_t](x,y,\tau_k)\\
&=\sum_{t_1\le\tau_k<t_2}\Delta[vu_t-uv_t](x,y,\tau_k)\\
&=\sum_{t_1\le\tau_k<t_2}\Big\{v(x,y,\tau_k)\Delta u_t(x,y,\tau_k)-u(x,y,\tau_k)\Delta v_t(x,y,\tau_k)\Big\}\\
&=\sum_{t_1\le\tau_k<t_2}\big[g_k(x,y,\tau_k)-f_k(x,y,\tau_k)\big]u(x,y,\tau_k)v(x,y,\tau_k). \qquad (49)
\end{aligned}$$

Using initial conditions (11) and imposing impulse terms obtained in (49) into (48), we obtain the following handy identity

$$\begin{aligned}
&\iiint_\Omega\big[(g-f)uv\big](x,y,t)\mathrm{d}V+\sum_{t_1\le\tau_k<t_2}\big[(g_k-f_k)uv\big](x,y,\tau_k)\\
&=\int_{t_1}^{t_2}\int_{y_1}^{y_2}\Big\{[r_1(t)-s_1(t)]u(x_1,y,t)v(x_1,y,t)\\
&\quad+[r_2(t)-s_2(t)]u(x_2,y,t)v(x_2,y,t)\Big\}\mathrm{d}y\mathrm{d}t\\
&\quad+\int_{t_1}^{t_2}\int_{x_1}^{x_2}\Big\{[r_3(t)-s_3(t)]u(x,y_1,t)v(x,y_1,t)\\
&\quad+[r_4(t)-s_4(t)]u(x,y_2,t)v(x,y_2,t)\Big\}\mathrm{d}x\mathrm{d}t\\
&\quad+\int_{y_1}^{y_2}\int_{x_1}^{x_2}\Big\{(vu_t)(x,y,t_2)-(vu_t)(x,y,t_1)\Big\}\mathrm{d}x\mathrm{d}y. \qquad (50)
\end{aligned}$$

Conditions (12), (13) and (14) of Theorem 3 imply that left-hand side of (50) is nonnegative which is possible only when

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2}\Big\{v(x,y,t_1)u_t(x,y,t_1)-v(x,y,t_2)u_t(x,y,t_2)\Big\}\mathrm{d}x\mathrm{d}y\le 0 \qquad (51)$$

for all $x\in\bar{\mathcal{I}}$ and $y\in\bar{\mathcal{J}}$. However, (51) is not possible since $u(x,y,t_1)=u(x,y,t_2)=0$ and $u(x,y,t)>0$ for $(x,y,t)\in\bar\Omega$, $u_t(x,y,t_1)>0$ and $u_t(x,y,t_2)<0$. This contradiction yields that v can not be a positive solution of problem (9)–(10) on $\bar\Omega$.

The proof of the case that $v<0$ in $\bar\Omega$, we let $v=-z$ in $\bar\Omega$. Then z becomes a positive solution of problem (9)–(10) in $\bar\Omega$. Repeating the same proof for z, we obtain that v can not be a negative solution of problem (9)–(10) on $\bar\Omega$. Therefore v must has a zero in $\bar\Omega$. The proof of Theorem 3 is complete. □

Remark 1. If the impulse effects are dropped from (7) and (9), i.e. $f_k(x,y,t)\equiv 0$ and $g_k(x,y,t)\equiv 0$, respectively, then Theorem 3 reduces to [17, Theorem 2.1].

Remark 2. If inequalities (12), (13) and (14) in Theorem 3 are replaced by the strict ones;

$$g(x,y,t)>f(x,y,t);\qquad (x,y,t)\in\Omega, \qquad (52)$$

$$s_j(t)>r_j(t);\qquad t\in\bar{\mathcal{K}}\qquad (j=1,2,3,4), \qquad (53)$$

and

$$g_k(x,y,t) > f_k(x,y,t); \qquad (x,y,t) \in \Omega_{\rm imp}, \qquad k \in \mathbb{N}, \tag{54}$$

Then it can be easily proved that v has a zero in interior of $\bar{\Omega}$.

Proposition 1 (Sturm comparison theorem)**.** *Let u be a positive solution of problem* (7)–(8) *on $\bar{\mathcal{I}}\times\bar{\mathcal{J}}\times\mathcal{K}$ satisfying the initial conditions* (11)*. If inequalities* (52), (53) *and* (54) *hold, then every solution v of problem* (9)–(10) *has a zero in* Ω.

Proof. The proof is similar with those of Theorem 3 up to inequality (50). Under conditions (11), (52), (53) and (54) on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$, left-hand side of (50) is positive, and possible only when

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2}\Big\{v(x,y,t_1)u_t(x,y,t_1) - v(x,y,t_2)u_t(x,y,t_2)\Big\}\mathrm{d}x\mathrm{d}y < 0 \tag{55}$$

for all $x \in \bar{\mathcal{I}}$ and $y \in \bar{\mathcal{J}}$. Then we have the analogous contradiction as in the proof of Theorem 3. Namely v must has a zero in Ω. □

Remark 3. Inequalities (52), (53) and (54) can be weakened and Proposition 1 can be commuted by the following result:

Proposition 2 (Sturm comparison theorem)**.** *Assume that inequalities* (12), (13) *and* (14) *hold. Let u be a positive solution of problem* (7)–(8) *on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$ satisfying the initial conditions* (11)*. If either*

$$\{(x,y,t) \in \Omega : g(x,y,t) - f(x,y,t) > 0\} \neq \emptyset \tag{56}$$

or

$$\{t \in \bar{\mathcal{K}} : s_j(t) - r_j(t) > 0, j = 1,2,3,4\} \neq \emptyset, \tag{57}$$

or that

$$g_{k_0}(x,y,\tau_{k_0}) > f_{k_0}(x,y,\tau_{k_0}) \tag{58}$$

for some $k_0 \in \mathbb{N}$ for which $(x,y,\tau_{k_0}) \in \Omega_{\rm imp}$, then every solution v of problem (9)–(10) *has a zero in* Ω.

Proof. Clearly conditions (12)–(14) and (56)–(58) imply inequality (55). □

The following oscillation criterion is immediate.

Corollary 1 (Sturm oscillation theorem)**.** *If the inequalities*

$$g(x,y,t) \geq f(x,y,t); \qquad (x,y,t) \in \Omega^*, \tag{59}$$

$$s_j(t) \geq r_j(t); \qquad t \in [t_*,\infty) \qquad (j = 1,2,3,4) \tag{60}$$

and

$$g_k(x,y,t) \geq f_k(x,y,t); \qquad (x,y,t) \in \Omega^*_{\rm imp} \qquad (k \in \mathbb{N}), \tag{61}$$

hold for every $t_* \geq t_0$, then every solution of problem (9)–(10) *is oscillatory whenever problem* (7)–(8) *is oscillatory, where*

$$\Omega^* = \{(x,y,t) : x \in \mathcal{I}, y \in \mathcal{J},\ t \in [t^*,\infty)\} \tag{62}$$

and

$$\Omega^*_{\rm imp} = \{(x,y,t) : x \in \mathcal{I}, y \in \mathcal{J},\ t \in [t^*,\infty),\ t = \tau_k, k \in \mathbb{N}\}. \tag{63}$$

3. Nonlinear Comparison Results

The results obtained for linear equations in previous section can be extended to the nonlinear hyperbolic equations of the form

$$\begin{cases} u_{tt} - \triangle u + \mathcal{F}(u, x, y, t) = 0; & (x, y, t) \in \Omega, \\ \Delta u_t(x, y, t) + \mathcal{F}_k(u, x, y, t) = 0; & (x, y, t) \in \Omega_{\text{imp}} \end{cases} \tag{64}$$

and

$$\begin{cases} v_{tt} - \triangle v + \mathcal{G}(v, x, y, t) = 0; & (x, y, t) \in \Omega, \\ \Delta v_t(x, y, t) + \mathcal{G}_k(v, x, y, t) = 0; & (x, y, t) \in \Omega_{\text{imp}} \end{cases} \tag{65}$$

satisfying the boundary conditions (8) and (10), respectively. The functions $r_j(t)$ and $s_j(t)$, $j = 1, 2, 3, 4$, are as previously defined. We assume without further mention that

(i) $u(x, y, t)$ and $v(x, y, t)$ are continuous functions for $(x, y, t) \in \bar{\Omega} \setminus \bar{\Omega}_{\text{imp}}$, and that $\mathcal{F}(u, x, y, t)$, $\mathcal{F}_k(u, x, y, t)$, $\mathcal{G}(v, x, y, t)$ and $\mathcal{G}_k(v, x, y, t)$, $k \in \mathbb{N}$ are real valued continuous functions defined on $\bar{\Omega} \setminus \bar{\Omega}_{\text{imp}}$;

(ii) $p(t), q(t) : \bar{\mathcal{K}} \to \mathbb{R}$ and $(\mu, x, y, t) \in \mathbb{R} \times \bar{\Omega}$ are continuous functions for which

$$\mu\mathcal{F}(\mu, x, y, t) \leq p(t)\mu^2 \quad \text{and} \quad \mu\mathcal{G}(\mu, x, y, t) \geq q(t)\mu^2;$$

(iii) $\{p_k\}$ and $\{q_k\}$ are sequences of real numbers for which

$$\mu\mathcal{F}_k(\mu, x, y, t) \leq p_k\mu^2 \quad \text{and} \quad \mu\mathcal{G}_k(\mu, x, y, t) \geq q_k\mu^2$$

for all $t \geq t_0$.

Now, we have the following nonlinear comparison result.

Theorem 4 (Sturm comparison theorem)**.** *Let u be a positive solution of problem* (64)–(8) *on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$ satisfying the initial conditions* (11)*. If the inequalities*

$$q(t) \geq p(t) \quad \textit{and} \quad s_j(t) \geq r_j(t) \qquad (j = 1, 2, 3, 4) \tag{66}$$

hold for $t \in \bar{\mathcal{K}}$, and that

$$q_k \geq p_k \tag{67}$$

for all $k \in \mathbb{N}$ for which $\tau_k \in \bar{\mathcal{K}}$, then every solution v of problem (65)–(10) *has a zero in $\bar{\Omega}$.*

Proof. The proof is based on the inequality

$$\begin{aligned} \left[uv_x - vu_x\right]_x + \left[uv_y - vu_y\right]_y + \left[vu_t - uv_t\right]_t &= \left[u\mathcal{G}(v, x, y, t) - v\mathcal{F}(u, x, y, t)\right] \\ &\geq \left[q(t) - p(t)\right]uv \end{aligned}$$

for $u \in \mathrm{C}\left(\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}, \mathbb{R}\right)$, $v \in \mathrm{C}\left(\bar{\Omega}, \mathbb{R}\right)$, and can be done following the same steps those in Theorem 3. Therefore it is left to the reader. □

Remark 4. If the inequalities given in (66) and (67) are replaced by the strict ones;

$$q(t) > p(t) \quad \text{and} \quad s_j(t) > r_j(t) \qquad (j = 1, 2, 3, 4), \tag{68}$$

and that

$$q_k > p_k \tag{69}$$

for all $k \in \mathbb{N}$ for which $\tau_k \in \bar{\mathcal{K}}$, then we have the following comparison result.

Proposition 3 (Sturm comparison theorem)**.** *Let u be a positive solution of problem* (64)–(8) *on $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$ satisfying the initial conditions* (11)*. If the inequalities in* (68) *and* (69) *hold for $t \in \bar{\mathcal{K}}$, then every solution v of problem* (65)–(10) *has a zero in Ω.*

Proposition 3 can be weakened by the following result.

Proposition 4 (Sturm comparison theorem)**.** *Assume that the inequalities in* (66) *and* (67) *hold for* $t \in \bar{\mathcal{K}}$*. Let* u *be a positive solution of problem* (64)–(8) *on* $\bar{\mathcal{I}} \times \bar{\mathcal{J}} \times \mathcal{K}$ *satisfying the initial conditions* (11)*. If either*

$$\left\{t \in \bar{\mathcal{K}} : q(t) - p(t) > 0\right\} \neq \emptyset$$

or

$$\left\{t \in \bar{\mathcal{K}} : s_j(t) - r_j(t) > 0, j = 1, 2, 3, 4\right\} \neq \emptyset$$

or that

$$q_{k_0} > p_{k_0}$$

for some $k_0 \in \mathbb{N}$*, then every solution* v *of problem* (65)–(10) *has a zero in* Ω*.*

The following oscillation criterion is immediate.

Corollary 2 (Sturm oscillation theorem)**.** *If the inequalities given in* (68) *and* (67) *are satisfied for* $t \in [t^*, \infty)$*, for every* $t^* \geq t_0$*, then every solution of problem* (65)–(10) *is oscillatory whenever problem* (64)–(8) *is oscillatory.*

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References

[1] Bainov, D., Kamont, Z., Minchev, E., On first order impulsive partial differential inequalities, *Appl. Math. Comput.*, 61 (2-3) (1994), 207–230. https://doi.org/10.1016/0096-3003(94)90048-5
[2] Bainov, D., Kamont, Z., Minchev, E., First-order impulsive partial differential inequalities, *Internat. J. Theoret. Phys.*, 33(6) (1994), 1341–1358. https://doi.org/10.1007/BF00670798
[3] Bainov, D. D., Kamont, Z., Minchev, E., Comparison principles for impulsive hyperbolic equations of first order, *J. Comput. Appl. Math.*, 60(3) (1995), 379–388. https://doi.org/10.1016/0377-0427(94)00046-4
[4] Bainov, D. D., Kamont, Z., Minchev, E., On the impulsive partial differential-functional inequalities of first order, *Utilitas Math.*, 48 (1995), 107–128.
[5] Bainov, D., Simeonov, P., Oscillation Theory of Impulsive Differential Equations, International Publications, 1998.
[6] Cui, B. T., Liu, Y., Deng, F., Some oscillation problems for impulsive hyperbolic differential systems with several delays, *Appl. Math. Comput.*, 146(2-3) (2003), 667–679. https://doi.org/10.1016/S0096-3003(02)00611-2
[7] Fu, X., Liu, X., Oscillation criteria for impulsive hyperbolic systems, *Dynam. Contin. Discrete Impuls. Systems*, 3(2) (1997), 225–244.
[8] Hernández, E., Existence results for a partial second order functional differential equation with impulses, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 14(2) (2007), 229–250.
[9] Kreith, K., Sturmian theorems for hyperbolic equations, *Proc. Amer. Math. Soc.*, 22 (1969), 277–281. https://doi.org/10.2307/2036969
[10] Kreith, K., Oscillation Theory, Springer-Verlag, New York, 1973. https://doi.org/10.1007/BFb0067537
[11] Lakshmikantham, V., Bainov, D. D., Simeonov, P. S., Theory of Impulsive Differential Equations, World Scientific Publishing Co., New Jersey, 1989. https://doi.org/10.1142/0906
[12] Luo, Q., Xiao, J., Deng, F., Oscillation of nonlinear partial functional differential equations with impulse, *Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms, Intelligent and complex systems*, (2003), 130–135.
[13] Luo, L. P., Zeng, Y. H., Luo, Z. G., Oscillation theorems for quasilinear hyperbolic systems with effect of impulse and delay, *Acta Math. Appl. Sin.*, 37(5) (2014), 824–834.
[14] Minchev, E., Oscillation of solutions of impulsive nonlinear hyperbolic differential-difference equations, *Math. Balkanica*, 12(1-2) (1998), 215–224.
[15] Ning, C., Baodan, T., Yang, C., Jiqian, C., Some oscillation theorems of solutions for hyperbolic differential equations system with impulsive case, *Far East J. Appl. Math.*, 62(2) (2012), 81–105.
[16] Özbekler, A., Uslu İşler, K., Sturm comparison criterion for impulsive hyperbolic equations, *A Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 114(86) (2020), 1–10. https://doi.org/10.1007/s13398-020-00813-7
[17] Özbekler, A., Uslu İşler, K., Alzabut, J., Sturmian comparison theorem for hyperbolic equations on a rectangular prism, *AIMS Mathematics*, 9(20) (2024), 4805–4815. https://doi.org/10.3934/math.2024232
[18] Zhu, X., Li, Y., Lu, L., Oscillation criteria for impulsive hyperbolic equations of neutral type, *Chinese Quart. J. Math.*, 21(2) (2006), 176–184.