GU J Sci 30(4): 404-411 (2017)

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

On B-Semigroups

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Article Info

Received:10/01/2017
Accepted: 18/09/2017

Keywords

B-algebra
semigroup
B-semigroup
B-homomorphism

Abstract

In this paper, we combined B-algebras and semigroups and introduced the notion of B-semigroups. We obtained some properties of B-semigroups. We also introduced the notion of the homomorphism and ideal on B-semigroups and studied their fundamental properties.

1. INTRODUCTION

Y. Imai and K. Iseki introduced two classes of abstract algebras called BCK-algebras and BCI-algebras ([7], [9]). It is known that the class of BCK-algebras is a proper subclass of BCI-algebras. In [4], [5], Hu and Li introduced a wide class of abstract algebras called BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of BCH-algebras. Neggers and Kim ([14]) introduced the notion of a d-algebra which is a generalization of BCK-algebras, and also they introduced the notion of B-algebras ([15], [16]). Jun, Roh and Kim ([10]) introduced a new notion called BH-algebra which is another generalization of BCH/BCI/BCK-algebras. Walendziak obtained another equivalent axioms for B-algebras ([18]). Kim and Kim ([11]) introduced the notion of BM-algebra which is a specialization of B-algebras. They proved that the class of BM-algebras is a proper subclass of B-algebras and also showed that a BM-algebra is equivalent to a 0-commutative B-algebra. In [12], Kim and Kim introduced the Notion of BE-algebra as a generalization of a BCK-algebra. Using the notion of upper sets they gave an equivalent condition of the filter in BE-algebras. Schein ([17]) considered systems of the form $(\phi;\circ,\backslash)$, where ϕ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\phi;\circ)$ is a function semigroup) and the set theoretic subtraction "\" (and hence $(\phi;\backslash)$ is a subtraction algebra in the sence of [1]. Zelinka ([20]) discussed a problem proposed by Schein concerning the structure of multiplication in a subtraction semigroup. In [2], Ahn and Kim combined BE-algebras and semigroups and introduced the notion of BE-semigroups and investigate some of their properties.
In this paper, we combined B-algebras and semigroups and introduced the notion of B-semigroups. We obtained some properties on B-semigroups. We also introduced the notion of the homomorphism and ideal on B-semigroups and studied their related properties.

2. PRELIMINARIES

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Definition 2.1 ([15]) A B-algebra is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x*x=0$,

(II) $x*0=x$,

(III) $(x*y)*z=x*(z*(0*y))$

for all $x,y,z\in X$.

Example 2.1 ([15]) Let $X=\{0,a,b\}$ be a set with the following table:

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Then $(X;*,0)$ is a B-algebra.

Proposition 2.2 ([3,15]) If $(X;*,0)$ is a B-algebra, then for all $x,y,z\in X$,

(i) $x*z=y*z$ implies $x=y$,

(ii) $z*x=z*y$ implies $x=y$,

(iii) $x*y=0$ implies $x=y$,

(iv) $0*(0*x)=x$.

Definition 2.3 A B-algebra $(X;*,0)$ is said to be 0-commutative if

$$x*(0*y)=y*(0*x)$$

for all $x,y\in X$.

Theorem 2.4 ([15]) If $(X;*,0)$ is 0-commutative B-algebra, then $x*(x*y)=y$ for all $x,y\in X$.

Definition 2.5 A B-algebra $(X;*,0)$ is said to be associative if

$$(x*y)*z=x*(y*z)$$

for all $x,y,z\in X$.

Example 2.2 Let $(G;*,0)$ be a group such that $x*x=0$ for all $x\in G$. Then G is an associative and commutative B-algebra.

Proposition 2.6 If $(X;*,0)$ is an associative B-algebra, then for all $x,y,z\in X$,

(i) $(x*y)*z=(x*z)*y$,

(ii) X commutative iff $x*y=y*x$.

Proof. (i) is clear by (III) and (II). (ii) follows from the definition of commutativity and associativity.

3. B-SEMİGROUPS, HOMOMORPHİSMS AND IDEALS ON B-SEMİGROUPS

Definition 3.1 An algebraic system $(X;*,\cdot,0)$ is called a B-semigroup if it satisfies the following:

(IV) $(X;*,0)$ is a B-algebra,

(V) $(X;\cdot)$ is a semigroup,

(VI) the operation "$\cdot$" is distributive (on both sides) over the operation "$*$", that is,

$$x\cdot(y*z)=(x\cdot y)*(x\cdot z)$$

for all $x,y,z\in X$.

Definition 3.2 Let $(X;*,\cdot,0)$ be a B-semigroup.

(i) If $x\cdot y=y\cdot x$ for all $x,y\in X$ then X is called a commutative B-semigroup,

(ii) If $x*(0*y)=y*(0*x)$ for all $x,y\in X$, then X is called a 0-commutative B-semigroup,

(iii) If there exists an element "1" such that $1\cdot y=y\cdot 1$ for all $x\in X$, then X is said to be a B-semigroup with unity.

Example 3.1 $(\mathbb{Z};-,\cdot,0)$ and $(\mathbb{Z}_2;+,\cdot,0)$ are commutative B-semigroups with unity.

Example 3.2 Define two operations "$*$" and "$\cdot$" on a set $X=\{0,1,2,3\}$ as follows:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	1	3
3	0	3	3	0

It is easy to see that $(X;*,\cdot,0)$ is a commutative B-semigroup with unity.

Example 3.3 Define two operations "$*$" and "$\cdot$" on a set $Y=\{0,1,2,3\}$ as follows:

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

It is easy to see that $(Y;*,\cdot,0)$ is a commutative B-semigroup with unity.

Proposition 3.3 Let $(X;*,\cdot,0)$ be a B-semigroup. Then

$$x\cdot 0=0\cdot x=0$$

for all $x\in X$.

Proof. For all $x\in X$, we have that $x\cdot 0=x\cdot(0*0)=(x\cdot 0)*(x\cdot 0)=0*0=0$ and $0\cdot x=(0*0)\cdot x=(0\cdot x)*(0\cdot x)=0*0=0$.

Definition 3.4 A nonzero element x in a B-semigroup $(X;*,\cdot,0)$ is said to be a left (resp., right) zero divisor if there exists a nonzero $y\in X$ such that $x\cdot y=0$ (resp., $y\cdot x=0$). A zero divisor is an element of X which is both a left and a right zero divisor.

Theorem 3.5 Let $(X;*,\cdot,0)$ be a B-semigroup. Then X has no zero divisors if and only if the left and right cancellation laws for the operation "$\cdot$" hold in X ; that is, for all $x,y,z\in X$ with $x\neq 0$

(i) $x\cdot y=x\cdot z$ implies $x=y$,

(ii) $y\cdot x=z\cdot x$ implies $x=y$.

Proof. Let X has no zero divisors. Assume that $x\cdot y=x\cdot z$ and $x\neq 0$. Then, we have $(x\cdot y)*(x\cdot z)=0$ by (I). This implies $x\cdot(y*z)=0$. Since X has no zero divisors and $x\neq 0$, we obtain $y*z=0$. By Proposition 2.2 (iii), we get $y=z$. Similarly, $y\cdot x=z\cdot x$ implies $y=z$. Conversely, let the left and right cancellation laws hold in X. Assume that $x\cdot y=0$ and $x\neq 0$. Then since, $x\cdot y=x\cdot 0$, we get $y=0$ using the left cancellation law.

In a B-semigroup X, for an element x and positive integer n, we define $x^n=x\cdots x$ (n factors).

Proposition 3.6 Let $(X;*,\cdot,0)$ be a B-semigroup and the operation "$*$" be associative. If $x\cdot y=y\cdot x$, then $(x*y)^2=x^2*y^2$.

Proof. Since $x\cdot y=y\cdot x$ and by (I), we have

$$\begin{aligned}(x*y)^2&=(x*y)\cdot(x*y)\\&=((x*y)\cdot x)*((x*y)\cdot y)\\&=\left(x^2*(y\cdot x)\right)*\left((x\cdot y)*y^2\right)\\&=x^2*(y\cdot x)*(y\cdot x)*y^2\\&=x^2*0*y^2\\&=x^2*y^2.\end{aligned}$$

Proposition 3.7 Let $(X;*,\cdot,0)$ be a B-semigroup and the operation "$*$" be associative. If $x\cdot y=y\cdot x$ then $(x*y)^{2^n}=x^{2^n}*y^{2^n}$.

Proof. It is easily seen by induction on n.

Definition 3.8 Let $(X;*,\cdot,0)$ be a B-semigroup and $A\subseteq X$. If, for all $x,y\in A$,

(i) $x*y\in A$,

(ii) $x\cdot y\in A$

then A is called a B-subsemigroup of X.

Example 3.4 In Example 3.2, the set $A=\{0,3\}$ and the set $B=\{0,2\}$ in Example 3.3 are the B-subsemigroups of X and Y, respectively.

Definition 3.9 Let $(X;*,\cdot,0)$ and $(Y;*',\cdot',0')$ be two B-semigroups. A mapping $f:X\to Y$ is called a B-homomorphism if for all $x,y\in X$,

(i) $f(x*y)=f(x)*'f(y)$,

(ii) $f(x\cdot y)=f(x)\cdot' f(y)$.

Proposition 3.10 Let $f:X\to Y$ be a B-homomorphism. Then $f(0)=0$.

Proof. Using (I), we get $f(0)=f(0*0)=f(0)*'f(0)=0.$

Let $f:X\to Y$ be a B-homomorphism. If f is injective, then f is said to be a B-monomorphism. If f is surjective, then f is called an B-epimorphism. If f is bijective, then f is called an B-isomorphism. For any B-homomorphism $f:X\to Y$, the kernel of f (denoted by $Kerf$) is the set $\{x\in X : f(x)=0\}$. If A is a subset of A, then $f(A)=\{f(x):x\in A\}$ is the image of A. The set $f(X)$ is called the image set of f and denoted $\operatorname{Im} f$. If B is a subset of Y, $f^{-1}(B)=\{x\in X : f(x)\in B\}$ is the inverse image of B.

Example 3.5 Consider the B-semigroups in Example 3.1. Define $f:\mathbb{Z}\to\mathbb{Z}_2$ such that

$$f(x)=\begin{cases}0 \;, & if\ \ x\ is\ an\ even\ number\\ 1 \;, & if\ \ x\ is\ an\ odd\ number.\end{cases}$$

Then it is easily seen that f is an B-homomorphism.

Example 3.6 For the B-semigroups in Example 3.2 and Example 3.3, the function $f:X\to Y$ defined by $f(0)=0, f(1)=1, f(2)=3$ and $f(3)=2$ is an B-homomorphism.

Example 3.7 Let $(X;*,\cdot,0)$ be a commutative B-semigroup and the operation $"*"$ be associative. The function $f:X\to X$ defined by $f(x)=x^{2^n}$ is a homomorphism.

Theorem 3.11 Let X and Y be B-semigroups and $f:X\to Y$ be a B-homomorphism. Then

(i) $Kerf$ is a B-subsemigroup of X,

(ii) If A is a B-subsemigroup of X, then $f(A)$ is a B-subsemigroup of Y,

(iii) If B is a B-subsemigroup of Y, then $f^{-1}(B)$ is a B-subsemigroup of X,

(iv) If X is a commutative and f is an epimorphism, then Y is also a commutative B-semigroup,

(v) If X is a B-semigroup with unity if f is an epimorphism, then Y is a B-semigroup with unity and $f(1)=1$.

Proof It is easily seen as similar the proofs of the ring theory.

Theorem 3.12 Let X and Y be B-semigroups and $f:X\to Y$ be a B-homomorphism. Then f is one-to-one if and only if $Kerf=\{0\}$.

Proof. Let f is one-to-one and $a\in Kerf$. Then $f(a)=0$. We know that $f(0)=0$. Hence $f(a)=f(0)$. Since f is one to one, we get $a=0$. So $Kerf=\{0\}$. Conversely, let $Kerf=\{0\}$ and $f(a)=f(b)$. Then $f(a)*f(b)=f(a*b)=0$. Hence we have $a*b\in Kerf$ and $a*b=0$. So we get $a=b$ by Proposition 2.2 (iii).

Proposition 3.13 Let $f:X\to Y$ and $g:Y\to Z$ be two B-homomorphisms. Then the function $g\circ f$ is a B-homomorphism.

Proof. It is easily seen as similar the proof of the ring theory.

Definition 3.14 Let X be a B-semigroup. A nonempty subset of I of X is called a left (resp. right) ideal of X if it satisfies:

(I1) $(\forall a\in I)(\forall x\in X)\ a*x\in I\ implies\ x\in I$,

(I2) $(\forall a\in I)(\forall x\in X)\ x\cdot a\in I$ (resp. $a\cdot x\in I$).

If I is both left and right ideal, then I is said to be an ideal of X .

Example 3.8 The set $2\mathbb{Z}$ for the B-semigroup $(\mathbb{Z};-,\cdot,0)$, the set $\{0,3\}$ for the B-semigroup X in Example 3.2 and the set $\{0,2\}$ for the B-semigroup Y in Example 3.3 are ideals.

Theorem 3.15 Let X and Y be B-semigroups and $f:X\to Y$ be a B-homomorphism. Then

(i) $Kerf$ is an ideal of X,

(ii) If I is an ideal of X and f is an epimorphism, then $f(I)$ is an ideal of Y.

Proof. (i) Let $a \in Kerf$ and $x \in X$. Assume that $a * x \in Kerf$. Since $f(a*x) = f(a) * f(x) = 0,$ and $a \in Kerf$, we have $f(x) = f(a) = 0.$ Then we get $x \in Kerf$. Secondly, since $f(x \cdot a) = f(x) \cdot f(a) = f(x) \cdot 0 = 0,$ and similarly $f(a \cdot x) = 0,$ we see that $x \cdot a, a \cdot x \in Kerf$.

(ii) Let $b \in f(I)$ and $y \in Y$. Assume that $b * y \in f(I)$. There exist $a, x \in X$ such that $f(a) = b$ and $f(x) = y$ since f is an epimorphism. Then since $b * y = f(a) * f(x) = f(a * x) \in f(I)$ we have $a * x \in I.$ Hence we get $x \in I$ since I is an ideal. So $f(x) = y \in f(I)$. Similarly it can be seen that $y \cdot b, b \cdot y \in f(I)$.

Let $(X;*,\cdot,0)$ be a B-semigroup and let $x \in X$. Define $nx = (n-1)x * (0 * x)$ if n is a positive integer, $nx = 0 * (-n)x$ if n is a negative integer and $0x = 0$. For example, $x = 1x = 0 * (0 * x)$, $2x = x * (0 * x)$, $(-1)x = 0 * x$, $(-2)x = 0 * 2x$.

Proposition 3.16 $(X;*,\cdot,0)$ be a B-semigroup and $x, a \in X$. Then $(nx) \cdot a = n(x \cdot a)$ and $a \cdot (nx) = n(a \cdot x)$ for all $n \in \mathbb{Z}$.

Proof. Let $n = 1$. Then we have $(1x) \cdot a = (x * (0 * x)) \cdot a = x \cdot a * (0 * x \cdot a) = 1(x \cdot a)$. Now let $1 < n \in \mathbb{N}$. Assume that, for $n = k-1$, the equality is true. That is, let $((k-1)x) \cdot a = (k-1)(x \cdot a)$. Then we get, for $n = k$,

$$\begin{aligned}
(kx) \cdot a &= ((k-1)x * (0 * x)) \cdot a \\
&= ((k-1)x) \cdot a * ((0 * x) \cdot a) \\
&= ((k-1)(x \cdot a)) * ((0 \cdot a) * (x \cdot a)) \\
&= ((k-1)(x \cdot a)) * (0 * (x \cdot a)) \\
&= k(x \cdot a).
\end{aligned}$$

Hence the equality $(nx) \cdot a = n(x \cdot a)$ is true for all $b \in \mathbb{Z}^+$. Now let $n < 0$. Then we have

$$\begin{aligned}
(nx) \cdot a &= (0 * (-n)x) \cdot a \\
&= (0 \cdot a) * ((-n)x) \cdot a, \quad -n > 0 \\
&= 0 * (-n)(x \cdot a) \\
&= n(x \cdot a).
\end{aligned}$$

The other equality can be prove similarly.

The following Theorem is obtained from Theorem 2.12 in [15].

Theorem 3.17 Let $(X;*,0)$ be a B-algebra and $x \in X$. Then

$$mx * nx = (m-n)x$$

where m and n are positive integers.

Definition 3.18 Let $(X;*,0)$ be a B-algebra and $x \in X$. The set $< x >$ defined by

$$< x > = \{nx : n \in \mathbb{Z}\}$$

is called the set generated by x.

Example 3.9 For the B-algebra $(\mathbb{Z};-,0)$, we see that $< 2 > = \{n2 : n \in \mathbb{Z}\} = 2\mathbb{Z}$. For the B-algebra in Example 3.2, $< 0 > = \{0\}, < 1 > = < 2 > = X$ and $< 3 > = \{0,3\}$.

Theorem 3.19 Let $(X;*,\cdot,0)$ be a 0-commutative B-semigroup and $x \in X$. Then set $< x >$ is an ideal of X.

Proof. Let $a \in < x >, y \in X$ and $a * y \in < x >$. Then for any $m, n \in \mathbb{Z}$, we have $a = mx$ and $a * y = nx$. Hence using Theorem 2.4 and Theorem 3.16, we get

$$y = a * (a * y) = a * nx = mx * nx = (m-n)x \in < x >.$$

Also for $a \in < x >, y \in X$, we obtain $a.y = (mx).y = m(x.y) \in < x >$ and similarly $y \cdot a \in < x >$.

CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

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