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A Review on Shrinkage Parameters in Ridge Regression

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Abstract

Multicollinearity problem arises in the case of near-linear dependencies among the independent (explanatory) variables. Ridge regression based on different shrinkage parameters is a commonly used method for solving this problem. There are many shrinkage parameters; therefore, the proper selection of the shrinkage parameters is important. In this study, we investigated the popular shrinkage parameters and compared them by means of a simulation study in terms of Mean Square Error. Furthermore, a test statistic for Ridge estimator was analyzed for comparing these parameters through simulation.

1. INTRODUCTION

The most common method to estimate the regression coefficients is the Least Squares (LS) method. We consider the following multiple linear regression model with n observations and h independent variables as;

$$Y = X\beta + \varepsilon \ , \qquad \varepsilon \sim N\left(0,\ \sigma^2 I_n\right), \qquad rank\left(X_{n\times q}\right) = q \le n. \tag{1}$$

Y, $(n\times 1)$ dimensional dependent variable vector; X, $(n\times q)$ dimensional non- stochastic input matrix $(q = h+1)$; β, $(q\times 1)$ dimensional unknown coefficient vector;

ε, is error vector providing $E(\varepsilon)=0$ and $E(\varepsilon\varepsilon')=\sigma^2 I_n$. The LS estimator can be given as $\hat{\beta} = \left(X'X\right)^{-1} X'Y$.

However, in order to have valid results, some assumptions need to be made for the LS method. One of these assumptions is that there is no relationship among independent variables. Nevertheless, LS estimator performs well in the presence of mild- multicollinearity. When the degree of multicollinearity gets higher, LS estimator begins to perform poorly. In order to overcome this insufficiency, alternative methods are referred to and one of them is the biased estimation method. The most widely used biased estimation methods are Principal Components regression, Ridge regression and their variations.

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Estimations related to these methods have a small bias but they have smaller variance values compared to the LS estimators. Therefore, more accurate results can be obtained with respect to the LS method.

One of the estimators under the class of biased estimators is called shrinkage estimator. Principal Components regression, Ridge regression and their variates belong to this class. Ridge regression is the most common method to overcome the multicollinearity problem. The Ridge estimator can be given as $\tilde{\beta}_R = \left(X'X + k\mathrm{I}\right)^{-1} X'Y$ (k>0). When $X'X$ is ill conditioned, adding k to diagonal of $X'X$ improve the ill conditioned situation. One of the measures of ill conditioned matrices is the condition number denoted by (CN) and given as ;

CN= (largest eigenvalue/ smallest eigenvalue)$^{1/2}$ [1].

Farebrother [2] formed a general structure for shrinkage estimators. Also, it is shown that Ridge, Principal Components and Conditioned-Minimum Mean Square Error biased estimators are shrinkage estimators. Liski [3] proposed the use of the strong Mean Square Error (MSE) as a criterion to choose between the LS estimator and shrinkage estimator. Liski [4] also used the weak MSE test to make a choice between the LS estimator and shrinkage estimator. Ebegil et al. [5] suggested test statistics for the Ridge and Liu estimators which are shrinkage estimators and these estimators are compared under different correlation structures among independent variables with a simulation study.

There are many different optimum *k* values described in the literature. Hoerl and Kennard [6] gave an optimum *k* value to minimize the MSE of Ridge estimator. After this study, many authors obtained the optimum *k* values according to different criteria. The other popular studies in this area can be given as in Theobald [7], Hoerl et al. [8], Lawless and Wang [9], Dempster et al. [10], Gibbons [11], Saleh and Kibria [12], Kibria [13], Khalaf and Shukur [14], Zhang and Ibrahim [15], Saleh [16], Alkhamisi et al. [17], Alkhamisi and Shukur [18] and Muniz and Kibria [19].

In this study, we compare these popular *k* values according to their MSE criteria and their accept rate of the testing procedure given by Liski [3,4], which is also described in Section 3. The second section of this study gives the necessary and sufficient condition and the testing procedure for the comparison of the Ridge and LS estimators. The third section of this study gives the hypothesis procedure of the shrinkage estimators. In the fourth section, popular optimum *k* values are given and finally in the fifth section, we compare these optimum *k* values by means of Monte Carlo simulation.

2. THE NECESSARY AND SUFFICIENT CONDITION FOR THE COMPARISON OF RIDGE ESTIMATOR AND LS ESTIMATOR

The canonical form of the model is settled in the eq. (1) is:

$$Y = XPP'\beta + \varepsilon = Z\alpha + \varepsilon \tag{2}$$

where $Z = XP$ and $\alpha = P'\beta$. Also here, *P* is a ($q \times q$) orthonormal matrix such that $P' X' X P = \Lambda$ is a ($q \times q$) diagonal matrix whose diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_q$ are the eigenvalues of $(X'X)$, assumed to be in descending order.

Concerning the class of linear admissible estimators of a regression parameter, one of the linear admissible estimators is that:

$$\tilde{\beta} = A(\hat{\beta} - b) + b \tag{3}$$

where $\hat{\beta}$ is the LS estimator of β, *A* is a matrix with a dimension of $(q \times q)$ and *b* is a fixed vector with a dimension of $(q \times 1)$ [20]. These estimators are in the class of linear admissible estimators. Other conditions for admissibility are that:

$$(X'X)A \text{ or } A(X'X)^{-1} \tag{4}$$

should be symmetrical and eigenvalues of *A* should be in the interval of [0, 1]. In addition to the conditions stated in eq. (3) and eq. (4), it is required for the admissibility that *A* is symmetrical. Since $(X'X)$ and A matrices are symmetrical, there is such an orthonormal matrix *P* that diagonalizes these two matrices. The diagonal elements of $P' X' X P = \Lambda$ are $\lambda_1 > \lambda_2 > \ldots > \lambda_q$ which are the positive eigenvalues of matrix $(X'X)$ and the diagonal elements of $P' AP = \Delta$ are $\delta_1, \delta_2, \ldots, \delta_q$ which are the eigenvalues of A in the interval of [0,1]. Λ and Δ are the diagonal matrices with the dimension of $(q \times q)$ [3, 21].

Under the model stated in eq. (2), when the estimator in eq. (3) is multiplied by P';

$$\begin{aligned} \tilde{\alpha} &= P'A(\hat{\beta} - b) + P'b \\ &= P'AP(\hat{\alpha} - a) + a \\ &= \Delta(\hat{\alpha} - a) + a \end{aligned} \tag{5}$$

where $\hat{\alpha} = P'\hat{\beta}$, $a = P'b$. This type of linear admissible estimators is called shrinkage estimators [3,4].

Liski [3] has obtained a necessary and sufficient condition by comparing the MSE matrices of shrinkage estimators of which general structure have been given in the study and the MSE matrices of the LS estimators. Provided that this condition is satisfied, shrinkage estimators are as efficient as the LS estimators. If the difference between these two matrices, $MSE(\hat{\beta}) - MSE(\tilde{\beta})$, are non-negative definite then it can be shown as $MSE(\hat{\beta}) - MSE(\tilde{\beta}) \geq 0$.

Lemma 2.1 The difference, $MSE(\hat{\beta}) - MSE(\tilde{\beta})$, is non-negative definite if and only if the following inequality is satisfied by Liski [3]:

$$(\beta - b)'(I + A)^{-1} X' X (I - A)(\beta - b) / \sigma^2 \leq 1. \tag{6}$$

Also, the canonical form of eq. (6), $MSE(\hat{\alpha}) - MSE(\tilde{\alpha}) \geq 0$, can be given as follows:

$$(\alpha - a)'(I + \Delta)^{-1} \Lambda (I - \Delta)(\alpha - a) / \sigma^2 \leq 1. \tag{7}$$

The form in eq. (7) can be stated as:

$$\sum_{i=1}^{q} \gamma_i \lambda_i (\alpha_i - a_i)^2 / \sigma^2 \leq 1, \tag{8}$$

where $\gamma_i = \dfrac{(1-\delta_i)}{(1+\delta_i)}$.

Ridge estimator, known as shrinkage estimator can be defined as:

$$\tilde{\beta}_R = (X'X + kI)^{-1} X'Y \tag{9}$$

or

$$\tilde{\beta}_R = A\hat{\beta}, \tag{10}$$

where $A = (X'X + kI)^{-1} X'X$ and k>0. The necessary and sufficiency conditions for the superiority of Ridge estimator to LS estimator are given as;

$$\frac{(\beta - b)' \left(\frac{2}{k} I + \left(X'X \right)^{-1} \right)^{-1} (\beta - b)}{\sigma^2} \leq 1 \tag{11}$$

or

$$\frac{(\alpha - a)'\left(\frac{2}{k}I + \Lambda^{-1}\right)^{-1}(\alpha - a)}{\sigma^2} \leq 1 \tag{12}$$

3. A TEST FOR THE SELECTION OF SHRINKAGE ESTIMATORS

In this section, we summarize the general testing procedure of the shrinkage estimator against the LS estimator and we refer this procedure for testing the Ridge estimator by using the necessary and sufficiency condition which is given in section 2.

It is possible to choose between these two estimators using the conditions of necessary and sufficiency given in eq. (11) and eq. (12). In this case, form of the test statistics to be used in selecting between the shrinkage estimator $\tilde{\beta}$ and the LS estimator $\hat{\beta}$ is based on that inequality.

Let H be $H = (I + A)^{-1} X^{'} X (I - A)$ and $rank(H) = m$ where $m \leq q$. The necessary and sufficient condition for general shrinkage estimators is $\sum_{i=1}^{m} \gamma_i \omega_i \leq 1$ where $w_i = \lambda_i \alpha_i^2 / \sigma^2$. Thus, the hypothesis tests can be written as:

$$\text{H}_o\text{: } \sum_{i=1}^{m} \gamma_i \omega_i \leq 1$$

$$\text{H}_1\text{: } \sum_{i=1}^{m} \gamma_i \omega_i > 1 \tag{13}$$

Liski [3] investigated the following test statistic to test hypothesis given in eq. (13).

$$\tilde{F} = \hat{\beta}' H \hat{\beta} \big/ m\hat{\sigma}^2 , \tag{14}$$

where $\hat{\sigma}^2 = (Y - X\hat{\beta})'(Y - X\hat{\beta}) \big/ (n - q)$, and $b = 0$. The test statistic $\tilde{F}_R$ for the Ridge estimator can be obtained by using the $A = (X'X + kI)^{-1} X'X$ matrix in eq. (14). eq. (14) can be rewritten as follows:

$$\tilde{F} = \frac{1}{m} \sum_{i=1}^{m} \gamma_i F_i , \tag{15}$$

where $F_i = \lambda_i \hat{\alpha}_i^2 / \hat{\sigma}^2$. In eq. (15), it is quite hard to define the closed form of distribution function of non-central F_i unless all weights of γ_i are not one or zero. Therefore, approximate results can be obtained [3,4,22]. Patnaik [23] studied on central-F approach to a non-central F distribution. In this approach, central-F distribution can be written by using the first two moments of $F(\vartheta, n-q)$ and non-central $\tilde{F}(m, n-q, \omega)$ distribution:

$$\tilde{F}(m, n-q, \omega) \approx r \; F(\vartheta, n-q).$$

Here the parameters r and ϑ can be obtained with the first two moments of F distribution. In other words, two moment approach of central-F can be achieved by equating first two moments of central-F and $\tilde{F}/r$. Thus, it can be obtained;

$$E\left(\tilde{F}\right) = \frac{n-q}{(n-q-2)m} \sum_{i=1}^{m} \gamma_{i(1+\omega_i)} = \frac{n-q}{n-q-2}$$

and

$$E\left(\tilde{F}^2\right) = \frac{(n-q)^2}{r^2 (n-q-2)(n-q-4)m^2} \left\{ \left[\sum_{i=1}^{m} \gamma_i (1+\omega_i) \right]^2 + 2 \sum_{i=1}^{m} \gamma_i^2 (1+2\omega_i) \right\}$$

$$= \frac{(n-q)^2}{(n-q-2)(n-q-4)} \frac{\vartheta+2}{\vartheta}.$$

By solving these equations, r and ϑ can be written as follows:

$$r = \frac{1}{m} \sum_{i=1}^{m} \gamma_i (1+\omega_i) \text{ and } \vartheta = \frac{\left[\sum_{i=1}^{m} \gamma_i (1+\omega_i) \right]^2}{\sum_{i=1}^{m} \gamma_i^2 (1+2\omega_i)}. \tag{16}$$

Eq. (16) can be rewritten as $r = (\gamma+1)/m$ since $\gamma = \sum_{i=1}^{q} \gamma_i$ and $\sum_{i=1}^{m} \gamma_i \omega_i = 1$. The upper and lower limits of corrected degrees of freedom ϑ are defined as:

$$\vartheta_{\min} \le \vartheta \le \vartheta_{\max}, \tag{17}$$

where

$$\vartheta_{\min} = (\gamma + 1)^2 / (\sum_{i=1}^{m} \gamma_i^2 + 2\gamma_{\max})$$

$$\vartheta_{\max} = (\gamma + 1)^2 / (\sum_{i=1}^{m} \gamma_i^2 + 2\gamma_{\min})$$

The critical values $F_{\alpha}(\vartheta_{\max}, n-q)$ and $F_{\alpha}(\vartheta_{\min}, n-q)$ are obtained from central-F distribution where $F_{\alpha}(\vartheta_{\max}, n-q) \leq F_{\alpha}(\vartheta_{\min}, n-q)$ for all $0 < \alpha < 1$. $\tilde{F} / r$ statistic is compared with these values. Using these critical values for the test statistic, the decision rule can be formed as given below:

$\tilde{F}/r > F_{\alpha}(\vartheta_{\min}, n-q)$ H_o hypothesis is rejected,

$\tilde{F}/r < F_{\alpha}(\vartheta_{\max}, n-q)$ H_o hypothesis is accepted and

$F_{\alpha}(\vartheta_{\max}, n-q) \leq \tilde{F}/r \leq F_{\alpha}(\vartheta_{\min}, n-q)$ inconclusive.

4.THE POPULAR *k* VALUES OF RIDGE ESTIMATOR

In this section, we give the popular *k* values for Ridge estimator which can be found in literature. The popular *k* values used in this study are those by Hoerl and Kennard [6], by Theobald [7], by Hoerl et al. [8], by Hocking et al. [24], by Lawless and Wang [9], by Kibria [13], by Khalaf and Shukur [14], by Zhang and Ibrahim [15], by Saleh [16], by Alkhamisi et al. [17], by Alkhamisi and Shukur [18], by Muniz and Kibria [19] and by Muniz et al. [25]. The description of each of these popular *k* values is given in the Table 1.

Table 1: *The popular k values for Ridge estimator*

Authors	***k* values**
Hoerl and Kennard (1970)	$\hat{k}_{HK} = \hat{\sigma}^2 / \hat{\alpha}^2_{\max}$
Theobald (1974)	$\hat{k}_T = 2\hat{\sigma}^2 / \hat{\beta}'\hat{\beta}$
Hoerl et. al.(1975)	$\hat{k}_{HKB} = q\hat{\sigma}^2 / \hat{\beta}'_{LS}\hat{\beta}_{LS}$
Lawless and Wang (1976)	$\hat{k}_{LW} = q\hat{\sigma}^2 / \hat{\beta}' X'X \hat{\beta}$
Hocking et al.(1976)	$\hat{k}_{HSL} = \hat{\sigma}^2 \sum_{i=1}^{q} (\lambda_i \hat{\alpha}_i)^2 \Big/ \left(\sum_{i=1}^{q} \lambda_i \hat{\alpha}_i^2 \right)^2$
Kibria(2003)	$\hat{k}_{AM} = \sum_{i=1}^{q} \left(\hat{\sigma}^2 / \hat{\alpha}_i^2 \right) \Big/ q$
Kibria(2003)	$\hat{k}_{GM} = \prod_{i=1}^{q} \left(\hat{\sigma}^2 / \hat{\alpha}_i^2 \right)^{1/q}$
Kibria(2003)	$\hat{k}_{MED} = Median\left\{ \hat{\sigma}^2 / \hat{\alpha}_i^2 : i = 1,...,q \right\}$
Khalaf and Shukur(2005)	$\hat{k}_{KS} = \lambda_{\max}\hat{\sigma}^2 \Big/ \left((n-q)\hat{\sigma}^2 + \lambda_{\max}\hat{\alpha}^2_{\max} \right)$
Alkhamisi et al. (2006)	$\hat{k}_{AKSMAX} = Max\left\{ \lambda_i \hat{\sigma}_i^2 \Big/ \left((n-q)\hat{\sigma}_i^2 + \lambda_i \hat{\alpha}_i^2 \right) : i = 1,...,q \right\}$
Alkhamisi et al. (2006)	$\hat{k}_{AKSMED} = Median\left(\lambda_i \hat{\sigma}_i^2 \Big/ \left((n-q)\hat{\sigma}_i^2 + \lambda_i \hat{\alpha}_i^2 \right) : i = 1,...,q \right)$
Alkhamisi et al. (2006)	$\hat{k}_{AKSAM} = \sum_{i=1}^{q} \left(\lambda_i \hat{\sigma}_i^2 \Big/ \left((n-q)\hat{\sigma}_i^2 + \lambda_i \hat{\alpha}_i^2 \right) \right) / q$
Muniz and Kibria (2009)	$\hat{k}_{MKGM1} = \left(\prod_{i=1}^{q} \left(\lambda_i \hat{\sigma}_i^2 \Big/ \left((n-q)\hat{\sigma}_i^2 + \lambda_i \hat{\alpha}_i^2 \right) \right) \right)^{1/q}$
Muniz and Kibria (2009)	$\hat{k}_{MKMAX} = Max\left(1 \Big/ \sqrt{\hat{\sigma}^2 / \hat{\alpha}_i^2} : i = 1,...,q \right)$
Muniz and Kibria (2009)	$\hat{k}_{MKGM2} = \left(\prod_{i=1}^{q} \left(1 \Big/ \sqrt{\hat{\sigma}^2 / \hat{\alpha}_i^2} \right) \right)^{1/q}$
Muniz and Kibria (2009)	$\hat{k}_{MKGM3} = \left(\prod_{i=1}^{q} \sqrt{\hat{\sigma}^2 / \hat{\alpha}_i^2} \right)^{1/q}$
Muniz and Kibria (2009)	$\hat{k}_{MKMED} = Median\left\{ 1 \Big/ \sqrt{\hat{\sigma}^2 / \hat{\alpha}_i^2} : i = 1,...,q \right\}$
Muniz et al. (2012)	$\hat{k}_{MKSMAX1} = Max\left\{ \left((n-q)\hat{\sigma}_i^2 + \lambda_{\max}\hat{\alpha}_i^2 \right) \Big/ \left(\lambda_{\max}\hat{\sigma}_i^2 \right) : i = 1,...,q \right\}$
Muniz et al. (2012)	$\hat{k}_{MKSMAX2} = Max\left\{ \left(\lambda_{\max}\hat{\sigma}_i^2 \right) \Big/ \left((n-q)\hat{\sigma}_i^2 + \lambda_{\max}\hat{\alpha}_i^2 \right) : i = 1,...,q \right\}$
Muniz et al. (2012)	$\hat{k}_{MKSGM1} = \left(\prod_{i=1}^{q} \left(\left((n-q)\hat{\sigma}_i^2 + \lambda_{\max}\hat{\alpha}_i^2 \right) \Big/ \left(\lambda_{\max}\hat{\sigma}_i^2 \right) \right) \right)^{1/q}$
Muniz et al. (2012)	$\hat{k}_{MKSGM2} = \left(\prod_{i=1}^{q} \left(\left(\lambda_{\max}\hat{\sigma}_i^2 \right) \Big/ \left((n-q)\hat{\sigma}_i^2 + \lambda_{\max}\hat{\alpha}_i^2 \right) \right) \right)^{1/q}$
Muniz et al. (2012)	$\hat{k}_{MKSMED} = Median\left\{ \left((n-q)\hat{\sigma}_i^2 + \lambda_{\max}\hat{\alpha}_i^2 \right) \Big/ \left(\lambda_{\max}\hat{\sigma}_i^2 \right) : i = 1,...,q \right\}$

5.SIMULATION STUDY

In this section, we compare the performance of the test statistic based on Ridge estimator with the optimum *k* values. The optimum values of *k* for Ridge estimator are given Table 1.

The testing procedure for Ridge estimator is given in Liski [3]. In the experiment, we choose two sets of sample sizes *n*=15 and 30, two sets of explanatory variables number, *h*= 4 and 8. The explanatory variables are generated by;

$$x_{ij} = (1-\kappa^2)^{1/2} u_{ij} + \kappa u_{ih+1} \quad i = 1,2,...,n; \; j = 1,2,...,h$$

where u_{ij} denotes independent standard normal pseudorandom numbers and κ^2 is the theoretical correlation between any two explanatory variables. We select four sets of correlation κ=0.75, 0.85, 0.95 and 0.99. We choose six sets of standard deviation of errors, σ=0.01, 0.05, 0.25, 1, 4, 16. Since there are many tables, it is neither easy nor convenient for the reader to follow all of them. We therefore shortened these tables and chose κ=0.85, 0.95, 0.99 and σ=1, 4. These tables are available upon request. The dependent variable is generated from;

$$y_i = \beta_1 x_{i1} + \beta_1 x_{i2} + ... + \beta_h x_{ih} + \varepsilon_i \quad i = 1,2,...,n$$

where ε_i is independent normal pseudonumbers with zero mean and standard deviation σ [26,27]. We initially generated a fixed explanatory set of variables. Afterwards 10.000 sets of dependent variables are generalized based on this set of explanatory variables. We generated 10.000 sets of explanatory and dependent variables and we used the testing procedure for each estimator. We determined the MSEs of the Ridge estimators using *k* values, average of *k* values, and reject (R), accept (A) and inconclusive (I) numbers of the hypothesis test for each procedure. The results are given through Tables 2 to 5.

Table 2: *The k, MSE and number of accepted hypothesis for n=15 and h=4.*

κ		σ=1					σ=4				
	Methods	***k***	**MSE**	**R**	**A**	**I**	***k***	**MSE**	**R**	**A**	**I**
0.85	LS	-	4.5567	-	-	-	-	52.6138	-	-	-
	HKB	0.6440	1.5076	0	10000	0	0.7159	16.2758	0	10000	0
	HK	0.3055	2.5781	0	10000	0	0.2310	29.7084	0	10000	0
	HSL	0.8281	0.8077	98	9752	150	1.3740	4.5412	116	9744	140
	AM	0.9324	0.4454	2914	6761	325	2.2300	1.7994	331	9559	110
	LW	0.0593	3.4297	0	10000	0	0.0867	20.7845	0	9985	15
	T	0.3220	2.2413	0	10000	0	0.3579	24.3776	0	10000	0
	GM	0.7850	0.6272	339	9439	222	1.4318	4.5761	120	9767	113
	MED	0.8487	0.6812	540	9251	209	2.0976	4.1620	145	9757	98
	KS	0.2866	2.6273	0	10000	0	0.2200	30.5527	0	10000	0
	AKSMAX	0.7025	0.9079	95	9760	145	1.1356	3.4112	199	9658	143
	AKSMED	0.1625	2.2496	20	9927	53	0.1438	30.4702	14	9963	23
	AKSAM	0.2635	1.7888	30	9906	64	0.3653	11.4672	96	9808	96
	MKGM1	0.1373	2.6469	8	9987	5	0.1714	27.3595	15	9960	25
	MKMAX	1.8094	0.2934	662	9060	278	2.0806	6.3386	224	9636	140
	MKGM2	1.1287	0.7573	328	9474	198	0.8357	17.2335	127	9777	96
	MKGM3	0.8860	0.5767	183	9615	202	1.1966	6.3816	126	9764	110
	MKMED	1.0857	0.6282	354	9408	238	0.7125	16.4718	131	9764	105
	MKSMAX1	3.4895	0.1791	1509	8179	312	4.5448	4.1919	332	9567	101
	MKSMAX2	1.2581	0.2719	506	9188	306	2.2806	2.8625	219	9638	143
	MKSGM1	1.5239	0.4898	511	9250	239	1.0077	12.4557	179	9693	128
	MKSGM2	0.6562	0.9204	61	9808	131	0.9923	9.3575	71	9841	88
	MKSMED	1.3947	0.5007	548	9197	255	0.7341	14.9793	177	9702	121
0.95	LS	-	5.5524	-	-	-	-	72.4463	-	-	-
	HKB	0.9619	1.9056	0	10000	0	1.1315	22.2052	0	10000	0
	HK	0.3710	3.1489	0	10000	0	0.5005	40.8161	0	10000	0
	HSL	1.7285	0.9003	123	9748	129	3.8285	4.7829	127	9717	156
	AM	5.0831	0.6308	2167	7555	278	98.2665	1.8394	299	9565	136
	LW	0.1230	4.2630	0	10000	0	1.3362	23.7121	3	9984	13
	T	0.4809	2.8337	0	10000	0	0.5657	33.5166	0	10000	0
	GM	1.9069	0.9473	213	9647	140	4.5000	5.5840	118	9757	125
	MED	1.4703	0.9609	322	9504	174	1.6146	5.0728	144	9731	125
	KS	0.3447	3.2037	0	10000	0	0.4527	41.6620	0	10000	0
	AKSMAX	1.2763	1.0325	104	9764	132	4.6823	2.9633	209	9648	143
	AKSMED	0.0951	3.8633	4	9984	12	0.1466	41.7069	12	9974	14
	AKSAM	0.3746	2.4648	15	9956	29	1.2508	11.8382	106	9769	125
	MKGM1	0.1232	3.7242	0	9997	3	0.2183	36.7773	14	9966	20
	MKMAX	1.6418	0.2519	713	9051	236	1.4135	5.3029	235	9621	144
	MKGM2	0.7242	0.7081	398	9408	194	0.4714	17.4257	132	9730	138
	MKGM3	1.3809	0.7409	192	9641	167	2.1213	6.9557	131	9746	123
	MKMED	0.8346	0.5865	436	9368	196	0.8534	15.8263	150	9719	131
	MKSMAX1	2.9012	0.1351	1663	8060	277	2.2090	3.1309	336	9560	104
	MKSMAX2	3.7820	0.3787	434	9326	240	4.6823	2.5296	219	9633	148
	MKSGM1	0.8875	0.4262	615	9169	216	0.7943	11.5978	189	9655	156
	MKSGM2	1.1268	1.2522	46	9860	94	1.2590	11.0398	69	9820	111
	MKSMED	0.9078	0.4362	665	9117	218	0.9783	13.8280	195	9651	154
0.99	LS	-	35.2266	-	-	-	-	404.8712	-	-	-
	HKB	0.0784	9.8534	0	10000	0	0.1618	129.8474	0	10000	0
	HK	0.0263	19.2665	0	10000	0	0.0664	230.7801	0	10000	0
	HSL	0.8962	0.6928	217	9607	176	1.9290	7.1506	281	9579	140
	AM	0.3962	1.0942	1227	8572	201	55.6712	1.5363	337	9534	129
	LW	0.0604	9.0601	22	9899	79	0.9461	31.7480	125	9746	129
	T	0.0392	15.1038	0	10000	0	0.0809	196.4820	0	10000	0
	GM	0.1906	2.4371	141	9718	141	1.0204	23.4271	140	9727	133
	MED	0.3303	1.7965	206	9654	140	0.3555	37.6065	133	9761	106
	KS	0.0261	19.3123	0	10000	0	0.0656	231.6869	0	10000	0
	AKSMAX	0.7618	0.8093	210	9613	177	4.9361	0.9871	323	9539	138
	AKSMED	0.0283	13.6561	31	9913	56	0.0148	305.3539	1	9997	2

	AKSAM	0.2053	3.2224	123	9748	129	1.2433	7.9757	257	9606	137
	MKGM1	0.0349	12.3793	20	9922	58	0.0505	201.3044	14	9964	22
	MKMAX	6.1713	0.0689	1355	8352	293	3.8797	0.6897	381	9501	118
	MKGM2	2.2908	0.2368	573	9208	219	0.9900	5.5729	326	9544	130
	MKGM3	0.4365	1.0371	201	9641	158	1.0101	10.3710	232	9631	137
	MKMED	2.2510	0.2078	589	9198	213	2.0642	3.7354	341	9539	120
	MKSMAX1	38.2844	0.1656	4575	5124	301	15.2499	0.3389	594	9321	85
	MKSMAX2	0.7618	0.4845	340	9455	205	4.9361	0.9135	327	9536	137
	MKSGM1	5.6495	0.1026	1158	8618	224	2.6329	1.8533	371	9507	122
	MKSGM2	0.1770	2.8024	105	9756	139	0.3798	37.6500	101	9770	129
	MKSMED	6.1048	0.1182	1213	8555	232	5.0326	2.4471	403	9475	122

Table 3: *The k, MSE and number of accepted hypothesis for n=30 and h=4.*

κ		σ=1					σ=4				
	Methods	***k***	**MSE**	**R**	**A**	**I**	***k***	**MSE**	**R**	**A**	**I**
0.85	LS	-	4.4307	-	-	-	-	14.7072	-	-	-
	HKB	5.4665	1.3649	0	10000	0	19.1368	4.5521	0	10000	0
	HK	2.3657	2.3924	0	10000	0	6.5978	8.1083	0	10000	0
	HSL	3.8575	0.7829	41	9796	163	6.6098	2.5822	27	9848	125
	AM	21.8503	0.5417	2288	7323	389	82.1816	1.1394	332	9532	136
	LW	0.1469	3.8161	0	10000	0	0.2766	11.5306	0	10000	0
	T	2.7333	2.0898	0	10000	0	9.5684	6.9215	0	10000	0
	GM	11.0524	0.6598	225	9556	219	39.8585	1.8731	68	9798	134
	MED	20.5412	0.6907	368	9413	219	43.1662	1.8096	107	9779	114
	KS	1.4734	2.6259	0	10000	0	2.3239	9.6069	0	10000	0
	AKSMAX	1.9402	1.3080	21	9871	108	2.3239	5.2990	29	9887	84
	AKSMED	0.2508	3.4413	0	9999	1	0.3838	12.2096	0	10000	0
	AKSAM	0.6227	2.5453	1	9986	13	0.7985	9.5208	3	9983	14
	MKGM1	0.2689	3.3929	0	10000	0	0.4315	11.8174	0	10000	0
	MKMAX	0.6502	1.7891	48	9827	125	0.3893	9.6908	22	9935	43
	MKGM2	0.3008	2.8949	5	9961	34	0.1584	12.6874	0	9996	4
	MKGM3	3.3245	1.1328	23	9870	107	6.3134	4.6340	23	9911	66
	MKMED	0.3366	2.7812	4	9960	36	0.1604	12.5514	2	9993	5
	MKSMAX1	0.6787	1.4728	123	9723	154	0.4303	8.8150	43	9881	76
	MKSMAX2	3.5873	0.8505	57	9778	165	3.5337	4.9178	32	9871	97
	MKSGM1	0.4075	2.5029	14	9922	64	0.3264	11.5528	4	9984	12
	MKSGM2	2.4538	1.4609	7	9938	55	3.0642	6.6163	5	9966	29
	MKSMED	0.3991	2.6809	9	9945	46	0.3054	11.9925	4	9988	8
0.95	LS	-	9.4870	-	-	-	-	56.6102	-	-	-
	HKB	14.8186	2.8476	0	10000	0	1.2287	15.7286	0	9999	1
	HK	4.2863	5.1555	0	10000	0	0.5388	30.5946	0	10000	0
	HSL	4.2865	0.7812	104	9695	201	6.0343	2.5301	128	9704	168
	AM	154.5434	0.7713	1556	8125	319	4.5473	1.4311	309	9526	165
	LW	0.1550	7.3620	0	10000	0	0.1979	26.7876	1	9997	2
	T	7.4093	4.4103	0	10000	0	0.6144	24.4495	0	10000	0
	GM	53.2040	1.2222	118	9708	174	2.2152	3.8471	94	9755	151
	MED	67.3848	1.1134	216	9599	185	2.2482	3.3733	119	9741	140
	KS	2.1325	5.4142	0	10000	0	0.4768	32.0916	0	10000	0
	AKSMAX	2.1325	1.4564	63	9769	168	3.1517	4.8453	115	9706	179
	AKSMED	0.0961	7.8195	0	10000	0	0.1178	40.9833	1	9997	2
	AKSAM	0.5874	4.0639	11	9937	52	0.8526	15.5452	35	9843	122
	MKGM1	0.1481	7.1702	0	10000	0	0.1769	37.2900	2	9996	2
	MKMAX	0.4830	1.8855	127	9700	173	1.3623	10.5525	120	9732	148
	MKGM2	0.1371	3.9043	42	9842	116	0.6719	26.1960	32	9872	96
	MKGM3	7.2941	1.7226	41	9811	148	1.4884	7.2931	70	9777	153
	MKMED	0.1290	3.7520	41	9851	108	0.7760	25.8485	27	9882	91
	MKSMAX1	0.4689	1.3189	245	9563	192	2.0971	7.7678	163	9698	139
	MKSMAX2	4.2068	0.9533	93	9709	198	3.1517	4.4625	120	9706	174
	MKSGM1	0.2905	3.1322	77	9781	142	0.8177	19.4059	65	9814	121
	MKSGM2	3.4421	2.2670	17	9872	111	1.2229	10.8072	35	9832	133
	MKSMED	0.2529	3.5269	60	9809	131	0.9030	23.5538	38	9854	108
0.99	LS	-	45.9169	-	-	-	-	501.3973	-	-	-
	HKB	0.3986	12.9053	0	10000	0	0.9766	142.2434	0	10000	0
	HK	0.1037	24.6852	0	10000	0	0.3213	271.6539	0	10000	0
	HSL	4.2388	0.3886	208	9628	164	13.6591	4.0841	228	9617	155
	AM	14.1205	1.1187	804	8960	236	4.5976	0.9235	298	9563	139
	LW	0.1295	19.1882	3	9955	42	0.4476	47.8556	61	9791	148
	T	0.1993	20.2578	0	10000	0	0.4883	222.8427	0	10000	0
	GM	3.4694	3.7469	70	9801	129	2.1458	20.2913	105	9730	165
	MED	5.7519	3.2966	123	9746	131	2.1966	36.0721	108	9775	117
	KS	0.1013	24.9459	0	10000	0	0.2996	273.3113	0	10000	0

	AKSMAX	2.1931	0.8539	175	9674	151	3.3499	1.3376	230	9613	157
	AKSMED	0.0186	37.5441	0	10000	0	0.0109	417.0774	0	10000	0
	AKSAM	0.5588	5.2880	80	9776	144	0.8437	12.4818	160	9668	172
	MKGM1	0.0437	30.5006	0	9995	5	0.0327	306.6294	3	9984	13
	MKMAX	3.1055	0.5197	264	9563	173	1.7642	1.4374	253	9604	143
	MKGM2	0.5369	2.3981	176	9676	148	0.6827	13.3995	204	9633	163
	MKGM3	1.8626	2.6886	91	9749	160	1.4649	13.0009	141	9685	174
	MKMED	0.4266	2.1337	193	9650	157	0.6747	8.8714	224	9612	164
	MKSMAX1	9.8706	0.1913	719	9038	243	3.3376	0.4483	374	9498	128
	MKSMAX2	4.0210	0.6566	187	9651	162	3.3499	1.2833	233	9613	154
	MKSGM1	0.8002	1.2442	232	9607	161	0.8243	4.4761	240	9604	156
	MKSGM2	1.2497	5.3664	50	9839	111	1.2132	41.5445	64	9789	147
	MKSMED	0.4111	1.5310	260	9558	182	0.6807	5.5052	263	9580	157

Table 4: *The k, MSE and number of accepted hypothesis for n=15 and h=8.*

κ		σ=1					σ=4				
	Methods	***k***	**MSE**	**R**	**A**	**I**	***k***	**MSE**	**R**	**A**	**I**
0.85	LS	-	0.7682	-	-	-	-	40.0172	-	-	-
	HKB	2.2851	0.3272	9	9916	75	0.3815	9.6368	1	9982	17
	HK	0.3731	0.5665	0	10000	0	0.0517	23.9700	0	10000	0
	HSL	0.3743	0.5239	2	9998	0	5.2332	2.3989	301	9603	96
	AM	280.4529	0.3609	9736	197	67	52.0983	0.8307	1631	8260	109
	LW	0.0419	0.7252	0	10000	0	0.3361	13.4138	28	9934	38
	T	0.5713	0.5481	0	10000	0	0.0954	20.2731	0	10000	0
	GM	20.9756	0.1237	7204	2084	712	5.6034	2.0569	325	9566	109
	MED	9.1145	0.1326	6920	2331	749	5.8798	2.2421	312	9584	104
	KS	0.3596	0.5691	0	10000	0	0.0515	24.0544	0	10000	0
	AKSMAX	1.6375	0.1893	2045	7182	773	3.7966	2.5918	326	9561	113
	AKSMED	0.5846	0.3458	478	9158	364	0.4395	8.5302	190	9709	101
	AKSAM	0.7154	0.3152	687	8876	437	0.8288	6.2487	217	9683	100
	MKGM1	0.5219	0.3896	265	9491	244	0.3527	9.5631	148	9753	99
	MKMAX	1.6370	0.1448	3436	5962	602	4.3977	2.4584	595	9325	80
	MKGM2	0.2183	0.3834	867	8842	291	0.4224	7.2608	342	9546	112
	MKGM3	4.5799	0.1593	2897	6068	1035	2.3671	3.0233	318	9565	117
	MKMED	0.3400	0.3758	907	8748	345	0.4134	6.9197	344	9545	111
	MKSMAX1	2.7809	0.0904	5542	4067	391	19.4178	1.4988	1036	8898	66
	MKSMAX2	9.8399	0.0654	7491	2106	403	12.3332	0.5666	760	9128	112
	MKSGM1	0.2609	0.3500	1047	8627	326	0.3987	7.0496	386	9502	112
	MKSGM2	3.8333	0.1788	2216	6688	1096	2.5082	3.1856	256	9620	124
	MKSMED	0.2186	0.4192	867	8863	270	0.2494	8.4619	372	9513	115
0.95	LS	-	5.5055	-	-	-	-	85.8178	-	-	-
	HKB	0.1054	1.5813	4	9980	16	0.1858	21.5468	3	9986	11
	HK	0.0195	3.4294	0	10000	0	0.0286	52.2564	0	10000	0
	HSL	0.1062	1.4301	114	9804	82	6.7601	2.5085	423	9438	139
	AM	6.3199	0.2514	6115	3647	238	13.6877	1.5919	1244	8639	117
	LW	0.0080	4.1835	0	10000	0	0.3528	22.3022	66	9867	67
	T	0.0263	3.1485	0	10000	0	0.0464	45.0352	0	10000	0
	GM	0.5356	0.4458	760	8937	303	2.2618	4.8584	290	9572	138
	MED	0.3821	0.5444	711	9038	251	4.7009	4.9060	293	9572	135
	KS	0.0195	3.4338	0	10000	0	0.0285	52.3353	0	10000	0
	AKSMAX	0.3265	0.9445	406	9374	220	4.9971	2.7762	425	9438	137
	AKSMED	0.1010	1.7520	142	9747	111	0.1477	22.9011	150	9753	97
	AKSAM	0.1192	1.5856	199	9677	124	0.7567	12.0158	210	9673	117
	MKGM1	0.0790	2.0277	107	9791	102	0.1568	23.8687	118	9787	95
	MKMAX	7.1640	0.0861	3475	6306	219	5.9160	1.6312	755	9173	72
	MKGM2	1.3664	0.3962	1360	8424	216	0.6649	7.6592	516	9352	132
	MKGM3	0.7318	0.3787	833	8896	271	1.5039	4.9953	372	9487	141
	MKMED	1.7285	0.3291	1531	8253	216	0.5479	7.2761	530	9343	127
	MKSMAX1	51.3897	0.0768	6382	3483	135	35.0665	0.6833	1443	8494	63
	MKSMAX2	11.1710	0.0578	6281	3408	311	11.9167	0.6301	785	9144	71
	MKSGM1	2.3251	0.3048	1823	7975	202	0.7166	6.9102	615	9289	96
	MKSGM2	0.4301	0.5587	423	9337	240	1.3955	6.2957	252	9603	145
	MKSMED	3.1854	0.3043	2031	7775	194	0.4012	8.2200	620	9280	100
0.99	LS	-	40.3658	-	-	-	-	454.3936	-	-	-
	HKB	0.5479	8.9025	3	9983	14	0.2376	122.1399	2	9989	9
	HK	0.1530	23.7519	0	10000	0	0.0730	282.7939	0	10000	0
	HSL	0.7402	1.9933	277	9594	129	4.5446	1.3087	609	9328	63
	AM	4.3487	0.4705	3440	6424	136	4.4078	3.9198	888	9036	76
	LW	0.0529	11.6717	41	9919	40	0.2914	38.7935	271	9618	111
	T	0.1370	19.6311	0	10000	0	0.0594	257.4630	0	10000	0
	GM	1.4726	1.5150	331	9535	134	0.9036	21.9509	344	9547	109

	MED	1.1265	1.6416	310	9557	133	0.9117	24.0760	342	9560	98
	KS	0.1516	23.7559	0	10000	0	0.0726	282.8748	0	10000	0
	AKSMAX	0.7069	1.8664	334	9544	122	3.5478	1.4385	606	9329	65
	AKSMED	0.0385	6.0403	188	9691	121	0.0275	141.7225	143	9774	83
	AKSAM	0.1237	4.4230	222	9665	113	0.4695	17.9690	406	9493	101
	MKGM1	0.0333	7.4503	146	9756	98	0.0333	140.5870	129	9787	84
	MKMAX	2.5562	0.0250	5466	4402	132	3.7017	0.1712	917	9032	51
	MKGM2	0.8241	0.1228	1785	8119	96	1.0520	1.8758	681	9276	43
	MKGM3	1.2135	0.6395	618	9239	143	0.9506	8.6894	514	9408	78
	MKMED	0.9422	0.0984	1889	8012	99	1.5731	1.6534	694	9260	46
	MKSMAX1	6.5981	0.2879	8910	1060	30	13.7660	0.2417	2748	7184	68
	MKSMAX2	8.6738	0.1227	4181	5618	201	7.5804	0.6764	707	9242	51
	MKSGM1	0.8358	0.0550	3043	6851	106	1.3645	0.8913	813	9139	48
	MKSGM2	1.1964	1.6808	300	9563	137	0.7329	24.5100	335	9559	106
	MKSMED	0.9515	0.0536	3272	6618	110	3.1673	1.1712	844	9102	54

Table 5: *The k, MSE and number of accepted hypothesis for n=30 and h=8.*

κ		σ=1					σ=4				
	Methods	***k***	**MSE**	**R**	**A**	**I**	***k***	**MSE**	**R**	**A**	**I**
	LS	-	0.6762	-	-	-	-	9.0571	-	-	-
	HKB	1.6263	0.2699	16	8718	1266	3.2453	2.3760	2	9942	56
	HK	0.2320	0.4956	0	10000	0	1.5803	5.3868	0	10000	0
	HSL	0.2320	0.4683	1	9985	14	7.4437	1.7038	91	9735	174
	AM	20.2773	0.2534	9993	0	7	17.4658	0.4677	4101	5570	329
	LW	0.0099	0.6629	0	10000	0	0.2171	7.5166	0	10000	0
	T	0.4066	0.4775	0	10000	0	0.8113	5.1720	0	10000	0
	GM	9.3241	0.0764	9728	26	246	5.3492	0.7882	245	9499	256
	MED	9.9941	0.0838	9479	80	441	3.0443	1.0420	202	9582	216
	KS	0.2258	0.4987	0	10000	0	1.3352	5.5629	0	10000	0
0.85	AKSMAX	0.7563	0.2968	311	8688	1001	4.0626	2.2157	71	9772	157
	AKSMED	0.2280	0.4723	3	9964	33	0.2960	6.3247	2	9989	9
	AKSAM	0.2773	0.4469	12	9892	96	0.7287	5.1059	3	9957	40
	MKGM1	0.2027	0.4919	3	9978	19	0.2926	6.4466	2	9996	2
	MKMAX	2.0763	0.1531	3424	4362	2214	0.7955	3.3386	98	9749	153
	MKGM2	0.3275	0.3907	196	9307	497	0.4324	6.1779	10	9964	26
	MKGM3	3.0535	0.1393	5062	2065	2873	2.3128	1.9129	74	9766	160
	MKMED	0.3166	0.3835	235	9204	561	0.5738	5.8794	12	9949	39
	MKSMAX1	4.4286	0.0835	6983	1760	1257	0.7490	2.8704	195	9641	164
	MKSMAX2	7.4208	0.0515	9778	60	162	7.9702	0.8013	307	9425	268
	MKSGM1	0.2892	0.3587	316	9064	620	0.3650	6.0735	13	9949	38
	MKSGM2	3.4575	0.1583	3514	2757	3729	2.7397	1.9380	65	9777	158
	MKSMED	0.2177	0.4218	202	9396	402	0.4457	6.4340	13	9963	24
	LS	-	1.6458	-	-	-	-	13.0400	-	-	-
	HKB	0.8206	0.5162	8	9825	167	2.8585	3.4951	4	9918	78
	HK	0.2212	1.0521	0	10000	0	0.6840	7.8460	0	10000	0
	HSL	0.2213	0.8165	17	9900	83	3.6214	1.7966	171	9657	172
	AM	2.5869	0.1858	9472	271	257	27.7905	0.5171	3480	6235	285
	LW	0.0082	1.5780	0	10000	0	0.1283	10.5765	0	10000	0
	T	0.2051	1.0285	0	10000	0	0.7146	7.6396	0	10000	0
	GM	1.4548	0.1491	5271	2974	1755	7.7654	1.0982	233	9562	205
	MED	1.7874	0.1776	4451	3906	1643	5.4316	1.4211	217	9607	176
	KS	0.2163	1.0577	0	10000	0	0.6370	8.0252	0	10000	0
	AKSMAX	0.2224	0.7830	41	9835	124	2.6095	2.4024	142	9686	172
0.95	AKSMED	0.1115	1.0699	9	9952	39	0.1787	9.8489	1	9990	9
	AKSAM	0.1213	1.0436	10	9953	37	0.4792	7.0881	19	9922	59
	MKGM1	0.0884	1.1603	1	9981	18	0.1957	9.6927	1	9992	7
	MKMAX	2.1260	0.1576	2877	5943	1180	1.2092	3.7352	199	9653	148
	MKGM2	0.8291	0.5014	420	8965	615	0.3589	7.7474	42	9873	85
	MKGM3	1.2062	0.2191	1527	6948	1525	2.7866	2.4165	128	9696	176
	MKMED	0.7499	0.4634	554	8785	661	0.4484	7.2939	39	9863	98
	MKSMAX1	4.6225	0.0662	6276	2895	829	1.5697	2.9386	324	9544	132
	MKSMAX2	5.0060	0.0412	9298	359	343	8.6931	0.7046	402	9389	209
	MKSGM1	0.8490	0.4403	665	8673	662	0.3141	7.6615	53	9856	91
	MKSGM2	1.1778	0.2580	724	7888	1388	3.1840	2.4170	112	9714	174
	MKSMED	0.6659	0.5114	664	8735	601	0.3146	8.1976	37	9895	68
	LS	-	7.9938	0	0	0	-	94.8748	-	-	-
	HKB	0.2055	1.9797	4	9923	73	0.5886	23.1207	0	9937	63
0.99	HK	0.0956	4.6810	0	10000	0	0.1169	56.1014	0	10000	0
	HSL	0.1177	1.5262	97	9720	183	6.5515	1.0477	345	9521	134
	AM	5.3997	0.2297	5617	4008	375	34.8868	1.4838	1600	8235	165
	LW	0.0041	6.6040	0	9999	1	0.2116	37.6261	12	9918	70

	T	0.0514	4.3504	0	10000	0	0.1472	52.0709	0	10000	0
	GM	0.6174	0.5746	486	9149	365	3.2861	5.6310	158	9677	165
	MED	0.3757	0.7887	380	9343	277	2.1471	6.3956	155	9686	159
	KS	0.0947	4.6899	0	10000	0	0.1156	56.2639	0	10000	0
	AKSMAX	0.1164	1.5507	95	9725	180	4.0169	1.5589	322	9540	138
	AKSMED	0.0224	4.9872	5	9977	18	0.0267	68.3965	0	9992	8
	AKSAM	0.0327	3.7653	11	9919	70	0.5255	16.8609	100	9736	164
	MKGM1	0.0203	5.0821	4	9985	11	0.0392	61.8384	0	9980	20
	MKMAX	3.2343	0.0532	3998	5502	500	2.9245	1.1367	443	9447	110
	MKGM2	1.2727	0.3230	1016	8637	347	0.5516	7.1359	287	9561	152
	MKGM3	0.7857	0.4255	416	9218	366	1.8128	5.2694	194	9631	175
	MKMED	2.0129	0.2411	1249	8354	397	0.7148	6.4143	314	9544	142
	MKSMAX1	10.5568	0.0264	7804	1972	224	8.6491	0.3070	1045	8798	157
	MKSMAX2	7.9247	0.0531	6616	2804	580	9.8345	0.5182	438	9427	135
	MKSGM1	2.1270	0.1890	1757	7838	405	0.6477	5.3700	359	9522	119
	MKSGM2	0.4701	0.7470	181	9563	256	1.5440	7.8527	126	9705	169
	MKSMED	4.6994	0.1742	2241	7395	364	0.6233	6.5376	362	9503	135

From Table 2, it is observed that MKSMAX2 method is better than the others in terms of both MSE and number of accepted hypothesis, when n and κ are small, regardless of σ value. When κ is getting larger, AKSMAX method performs well compared the others regardless of σ value.

From Table 3, in the case of large n and small κ, HSL, GM, MED methods are better than the others. AKSMAX, MKMAX, MKSMAX1 and MKSMAX2 methods are better than the others when κ is getting larger.

From Table 4 and Table 5, HSL method performs well in terms of both MSE and the number of accepted hypothesis regardless of σ and n. Furthermore, as seen from all tables, the obtained values of all methods are affected by both κ and σ values. Besides, it can be said that σ values are more influential than κ on the k, MSE and number of accepted hypothesis.

In general, it can be said that regardless of the values of h, n and σ, in the case of bigger κ, AKSMAX method is better than the others. In the case of smaller κ, especially for bigger n, HSL method is better than the others.

3. REAL DATA EXAMPLE

In this section, we illustrate the theoretical finding in the previous sections with the dataset on Portland cement given by Woods et al. [28]. This data came from an experimental study into the heat evolved during the setting and hardening of Portland cements of varied composition and dependence of this heat on the percentages of four compounds in the clinkers from which the cement was produced. The four compounds are tricalium aluminate (X_1), tricalcium silicate (X_2), tetracalcium aluminaferrite (X_3) and beta-dicalcium silicate (X_4) respectively. The dependent variable Y is the heat evolved in calories per gram of cement after 180 days of curing. Hence, the data are given in Table 6.

Table 6. *The dataset on Portland cement*

X_1	X_2	X_3	X_4	Y
7	26	6	60	78,5
1	29	15	52	74,3
11	56	8	20	104,3
11	31	8	47	87,6
7	52	6	33	95,9
11	55	9	22	109,2
3	71	17	6	102,7
1	31	22	44	72,5
2	54	18	22	93,1
21	47	4	26	115,9
1	40	23	34	83,8
11	66	9	12	113,3
10	68	8	12	109,4

To detect multicollinearity we obtained CN value as $\text{CN} = \left(\frac{largest\ eigenvalue(X'X)}{smallest\ eigenvalue(X'X)}\right)^{1/2}$=37.1063. Because CN value is greater than 10, there are dependencies among explanatory variables. Thus, ridge estimator should be used instead of the LS estimator. First, the data is standardized and some of the required matrices and vectors for this standardized data set are given below. $X'X$ and $(X'X)^{-1}$ matrices are:

$$X'X = \begin{bmatrix} 12.000 & 2.743 & -9.890 & -2.945 \\ 2.743 & 12.000 & -1.671 & -11.676 \\ -9.890 & -1.671 & 12.000 & 0.354 \\ -2.945 & -11.676 & 0.354 & 12.000 \end{bmatrix}, \quad (X'X)^{-1} = \begin{bmatrix} 3.208 & 7.843 & 3.490 & 8.316 \\ 7.843 & 21.202 & 8.758 & 22.295 \\ 3.490 & 8.758 & 3.906 & 9.262 \\ 8.316 & 22.295 & 9.262 & 23.543 \end{bmatrix}.$$

The elements of Λ and Λ^{-1} are calculated as:

$$\Lambda = \begin{bmatrix} 0.020 & 0 & 0 & 0 \\ 0 & 2.239 & 0 & 0 \\ 0 & 0 & 18.913 & 0 \\ 0 & 0 & 0 & 26.828 \end{bmatrix}, \quad \Lambda^{-1} = \begin{bmatrix} 51.322 & 0 & 0 & 0 \\ 0 & 0.447 & 0 & 0 \\ 0 & 0 & 0.053 & 0 \\ 0 & 0 & 0 & 0.037 \end{bmatrix}.$$

$\hat{\beta}$ and $\hat{\sigma}^2$ are calculated as:

$$\hat{\beta}' = [9.124 \quad 7.939 \quad 0.653 \quad -2.411], \quad \hat{\sigma}^2 = \frac{\left(Y'Y - Y'X\,(X'X)^{-1}\,X'Y\right)}{(n-q)} = 5.318.$$

For parameter estimation, we obtained the Ridge estimators given in Table 7.

Table 7. *The obtained k values*

Methods	*k*	Methods	*k*
HKB	0.1395	AKSAM	3.6668
HK	0.0639	MKGM1	0.2727
HSL	85.2175	MKMAX	0.0741
AM	0.9282	MKGM2	4.2856
LW	0.0545	MKGM3	1.0381
T	0.0535	MKMED	0.9633
GM	2.0885	MKSMAX1	2.2532
MED	0.0899	MKSMAX2	18.7018
KS	0.5676	MKSGM1	2.9551
AKSMAX	0.0080	MKSGM2	5.4897
AKSMED	0.2062	MKSMED	0.0697

7. CONCLUSION

There are many *k* values in the literature suggested by various researchers for ridge estimators. In general, these estimators are compered in terms of the MSE criteria. However, this criterion for the comparison of these estimators cannot be sufficient. For this reason, in this study, these estimators were compared in terms of both the MSE criteria and the testing procedure given in Liski [3,4]. By performing a simulation study, we investigated which values of *k* are better than the others in which cases. The obtained results show that AKSMAX method is better than the others when κ is getting bigger regardless of the values of *h*, *n* and σ. Furthermore, HSL method is better than the others when κ is small, especially for large *n*.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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