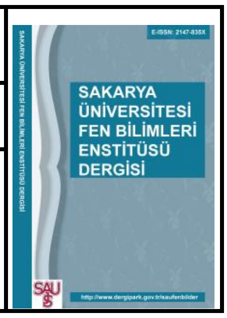
SAKARYA ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ DERGİSİ
SAKARYA UNIVERSITY JOURNAL OF SCIENCE

e-ISSN: 2147-835X
Dergi sayfası:http://dergipark.gov.tr/saufenbilder

Geliş/Received
Sep 19, 2017
Kabul/Accepted
Nov 28, 2017

Doi
10.16984/saufenbilder.338899



Some New Equalities On The Intuitionistic Fuzzy Modal Operators

Mehmet Çitil[*1], Feride Tuğrul[2]

ABSTRACT

In this study, properties of the two modal operator ($\square$, $\Diamond$) defined on intuitionistic fuzzy sets were investigated. Afterwards, some intuitionistic fuzzy operations ($\rightarrow$, @ , $\cup$, $\cap$, \$, #, $*$) were researched with modal operators ($\square$, $\Diamond$) . New equalities were obtained and proved.

Keywords: intuitionistic fuzzy sets, modal operators, operations.

[*] Sorumlu Yazar / Corresponding Author
[1] Kahramanmaraş Sütçü İmam Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, Kahramanmaraş, citil@ksu.edu.tr
[2] Kahramanmaraş Sütçü İmam Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, Kahramanmaraş, feridetugrul@gmail.com

1. INTRODUCTION

The notion of fuzzy logic was firstly defined by L.A.Zadeh in 1965. Then, intuitionistic fuzzy sets (shortly ifs) were defined by K.Atanassov in 1986. Intuitionistic fuzzy sets form a generalization of the notion of fuzzy set. In intuitionistic fuzzy set theory, sum of the membership function and the non-membership function is a value between 0 and 1. The intuitionistic fuzzy set theory is useful in various application areas, such as algebraic structures, robotics, control systems, agriculture areas, computer, irrigation, economy and various engineering fields. The knowledge and semantic representation of intuitionistic fuzzy set become more meaningful, resourceful and applicable since it include the membership degree, the non-membership degree and the hesitation margin.

The notion of Intuitionistic Fuzzy Operator (IFO) was defined firstly by K.Atanassov[3]. Several operators are defined in Intuitionistic Fuzzy Sets Theory.They are classified in three groups: modal, topological and level operators.K.Atanassov defined some first type modal operator ($\boxplus$,$\boxtimes$) on intuitionistic fuzzy sets [3]. $\boxplus_{\alpha}$ and $\boxtimes_{\alpha}$operators were defined by K.Dencheva in 2004[5]. $\boxplus_{\alpha\beta}$ and $\boxtimes_{\alpha\beta}$operators were defined by K.Atanassov in 2006[6]. $\boxplus_{\alpha\beta\gamma}$ and $\boxtimes_{\alpha\beta\gamma}$operators which are expansion of$\boxplus_{\alpha\beta}$ and $\boxtimes_{\alpha\beta}$operators respectively were defined K.Atanassov[7]. G.Cuvalcioglu was defined $E_{\alpha\beta}$ operator which is expansion of $\boxplus_{\alpha}$and $\boxtimes_{\alpha}$operators [8]. K.Atanassov produced that it created a diagram of first type modal operators in 2008. K.Atanassov defined $\boxdot_{\alpha,\beta,\gamma,\delta,\varepsilon,\tau}$operator in 2009 and he produced that this operator is most general form of operators in the diagram. $Z_{\alpha,\beta}^{\omega,\theta}$operator was defined by G.Cuvalcioglu in 2010 [9]. This operator is expansion of$E_{\alpha\beta}$, $\boxplus_{\alpha\beta}$ and $\boxtimes_{\alpha\beta}$ operators. Thisoperator settled in the diagram and expanded on the diagram [9]. G.Cuvalcioglu defined $Z_{\alpha,\beta}^{\omega,\theta}$ operator which is expansion of$Z_{\alpha,\beta}^{\omega}$,$\boxplus_{\alpha\beta}$ and $\boxtimes_{\alpha\beta}$operators in 2012 [10]. So, the diagram took its final state when this operator was defined. Some properties of first type modal operators were researched by many researchers[7][11][12].

Modal operators ($\Box$,$\Diamond$) defined over the set of all IFS's transform every IFS into a FS. They are similar to the operators 'necessity' and 'possibility' defined in some modal logics. The notion of modal operator ($\Box$,$\Diamond$) introduced on intuitionistic fuzzy sets were defined by K.Atanassov in 1986 [2].
The aim of this paper is to obtain new equalities by means of modal operators ($\Box$,$\Diamond$) and some intuitionistic fuzzy operations. These equalities make it easy application areas of operators. Shorter equalities have obtained using features of some intuitionistic fuzzy operations with modal operators. In this paper; for every IFS M,N in X, we get $M=\{\langle\mu_M(x),\nu_M(x)\rangle\}$ instead of $M=\left\{\langle x,\mu_M(x),\nu_M(x)\rangle \mid x\in X\right\}$ and $N=\{\langle\mu_N(x),\nu_N(x)\rangle\}$ instead of $N=\left\{\langle x,\mu_N(x),\nu_N(x)\rangle \mid x\in X\right\}$ for simplicity in proofs. Some basic definitions that we build on our work are given as follow.

2. PRELIMINARIES

Definition 1 [1] Let X be a nonempty set, a fuzzy set A drawn from X is defined as $A=\{\langle x,\mu_A(x)\rangle \mid x\in X\}$,where
$\mu_A(x):X\to[0,1]$ is the membership function of the fuzzy set A.

Definition 2 [2],[3] Let X be a nonempty set , an intuitionistic fuzzy set A in X is an object having the form
$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle \mid x\in X\}$,
where the function
$\mu_A(x),\nu_A(x):X\to[0,1]$
define respectively, the degree of membership and degree of nonmembership of the element $x\in X$, to the set A, which is a subset of X, and for every element $x\in X$,
$0\le\mu_A(x)+\nu_A(x)\le 1.$
Furthermore, we have
$\pi_A(x)=1-\mu_A(x)-\nu_A(x)$
called the intuitionistic fuzzy set index or hesitation on margin of x in A. $\pi_A(x)$ is degree of indeterminacy of $x\in X$ to the IFS A and $\pi_A(x)\in[0,1]$ i.e.,$\pi_A:X\to[0,1]$
for every $x\in X$. $\pi_A(x)$ expresses the lack of knowledge of whether x belongs to IFS A or not.

Definition 3 [2],[3] Let X be a nonempty set for every IFS M,N in X. For every two IFS's M and N the following operations and relations are valid.

$$M = \left\{\langle x, \mu_M(x), \nu_M(x)\rangle \mid x \in X\right\}$$
$$M^c = \left\{\langle x, \nu_M(x), \mu_M(x)\rangle \mid x \in X\right\}$$

$$M@N = \{\langle x, \tfrac{\mu_M(x)+\mu_N(x)}{2}, \tfrac{\nu_M(x)+\nu_N(x)}{2}\rangle \mid x \in X\}$$

$$M \to N = \left\{\langle x, \max(\nu_M(x), \mu_N(x)), \min(\mu_M(x), \nu_N(x))\rangle \mid x \in X\right\}$$

$$M \cap N = \{\langle x, \min(\mu_M(x), \mu_N(x)), \max(\nu_M(x), \nu_N(x))\rangle \mid x \in X\}$$

$$M \cup N = \{\langle x, \max(\mu_M(x), \mu_N(x)), \min(\nu_M(x), \nu_N(x))\rangle \mid x \in X\}$$

$$M \oplus N = \{\langle x, \mu_M(x) + \mu_N(x) - \mu_M(x).\mu_N(x), \nu_M(x).\nu_N(x)\rangle \mid x \in X\}$$

$$M \otimes N = \{\langle x, \mu_M(x).\mu_N(x), \nu_M(x) + \nu_N(x) - \nu_M(x).\nu_N(x)\rangle \mid x \in X\}$$

$$M \$ N = \{\langle x, \sqrt{\mu_M(x).\mu_N(x)}, \sqrt{\nu_M(x).\nu_N(x)}\rangle \mid x \in X\}$$

$$M \# N = \{\langle x, \frac{2\mu_M(x).\mu_N(x)}{\mu_M(x)+\mu_N(x)}, \frac{2\nu_M(x).\nu_N(x)}{\nu_M(x)+\nu_N(x)}\rangle \mid x \in X\}$$

$$M * N = \{\langle x, \frac{\mu_M(x)+\mu_B(x)}{2(\mu_M(x)\mu_B(x)+1)}, \frac{\nu_M(x)+\nu_B(x)}{2(\nu_M(x)\nu_B(x)+1)}\rangle \mid x \in X\}$$

Definition 4 [2],[3] Let X be a nonempty set, for every IFS M,N in X. $\Box$ and $\Diamond$ modal operators are defined as;

$$\Box M = \left\{\langle x, \mu_M(x)\rangle \mid x \in X\right\} = \left\{\langle x, \mu_M(x), 1-\mu_M(x)\rangle \mid x \in X\right\}$$
$$\Diamond M = \left\{\langle x, 1-\nu_M(x)\rangle \mid x \in X\right\} = \left\{\langle x, 1-\nu_M(x), \nu_M(x)\rangle \mid x \in X\right\} [1]$$
$$(\Box M)^c = \left\{\langle x, 1-\mu_M(x), \mu_M(x)\rangle \mid x \in X\right\}$$
$$(\Diamond M)^c = \left\{\langle x, \nu_M(x), 1-\nu_M(x)\rangle \mid x \in X\right\}$$

Theorem 1 [2],[3] Let X be a nonempty set, for every IFS M,N in X. The following equalities are ensured.

(a) $\Box\Box M = \Box M$

(b) $\Box\Diamond M = \Diamond M$

(c) $\Diamond\Box M = \Box M$

(d) $\Diamond\Diamond M = \Diamond M$

Theorem 2 [2],[3] Let X be a nonempty set, for every IFS M,N in X. The following equality could be provided by means of definition of @ operation.

$$\Box M @ \Box N = \Box(M@N)$$

Theorem 3 [2],[3] Let X be a nonempty set, for every IFS M,N in X ;

$$\Diamond M @ \Diamond N = \Diamond(M@N)$$

3. MAIN RESULTS

In this section, new equalities were obtained and proved by means of some intuitionistic fuzzy operations ($\to$, @, $\cup, \cap$, \$, #, $*$) and modal operators ($\Box$, $\Diamond$).

Theorem 4 [4] Let X be a nonempty set, for every IFS M,N in X;

$$(\Box M \oplus \Box N) @ (\Box M \otimes \Box N) = \Box M @ \Box N$$

Theorem 5 [4] Let X be a nonempty set, for every IFS M,N in X;

$$(\Diamond M \oplus \Diamond N) @ (\Diamond M \otimes \Diamond N) = \Diamond M @ \Diamond N$$

Theorem 6 [4] Let X be a nonempty set, for every IFS M,N in X;

$$[(\Box M @ \Box N)\$(\Box M \# \Box N)] = \Box M \$ \Box N$$

Theorem 7 [4] Let X be a nonempty set, for every IFS M,N in X;

$$[(\Diamond M @ \Diamond N) \$ (\Diamond M \# \Diamond N)] = \Diamond M \$ \Diamond N$$

Theorem 8 [4] Let X be a nonempty set, for every IFS M,N in X;

$$\Box[(\Diamond M @ \Diamond N)^c] = [\Diamond(M @ N)]^c$$

Theorem 9 [4] Let X be a nonempty set, for every IFS M,N in X;

$$[(\Box M \oplus \Diamond N)^c @ ((\Box M)^c \otimes \Diamond N)] \cup (\Box M)^c = (\Box M)^c$$

Theorem 10 Let X be a nonempty set, for every IFS M,N in X. The following equality is holds for;

$$[(\Diamond M \oplus \Box N) @ ((\Diamond M)^c \otimes \Box N)] \cup (\Diamond M) = (\Diamond M)$$

Proof.

$$\begin{aligned}
(\Diamond M \oplus \Box N) &= \{\langle 1 - \nu_M(x) + \mu_N(x) - ((1 - \nu_M(x)\mu_N(x), \nu_M(x)(1 - \mu_N(x))\rangle\} \\
&= \{\langle 1 - \nu_M(x) + \mu_N(x) - \mu_N(x) + \nu_M(x)\mu_N(x), \nu_M(x) - \nu_M(x)\mu_N(x))\rangle\} \\
&= \left\{\langle 1 - \nu_M(x) + \nu_M(x)\mu_N(x), \nu_M(x) - \nu_M(x)\mu_N(x))\rangle | x \in X\right\} \\
((\Diamond M)^c \otimes \Box N) &= \{\langle \nu_M(x)\mu_N(x), 1 - \mu_N(x) + 1 - \nu_M(x) - ((1 - \nu_M(x))(1 - \mu_N(x)))\rangle\} \\
&= \{\langle \nu_M(x)\mu_N(x), 1 - \mu_N(x) + 1 - \nu_M(x) - 1 + \nu_M(x) + \mu_N(x) - \nu_M(x)\mu_N(x))\rangle\} \\
&= \{\langle \nu_M(x)\mu_N(x), 1 - \nu_M(x)\mu_N(x))\rangle\}
\end{aligned}$$

$$\begin{aligned}
[(\Diamond M \oplus \Box N) @ ((\Diamond M)^c \otimes \Box N)] &= \left\{\langle \tfrac{1-\nu_M(x)+2\nu_M(x)\mu_N(x)}{2}, \tfrac{1+\nu_M(x)-2\nu_M(x)\mu_N(x)}{2} \rangle\right\} \\
[(\Diamond M \oplus \Box N) @ ((\Diamond M)^c \otimes \Box N)] \cup (\Diamond M) &= \{\langle \max(\tfrac{1-\nu_M(x)+2\nu_M(x)\mu_N(x)}{2}, 1 - \nu_M(x)), \\
&\quad \min(\tfrac{1+\nu_M(x)-2\nu_M(x)\mu_N(x)}{2}, \nu_M(x))\rangle | \} \\
&= \{\langle 1 - \nu_M(x), \nu_M(x)\rangle\} \\
&= \Diamond M \\
[(\Diamond M \oplus \Box N) @ ((\Diamond M)^c \otimes \Box N)] \cup (\Diamond M) &= (\Diamond M)
\end{aligned}$$

Theorem 11 Let X be a nonempty set, for every IFS M, N in X. We get the following equality;

$$[(\Diamond M \oplus \Box N)^c @ ((\Diamond M)^c \otimes \Box N)] \cup (\Diamond M)^c = (\Diamond M)^c$$

Proof.

$$\begin{aligned}
(\Diamond M \oplus \Box N) &= \{\langle 1 - \nu_M(x) + \mu_N(x) - ((1 - \nu_M(x)\mu_N(x), \nu_M(x)(1 - \mu_N(x))\rangle\} \\
&= \{\langle 1 - \nu_M(x) + \mu_N(x) - \mu_N(x) + \nu_M(x)\mu_N(x), \nu_M(x) - \nu_M(x)\mu_N(x))\rangle\} \\
&= \{\langle 1 - \nu_M(x) + \nu_M(x)\mu_N(x), \nu_M(x) - \nu_M(x)\mu_N(x))\rangle\} \\
(\Diamond M \oplus \Box N)^c &= \{\langle \nu_M(x) - \nu_M(x)\mu_N(x), 1 - \nu_M(x) + \nu_M(x)\mu_N(x)\rangle\}
\end{aligned}$$

$$\begin{aligned}
((\Diamond M)^c \otimes \Box N) &= \{\langle \nu_M(x)\mu_N(x), \\
&\quad 1 - \mu_N(x) + 1 - \nu_M(x) - ((1 - \nu_M(x))(1 - \mu_N(x)))\rangle\} \\
&= \{\langle \nu_M(x)\mu_N(x), 1 - \mu_N(x) + 1 - \nu_M(x) - 1 + \nu_M(x) + \mu_N(x) - \nu_M(x)\mu_N(x))\rangle\} \\
&= \{\langle \nu_M(x)\mu_N(x), 1 - \nu_M(x)\mu_N(x)\rangle\}
\end{aligned}$$

$$\begin{aligned}
[(\Diamond M \oplus \Box N)^c @ ((\Diamond M)^c \otimes \Box N)] &= \left\{\langle \frac{\nu_M(x)}{2}, 1 - \frac{\nu_M(x)}{2} \rangle\right\} \\
[(\Diamond M \oplus \Box N)^c @ ((\Diamond M)^c \otimes \Box N)] \cup (\Diamond M)^c &= \{\langle \max(\nu_M(x), \tfrac{\nu_M(x)}{2}), \min(1 - \nu_M(x), 1 - \tfrac{\nu_M(x)}{2})\rangle\} \\
&= \{\langle \nu_M(x), 1 - \nu_M(x)\rangle\} \\
&= (\Diamond M)^c \\
[(\Diamond M \oplus \Box N)^c @ ((\Diamond M)^c \otimes \Box N)] \cup (\Diamond M)^c &= (\Diamond M)^c
\end{aligned}$$

Theorem 12 Let X be a nonempty set, for every IFS M, N in X;

$$[(\Box M \otimes \Diamond N)^c @ ((\Box M)^c \oplus \Diamond N)] \cap (\Box M)^c = (\Box M)^c$$

The above equality is obtained.

Proof.

$$\begin{aligned}
(\Box M \otimes \Diamond N) &= \{\langle \mu_M(x)(1 - \nu_N(x)), 1 - \mu_M(x) + \nu_N(x) - (1 - \mu_M(x))\nu_N(x)\rangle\} \\
&= \{\langle \mu_M(x) - \mu_M(x)\nu_N(x), 1 - \mu_M(x) + \nu_N(x) - \nu_N(x) + \mu_M(x)\nu_N(x)\rangle\} \\
(\Box M \otimes \Diamond N)^c &= \{\langle 1 - \mu_M(x) + \mu_M(x)\nu_N(x), \mu_M(x) - \mu_M(x)\nu_N(x)\rangle\} \\
((\Box M)^c \oplus \Diamond N) &= \{\langle 1 - \mu_M(x) + 1 - \nu_N(x) - (1 - \mu_M(x))(1 - \nu_N(x)), \mu_M(x)\nu_N(x)\rangle\} \\
&= \{\langle 1 - \mu_M(x) + 1 - \nu_N(x) - 1 + \mu_M(x) + \nu_N(x) - \mu_M(x)\nu_N(x), \mu_M(x)\nu_N(x)\rangle\} \\
&= \{\langle 1 - \mu_M(x)\nu_N(x), \mu_M(x)\nu_N(x)\rangle\}
\end{aligned}$$

$$\begin{aligned}
[(\Box M \otimes \Diamond N)^c @ ((\Box M)^c \oplus \Diamond N)] &= \left\{\langle \tfrac{2-\mu_M(x)}{2}, \mu_M(x)\rangle\right\} \\
[(\Box M \otimes \Diamond N)^c @ ((\Box M)^c \oplus \Diamond N)] \cap (\Box M)^c &= \left\{\langle \min(\tfrac{2-\mu_M(x)}{2}, 1 - \mu_M(x)), \max(\mu_M(x), \mu_M(x))\rangle\right\} \\
&= \{\langle 1 - \mu_M(x), \mu_M(x)\rangle\} \\
&= (\Box M)^c \\
[(\Box M \otimes \Diamond N)^c @ ((\Box M)^c \oplus \Diamond N)] \cap (\Box M)^c &= (\Box M)^c
\end{aligned}$$

Theorem 13 Let X be a nonempty set, for every IFS M in X. We get;

$$(\Box M \oplus \Diamond M) @ (\Box M \otimes \Diamond M) = \Box M @ \Diamond M$$

Proof.

$$\begin{aligned}
(\Box M \oplus \Diamond M) &= \{\langle \mu_M(x) + 1 - \nu_M(x) - (\mu_M(x)(1 - \nu_M(x))), (1 - \mu_M(x))\nu_M(x)\rangle\} \\
&= \{\langle \mu_M(x) + 1 - \nu_M(x) - \mu_M(x) + \mu_M(x)\nu_M(x), \nu_M(x) - \mu_M(x)\nu_M(x))\rangle\} \\
&= \{\langle 1 - \nu_M(x) + \mu_M(x)\nu_M(x), \nu_M(x) - \mu_M(x)\nu_M(x))\rangle\}
\end{aligned}$$

$$\begin{aligned}
(\Box M \otimes \Diamond M) &= \{\langle \mu_M(x)(1 - \nu_M(x)), 1 - \mu_M(x) + \nu_M(x) - (\nu_M(x)(1 - \mu_M(x)))\rangle\} \\
&= \{\langle \mu_M(x) - \mu_M(x)\nu_M(x), 1 - \mu_M(x) + \nu_M(x) - \nu_M(x) + \mu_M(x)\nu_M(x))\rangle\} \\
&= \{\langle \mu_M(x) - \mu_M(x)\nu_M(x), 1 - \mu_M(x) + \mu_M(x)\nu_M(x)\rangle\}
\end{aligned}$$

$$\begin{aligned}
(\Box M \oplus \Diamond M) @ (\Box M \otimes \Diamond M) &= \{\langle \frac{1 - \nu_M(x) + \mu_M(x)}{2}, \frac{1 + \nu_M(x) - \mu_M(x)}{2}\rangle\} \\
&= \Box M @ \Diamond M \\
(\Box M \oplus \Diamond M) @ (\Box M \otimes \Diamond M) &= \Box M @ \Diamond M
\end{aligned}$$

Theorem 14 Let X be a nonempty set, for every IFS M, N in X. We obtain the following equality;

$$(\Box M \oplus \Diamond N) @ (\Box M \otimes \Diamond N) = \Box M @ \Diamond N$$

Proof.
The proof is similar to proof of the Theorem 13.

$$
\begin{aligned}
(\Box M \oplus \Diamond N) &= \{\langle \mu_M(x)+1-\nu_N(x)-(\mu_M(x)(1-\nu_N(x))), (1-\mu_M(x))\nu_N(x)\rangle\} \\
&= \{\langle \mu_M(x)+1-\nu_N(x)-\mu_M(x)+\mu_M(x)\nu_N(x), \nu_N(x)-\mu_M(x)\nu_N(x)\rangle\} \\
&= \{\langle 1-\nu_N(x)+\mu_M(x)\nu_N(x), \nu_N(x)-\mu_M(x)\nu_N(x)\rangle\} \\
(\Box M \otimes \Diamond N) &= \{\langle \mu_M(x)(1-\nu_N(x)), 1-\mu_M(x)+\nu_N(x)-(\nu_N(x)(1-\mu_M(x)))\rangle\} \\
&= \{\langle \mu_M(x)-\mu_M(x)\nu_N(x), 1-\mu_M(x)+\nu_N(x)-\nu_N(x)+\mu_M(x)\nu_N(x)\rangle\} \\
&= \{\langle \mu_M(x)-\mu_M(x)\nu_N(x), 1-\mu_M(x)+\mu_M(x)\nu_N(x)\rangle\}
\end{aligned}
$$

$$
\begin{aligned}
(\Box M \oplus \Diamond N) @ (\Box M \otimes \Diamond N) &= \{\langle \frac{1-\nu_N(x)+\mu_M(x)}{2}, \frac{1+\nu_N(x)-\mu_M(x)}{2}\rangle\} \\
&= \Box M @ \Diamond N \\
(\Box M \oplus \Diamond N) @ (\Box M \otimes \Diamond N) &= \Box M @ \Diamond N
\end{aligned}
$$

Theorem 15 Let X be a nonempty set, for every IFS M in X;

$$(\Box M \cup \Diamond M) @ (\Box M \cap \Diamond M) = \Box M @ \Diamond M$$

The above equality is holds for.
Proof.

$$
\begin{aligned}
(\Box M \cup \Diamond M) &= \{\langle \max(\mu_M(x), 1-\nu_M(x)), \min(1-\mu_M(x), \nu_M(x))\rangle\} \\
(\Box M \cap \Diamond M) &= \{\langle \min(\mu_M(x), 1-\nu_M(x)), \max(1-\mu_M(x), \nu_M(x))\rangle\}
\end{aligned}
$$

$$
\begin{aligned}
(\Box M \cup \Diamond M) @ (\Box M \cap \Diamond M) &= \{\langle \frac{\max(\mu_M(x),1-\nu_M(x)) + \min(\mu_M(x),1-\nu_M(x))}{2}, \\
&\quad \frac{\min(1-\mu_M(x),\nu_M(x)) + \max(1-\mu_M(x),\nu_M(x))}{2}\rangle\} \\
&= \{\langle \frac{\mu_M(x)+1-\nu_M(x)}{2}, \frac{1-\mu_M(x)+\nu_M(x)}{2}\rangle\} \\
&= \Box M @ \Diamond M \\
(\Box M \cup \Diamond M) @ (\Box M \cap \Diamond M) &= \Box M @ \Diamond M
\end{aligned}
$$

Theorem 16 Let X be a nonempty set, for every IFS M, N in X ;

$$(\Box M \cup \Diamond N) @ (\Box M \cap \Diamond N) = \Box M @ \Diamond N$$

It is easily seen similar to Theorem 15.

Proof.

$$
\begin{aligned}
(\Box M \cup \Diamond N) &= \{\langle \max(\mu_M(x), 1-\nu_N(x)), \min(1-\mu_M(x), \nu_N(x))\rangle\} \\
(\Box M \cap \Diamond N) &= \{\langle \min(\mu_M(x), 1-\nu_N(x)), \max(1-\mu_M(x), \nu_N(x))\rangle\}
\end{aligned}
$$

$$
\begin{aligned}
(\Box M \cup \Diamond N) @ (\Box M \cap \Diamond N) &= \{\langle \frac{\max(\mu_M(x),1-\nu_N(x)) + \min(\mu_M(x),1-\nu_N(x))}{2} \\
&\quad , \frac{\min(1-\mu_M(x),\nu_N(x)) + \max(1-\mu_M(x),\nu_N(x))}{2}\rangle\} \\
&= \{\langle \frac{\mu_M(x)+1-\nu_N(x)}{2}, \frac{1-\mu_M(x)+\nu_N(x)}{2}\rangle\} \\
&= \Box M @ \Diamond N \\
(\Box M \cup \Diamond N) @ (\Box M \cap \Diamond N) &= \Box M @ \Diamond N
\end{aligned}
$$

Theorem 17 Let X be a nonempty set, for every IFS M in X. We get the following equality;

$$(\Box M @ \Diamond M)\ \$\ (\Box M \# \Diamond M) = \Box M\ \$\ \Diamond M$$

Proof.

$$
\begin{aligned}
(\Box M @ \Diamond M) &= \{\langle \frac{\mu_M(x)+1-\nu_M(x)}{2}, \frac{1-\mu_M(x)+\nu_M(x)}{2}\rangle\} \\
(\Box M \# \Diamond M) &= \{\langle \frac{2\mu_M(x)(1-\nu_M(x))}{\mu_M(x)+1-\nu_M(x)}, \frac{2(1-\mu_M(x))\nu_M(x)}{1-\mu_M(x)+\nu_M(x)}\rangle\}
\end{aligned}
$$

$$
\begin{aligned}
(\Box M @ \Diamond M)\ \$\ (\Box M \# \Diamond M) &= \{\langle \sqrt{\frac{\mu_M(x)+1-\nu_M(x)}{2} \frac{2\mu_M(x)(1-\nu_M(x))}{\mu_M(x)+1-\nu_M(x)}}, \\
&\quad \sqrt{\frac{1-\mu_M(x)+\nu_M(x)}{2} \frac{2(1-\mu_M(x))\nu_M(x)}{1-\mu_M(x)+\nu_M(x)}}\rangle\} \\
&= \{\langle \sqrt{\mu_M(x)(1-\nu_M(x))}, \sqrt{(1-\mu_M(x))\nu_M(x)}\rangle\} \\
&= \Box M\ \$\ \Diamond M \\
(\Box M @ \Diamond M)\ \$\ (\Box M \# \Diamond M) &= \Box M\ \$\ \Diamond M
\end{aligned}
$$

Theorem 18 Let X be a nonempty set, for every IFS M, N in X;

$$(\Box M @ \Diamond N)\ \$\ (\Box M \# \Diamond N) = \Box M\ \$\ \Diamond N$$

It is easily seen similar to Theorem 17.
Proof.

$$
\begin{aligned}
(\Box M @ \Diamond N) &= \{\langle \frac{\mu_M(x)+1-\nu_N(x)}{2}, \frac{1-\mu_M(x)+\nu_N(x)}{2}\rangle\} \\
(\Box M \# \Diamond N) &= \{\langle \frac{2\mu_M(x)(1-\nu_N(x))}{\mu_M(x)+1-\nu_N(x)}, \frac{2(1-\mu_M(x))\nu_N(x)}{1-\mu_M(x)+\nu_N(x)}\rangle\}
\end{aligned}
$$

$$(\Box M \,@\, \Diamond N)\ \$\ (\Box M \,\#\, \Diamond N) = \{\langle \sqrt{\frac{\mu_M(x)+1-\nu_N(x)}{2}\frac{2\mu_M(x)(1-\nu_N(x))}{\mu_M(x)+1-\nu_N(x)}}, \sqrt{\frac{1-\mu_M(x)+\nu_N(x)}{2}\frac{2(1-\mu_M(x))\nu_N(x)}{1-\mu_M(x)+\nu_N(x)}}\rangle\}$$

$$= \{\langle \sqrt{\mu_M(x)(1-\nu_N(x))}, \sqrt{(1-\mu_M(x))\nu_N(x)}\rangle\}$$

$$= \Box M\ \$\ \Diamond N$$

$$(\Box M \,@\, \Diamond N)\ \$\ (\Box M \,\#\, \Diamond N) = \Box M\ \$\ \Diamond N$$

Theorem 19 Let X be a nonempty set, for every IFS M, N in X. The equality is obtained.

$$(\Box M \cup \Box N) * (\Box M \cap \Box N) = \Box M * \Box N$$

Proof.

$$(\Box M \cup \Box N) = \{\langle \max(\mu_M(x), \mu_N(x)), \min(1-\mu_M(x), 1-\mu_N(x))\rangle\}$$

$$(\Box M \cap \Box N) = \{\langle \min(\mu_M(x), \mu_N(x)), \max(1-\mu_M(x), 1-\mu_N(x))\rangle\}$$

$$(\Box M \cup \Box N) * (\Box M \cap \Box N) = \{\langle \frac{\max(\mu_M(x), \mu_N(x)) + \min(\mu_M(x), \mu_N(x))}{2(\mu_M(x)\mu_N(x)+1)}, \frac{\min(1-\mu_M(x), 1-\mu_N(x)) + \max(1-\mu_M(x), 1-\mu_N(x))}{2((1-\mu_M(x))(1-\mu_N(x))+1)}\rangle\}$$

$$= \{\langle \frac{\mu_M(x)+\mu_N(x)}{2(\mu_M(x)\mu_N(x)+1)}, \frac{1-\mu_M(x)+1-\mu_N(x)}{2((1-\mu_M(x))(1-\mu_N(x))+1)}\rangle\}$$

$$= \Box M * \Box N$$

$$(\Box M \cup \Box N) * (\Box M \cap \Box N) = \Box M * \Box N$$

Theorem 20 Let X be a nonempty set, for every IFS M, N in X. We get the following equalities;

$$(\Box M \cup \Box N) * (\Box M \cap \Box N) = \Box M * \Box N$$

$$(\Box M * \Box N) \,@\, (\Box M * \Box N) = \Box M * \Box N$$

$$(\Box M * \Box N)\ \$\ (\Box M * \Box N) = \Box M * \Box N$$

Conclusion 1

This conclusion is easy to see from Theorem 20.

$$(\Box M \cup \Box N) * (\Box M \cap \Box N) = (\Box M * \Box N) \,@\, (\Box M * \Box N) = (\Box M * \Box N)\ \$\ (\Box M * \Box N) = \Box M * \Box N$$

Theorem 21 Let X be a nonempty set, for every IFS M, N in X. We obtain;

$$(\Diamond M \cup \Diamond N) * (\Diamond M \cap \Diamond N) = \Diamond M * \Diamond N$$

Proof.

$$(\Diamond M \cup \Diamond N) = \{\langle \max(1-\nu_M(x)), (1-\nu_N(x)), \min(\nu_M(x), \nu_N(x))\rangle\}$$

$$(\Diamond M \cap \Diamond N) = \{\langle \min(1-\nu_M(x)), (1-\nu_N(x)), \max(\nu_M(x), \nu_N(x))\rangle\}$$

$$(\Diamond M \cup \Diamond N) * (\Diamond M \cap \Diamond N) = \{\langle \frac{\max(1-\nu_M(x), 1-\nu_N(x)) + \min(1-\nu_M(x), 1-\nu_N(x))}{2((1-\nu_M(x))(1-\nu_N(x))+1)}, \frac{\min(\nu_M(x), \nu_N(x))}{2(\nu_M(x)\nu_N(x)+1)}\rangle\}$$

$$= \{\langle \frac{1-\nu_M(x)+1-\nu_N(x)}{2((1-\nu_M(x))(1-\nu_N(x))+1)}, \frac{\nu_M(x)+\nu_N(x)}{2(\nu_M(x)\nu_N(x)+1)}\rangle\}$$

$$= \Diamond M * \Diamond N$$

$$(\Diamond M \cup \Diamond N) * (\Diamond M \cap \Diamond N) = \Diamond M * \Diamond N$$

Theorem 22 Let X be a nonempty set, for every IFS M, N in X. The following equalities are hold for.

$$(\Diamond M \cup \Diamond N) * (\Diamond M \cap \Diamond N) = \Diamond M * \Diamond N$$

$$(\Diamond M * \Diamond N)\ \$\ (\Diamond M * \Diamond N) = \Diamond M * \Diamond N$$

$$(\Diamond M * \Diamond N) \,@\, (\Diamond M * \Diamond N) = \Diamond M * \Diamond N$$

Conclusion 2

This conclusion is easy to see from Theorem 22.

$$(\Diamond M \cup \Diamond N) * (\Diamond M \cap \Diamond N) = (\Diamond M * \Diamond N)\ \$\ (\Diamond M * \Diamond N) = (\Diamond M * \Diamond N) \,@\, (\Diamond M * \Diamond N) = \Diamond M * \Diamond N$$

Theorem 23 Let X be a nonempty set, for every IFS M, N in X. We get;

$$[(\Diamond M\,\$\,\Diamond N) \cup (\Diamond M\,\$\,\Diamond N)^c] \,@\, [(\Diamond M\,\$\,\Diamond N) \cap (\Diamond M\,\$\,\Diamond N)^c] = (\Diamond M\,\$\,\Diamond N) \,@\, (\Diamond M\,\$\,\Diamond N)^c$$

Proof.

$$(\Diamond M\,\$\,\Diamond N) = \{\langle \sqrt{(1-\nu_M(x))(1-\nu_N(x))}, \sqrt{\nu_M(x)\nu_N(x)}\rangle\}$$

$$(\Diamond M\,\$\,\Diamond N)^c = \{\langle \sqrt{\nu_M(x)\nu_N(x)}, \sqrt{(1-\nu_M(x))(1-\nu_N(x))}\rangle\}$$

$$[(\Diamond M\,\$\,\Diamond N) \cup (\Diamond M\,\$\,\Diamond N)^c] = \{\langle \max(\sqrt{(1-\nu_M(x))(1-\nu_N(x))}, \sqrt{\nu_M(x)\nu_N(x)}), \min(\sqrt{(1-\nu_M(x))(1-\nu_N(x))}, \sqrt{\nu_M(x)\nu_N(x)})\rangle\}$$

$$[(\Diamond M\,\$\,\Diamond N) \cap (\Diamond M\,\$\,\Diamond N)^c] = \{\langle \min(\sqrt{(1-\nu_M(x))(1-\nu_N(x))}, \sqrt{\nu_M(x)\nu_N(x)}), \max(\sqrt{(1-\nu_M(x))(1-\nu_N(x))}, \sqrt{\nu_M(x)\nu_N(x)})\rangle\}$$

$$[(\Diamond M \$ \Diamond N) \cup (\Diamond M \$ \Diamond N)^c] @ [(\Diamond M \$ \Diamond N) \cap (\Diamond M \$ \Diamond N)^c]$$

$$= \{\langle \frac{\max\left(\sqrt{(1-\nu_M(x))(1-\nu_N(x))}, \sqrt{\nu_M(x)\nu_N(x)}\right) + \min\left(\sqrt{(1-\nu_M(x))(1-\nu_N(x))}, \sqrt{\nu_M(x)\nu_N(x)}\right)}{2},$$

$$\frac{\max\left(\sqrt{(1-\nu_M(x))(1-\nu_N(x))}, \sqrt{\nu_M(x)\nu_N(x)}\right) + \min\left(\sqrt{(1-\nu_M(x))(1-\nu_N(x))}, \sqrt{\nu_M(x)\nu_N(x)}\right)}{2} \rangle\}$$

$$= \{\langle \frac{\sqrt{(1-\nu_M(x))(1-\nu_N(x))} + \sqrt{\nu_M(x)\nu_N(x)}}{2}, \frac{\sqrt{(1-\nu_M(x))(1-\nu_N(x))} + \sqrt{\nu_M(x)\nu_N(x)}}{2} \rangle\}$$

Theorem 24 Let X be a nonempty set, for every IFS M, N in X. The equality is obtained;

$$[(\Box M \$ \Box N) \cup (\Box M \$ \Box N)^c] @ [(\Box M \$ \Box N) \cap (\Box M \$ \Box N)^c] = (\Box M \$ \Box N) @ (\Box M \$ \Box N)^c$$

Proof. The proof is similar to proof of the Theorem 23.

Theorem 25 Let X be a nonempty set, for every IFS M, N in X. We get the following equality;

$$\left[(\Box M \cup \Box N) \# (\Box M \cap \Box N)\right] \$ \left[(\Box M \cup \Box N) @ (\Box M \cap \Box N)\right] = \left(\Box M \$ \Box N\right)$$

Proof.

$$(\Box M \cup \Box N) = \{\langle \max(\mu_M(x), \mu_N(x)), \min(1-\mu_M(x)), 1-\mu_N(x)) \rangle\}$$

$$(\Box M \cap \Box N) = \{\langle \min(\mu_M(x), \mu_N(x)), \max(1-\mu_M(x)), 1-\mu_N(x)) \rangle\}$$

$$\left[(\Box M \cup \Box N) \# (\Box M \cap \Box N)\right] = \{\langle \frac{2\max(\mu_M(x), \mu_N(x)) \min(\mu_M(x), \mu_N(x))}{\max(\mu_M(x), \mu_N(x)) + \min(\mu_M(x), \mu_N(x))}, \frac{2\max(1-\mu_M(x)) \min(1-\mu_N(x))}{\max(1-\mu_M(x)) + \min(1-\mu_N(x))} \rangle\}$$

$$= \{\langle \frac{2\mu_M(x)\mu_N(x)}{\mu_M(x)+\mu_N(x))}, \frac{2(1-\mu_M(x))(1-\mu_N(x))}{(1-\mu_M(x)) + (1-\mu_N(x))} \rangle\}$$

$$\left[(\Box M \cup \Box N) @ (\Box M \cap \Box N)\right] = \{\langle \frac{\max(\mu_M(x), \mu_N(x)) + \min(\mu_M(x), \mu_N(x))}{2}, \frac{\max(1-\mu_M(x)) + \min(1-\mu_N(x))}{2} \rangle\}$$

$$= \{\langle \frac{\mu_M(x)+\mu_N(x)}{2}, \frac{(1-\mu_M(x)) + (1-\mu_N(x))}{2} \rangle\}$$

$$\left[(\Box M \cup \Box N) \# (\Box M \cap \Box N)\right] \$ \left[(\Box M \cup \Box N) @ (\Box M \cap \Box N)\right] = \{\langle \sqrt{\mu_M(x)\mu_N(x)}, \sqrt{(1-\mu_M(x))(1-\mu_N(x))} \rangle\}$$

$$= \left(\Box M \$ \Box N\right)$$

$$\left[(\Box M \cup \Box N) \# (\Box M \cap \Box N)\right] \$ \left[(\Box M \cup \Box N) @ (\Box M \cap \Box N)\right] = \left(\Box M \$ \Box N\right)$$

Theorem 26 Let X be a nonempty set, for every IFS M, N in X. The following equality is holds for;

$$\left[(\Diamond M \cup \Diamond N) \# (\Diamond M \cap \Diamond N)\right] \$ \left[(\Diamond M \cup \Diamond N) @ (\Diamond M \cap \Diamond N)\right] = \left(\Diamond M \$ \Diamond N\right)$$

Proof. The proof is similar to proof of the Theorem 25.

Theorem 27 Let X be a nonempty set, for every IFS M, N in X.

$$\left[\left((\Box M \oplus \Box N)^c \to \left(\Box M \$ \Box N\right)\right) \to \left((\Box M \otimes \Box N) \to \left(\Box M \$ \Box N\right)^c\right)^c\right] \cap \left[\Box M \$ \Box N\right] = \Box M \$ \Box N$$

The above equality is obtained.

Proof.

$$(\Box M \oplus \Box N)^c \to \left(\Box M \$ \Box N\right) = \{\langle \mu_M(x) + \mu_N(x) - \mu_M(x)\mu_N(x), \sqrt{1 - \mu_M(x) - \mu_N(x) + \mu_M(x)\mu_N(x)} \rangle\}$$

$$\left((\Box M \otimes \Box N) \to \left(\Box M \$ \Box N\right)^c\right)^c = \{\langle \sqrt{\mu_M(x)\mu_N(x)}, 1 - \mu_M(x)\mu_N(x) \rangle\}$$

Theorem 28 Let X be a nonempty set, for every IFS M, N in X. We obtain the following equality;

$$\left[\left((\Diamond M \oplus \Diamond N)^c \to \left(\Diamond M \$ \Diamond N\right)\right) \to \left((\Diamond M \otimes \Diamond N) \to \left(\Diamond M \$ \Diamond N\right)^c\right)^c\right] \cap \left[\Diamond M \$ \Diamond N\right] = \Diamond M \$ \Diamond N$$

Proof. The proof is similar to proof of the Theorem 27.

4. CONCLUSION

In this paper, properties of modal operators defined on intuitionistic fuzzy sets have been investigated. Then, some intuitionistic fuzzy operations have been researched with modal operators. New equalities have been obtained. These equalities make it easy application areas of operators. Shorter equalities have been obtained using features of some intuitionistic fuzzy operations with modal operators. These equalities are more useful because they are shorter and more practical. These equalities could be made use of in many application areas of operators and these offer more practical solutions as they provide simplicity. Robotic, economy, control systems, computer and algebraic structures are some of these application areas.

REFERENCES

[1] L.A.Zadeh , "Fuzzy Sets*", *Information and Control*,vol.8, pp. 338-353, 1965.

[2] K.T.Atanassov, " Intuitionistic fuzzy sets", *Fuzzy Sets and Systems* , vol.20, no.1, pp. 87-96, 1986.

[3] K.T.Atanassov, " Intuitionistic Fuzzy Sets,Theory and Applications" , *Physica-Verlag Heidelberg*, Germany, pp. 319, 1999.

[4] E. Eker, F. Tuğrul and M. Çitil, " New equalities on the intuitionistic fuzzy operators and operations" , *Notes on Intuitionistic Fuzzy Sets*, vol.21, no.4, pp. 124-128, 2015.

[5] K.Dencheva, " Extension of intuitionistic fuzzy modal operators" *Proc.of the Second Int. IEEE Symp. Intelligent Systems*, vol.3, pp. 22-24, 2004.

[6] K.T. Atanassov , " The most general form of one type of intuitionistic fuzzy modal operators" ,*Notes on Intuitionistic Fuzzy Sets*, vol.12, no.2, pp. 36-38, 2006.

[7] K.T. Atanassov, " Some Properties of the operators from one type of intuitionistic fuzzy modal operators" , *Advanced Studies on Contemporary Mathematics*, vol.15, no.1, pp.13-20, 2007.

[8] G. Çuvalcıoğlu, " Some Properties of $E_{\alpha,\beta}$ operator" , *Advanced Studies on Contemporary Mathematics*, vol. 14, no.2, pp. 305-310, 2007.

[9] G. Çuvalcıoğlu, " Expand the modal operator diagram with $Z^{\omega}_{\alpha,\beta}$" , *Jangjeon Math. Soc.*, vol.13, no.3, pp. 403-412, 2010.

[10] G. Çuvalcıoğlu, " On the Diagram of One Type Modal Operators on Intuitionistic Fuzzy Sets: Last Expanding with $Z^{\omega,\theta}_{\alpha,\beta}$" ,*Iranian J. of Fuzzy Systems*, vol.10, no.1, pp. 89-106, 2013.

[11] G. Çuvalcıoğlu and, S.Yılmaz, " Some properties of OTMOs on IFSs" , *Advanced Studies in Contemporary Mathematics*, vol.20, no.4, pp. 621-628, 2010.

[12] B. Doycheva , " Inequalities with intuitionistic fuzzy topological and Gökhan Çuvalcıoglu's operators" , *Notes on Intuitionistic Fuzzy Sets*, vol.14, no.1, pp. 20-22, 2008.