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Global Behavior of Solutions of a Two-dimensional System of Difference Equations

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Abstract — In this paper, we mainly investigate the qualitative and quantitative behavior of the solutions of a discrete system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{ax_{n-1} + by_{n-1}}, \quad n = 0, 1, \ldots,$$

where a, b and the initial values x_{-1}, x_0, y_{-1}, y_0 are non-zero real numbers. For $a \in \mathbb{R}_+ - \{1\}$, we show any admissible solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is either entirely located in a certain quadrant of the plane or there exists a natural number $N > 0$ (we calculate its value) such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is located. Besides, some numerical simulations with graphs are given in the article to emphasize the efficiency of our theoretical results.

Subject Classification (2020): 39A06, 39A20.

1. Introduction

In case of interruption of events developing over time, mathematical models are established with difference equations using discrete variables. In this way, difference equations have an important place in research on real-life problems, especially in fields such as economics, medicine, chemistry and biology. In addition to its importance in practice, difference equations are also used in theoretical research, that is, to obtain solutions of differential equations, delayed differential equations, and fractional differential equations. It is very difficult most of the time to obtain solutions to rational difference equations. Additionally, there is no general technique to obtain or qualitatively investigate solutions. For this reason, the study of non-linear difference equations of order greater than one is truly remarkable and every qualified study in this field is valuable.

Difference equations have a very old history. However, its research has progressed rapidly, especially in the last thirty years. Research in this field can be carried out under three headings: quantitative, qualitative and numerical. Quantitative research is carried out by determining the analytical solutions of the equation, qualitative research is carried out by examining the behavior of the solutions of the equation, and numerical research is carried out by determining the approximate values of the solution of the equation by various

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methods.

Therefore, this paper can be viewed as both a qualitative and quantitative investigation of a system of difference equations. Now let's give a detailed background of the system we discuss in this article:

In [29], the authors studied the global dynamics of the system

$$x_{n+1} = \frac{\beta_1 x_n}{A_1 + y_n},\ y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n + y_n}, \quad n = 0, 1, \ldots,$$

where the parameters $\gamma_2, A_1, \beta_1, \beta_2$ are positive numbers and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers such that $x_0 + y_0 > 0$.

Camouzis et al. [10], studied the global behavior of the system of difference equations

$$x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{x_n},\ y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{B_2 x_n + C_2 y_n}, \quad n = 0, 1, \ldots, \tag{1.1}$$

with nonnegative parameters and positive initial conditions. They studied the boundedness character of the system (1.1) in its special cases.

In [9], Camouzis et al. conjectured that:

Every positive solution of the system

$$x_{n+1} = \frac{y_n}{x_n},\ y_{n+1} = \frac{\gamma_2 y_n}{A_2 + B_2 x_n + y_n}, \quad n = 0, 1, \ldots,$$

with nonnegative parameters and positive initial conditions, converges to a finite limit.

Bekker et al. [8] confirmed that conjecture.

In [28], Kudlak et al. studied the existence of unbounded solutions of the system of difference equations

$$x_{n+1} = \frac{x_n}{y_n},\ y_{n+1} = x_n + \gamma_n y_n,\ n = 0, 1, \ldots,$$

where $0 < \gamma_n < 1$ and the initial values are positive real numbers.

There is an increasing interest in the applications of difference and systems of difference equations in various fields. Even if a difference equation appears very plain and simple, its solutions can exhibit very complex behavior. In this paper, we study the global behavior of the admissible solutions of the system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{a x_{n-1} + b y_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where a, b, and the initial values x_{-1}, x_0, y_{-1}, y_0 are nonzero real numbers.

We shall study here, the behavior of the solutions of system (1.2) using their closed form. Other relevant qualitative and quantitative theories of difference equations can be obtained in references ([1]-[7], [12], [15], [16], [22], [26], [30], [32]-[34] and the references therein). For more on discrete systems of difference equations that are solved in closed form in references (see [11], [13], [14], [17]-[21], [23]-[25], [31], [35]-[38]).

2. Linearized Stability and Solution of the System (1.2)

In this section, we investigate the local asymptotic behavior of the equilibrium point of the system (1.2) and derive its solution.

It is clear that the system (1.2) has no equilibrium points when $a = 1$ and it has a unique equilibrium point

$(\frac{b}{1-a},1)$ when $a\neq 1$. To study the linearized stability of the unique equilibrium point of the system (1.2), we consider the transformation

$$F\begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1}\end{pmatrix}=\begin{pmatrix} \frac{x_{n-1}}{y_{n-1}} \\ x_n \\ \frac{x_{n-1}}{ax_{n-1}+by_{n-1}} \\ y_n\end{pmatrix}.$$

The linearized system associated with the system (1.2) about an equilibrium point $(\bar{x},\bar{y})$ is

$$Z_{n+1}=J_F(\bar{x},\bar{y})Z_n,\ n=0,1,...,$$

where

$$Z_n=\begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1}\end{pmatrix} \text{ and } J_F(\bar{x},\bar{y})=\begin{pmatrix} 0 & \frac{1}{\bar{y}} & 0 & -\frac{\bar{x}}{\bar{y}^2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{b\bar{y}^3}{\bar{x}^2} & 0 & -\frac{b\bar{y}^2}{\bar{x}} \\ 0 & 0 & 1 & 0\end{pmatrix}.$$

For more results on the stability of difference equations, see [27].

Theorem 2.1. Assume that $a\neq 1$. Then the equilibrium point $(\frac{b}{1-a},1)$ of the system (1.2) is

1. locally asymptotically stable if $|a|<1$,
2. unstable (saddle point) if $|a|>1$.

Proof.

The Jacobian matrix about the equilibrium point $(\frac{b}{1-a},1)$ becomes

$$J_F\left(\frac{b}{1-a},1\right)=\begin{pmatrix} 0 & 1 & 0 & -\frac{b}{1-a} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{(1-a)^2}{b} & 0 & -(1-a) \\ 0 & 0 & 1 & 0\end{pmatrix}. \tag{2.1}$$

It is enough to see that the eigenvalues of the matrix (2.1) are $0,0,\sqrt{|a|},-\sqrt{|a|}$, and the result follows.

Now, returning to the system (1.2), we can write

$$u_{n+1}=au_{n-1}+b,\ n=0,1,..., \tag{2.2}$$

where

$$u_n=\frac{x_n}{y_n},\text{ with } u_{-1}=\frac{x_{-1}}{y_{-1}},\text{ and } u_0=\frac{x_0}{y_0}.$$

Solving (2.2), we obtain the following:

1. If $a\neq 1$, then

$$x_n=\begin{cases} \frac{a^{\frac{n-1}{2}}\alpha_1+b}{1-a} & ,n=1,3,..., \\ \frac{a^{\frac{n}{2}-1}\alpha_2+b}{1-a} & ,n=2,4,..., \end{cases} \tag{2.3}$$

and

$$y_n = \begin{cases} \frac{a^{\frac{n-1}{2}}\alpha_1+b}{a^{\frac{n+1}{2}}\alpha_1+b} & ,n=1,3,..., \\ \frac{a^{\frac{n}{2}}\alpha_2+b}{a^{\frac{n}{2}}\alpha_2+b} & ,n=2,4,..., \end{cases} \quad (2.4)$$

where $\alpha_i = \frac{x_{-2+i}}{y_{-2+i}}(1-a)-b,\ i=1,2.$

2. If $a=1$, then

$$x_n = \begin{cases} \beta_1 + b(\frac{n-1}{2}) & ,n=1,3,..., \\ \beta_2 + b(\frac{n}{2}-1) & ,n=2,4,..., \end{cases} \quad (2.5)$$

and

$$y_n = \begin{cases} \frac{\beta_1+b(\frac{n-1}{2})}{\beta_1+b(\frac{n+1}{2})} & ,n=1,3,..., \\ \frac{\beta_2+b(\frac{n}{2}-1)}{\beta_2+b(\frac{n}{2})} & ,n=2,4,..., \end{cases} \quad (2.6)$$

where $\beta_i = \frac{x_{-2+i}}{y_{-2+i}},\ i=1,2.$

The forbidden set for the system (1.2) depends on the value of a. For the system (1.2) we have the following:

- If $a \neq 1$, then the forbidden set of the system (1.2) is

$$F_1 = \bigcup_{m=0}^{\infty}\{(x,y)\in\mathbb{R}^2 : y = -\frac{a^m}{b}x\}.$$

- If $a=1$, then the forbidden set of the system (1.2) is

$$F_2 = \bigcup_{m=1}^{\infty}\{(x,y)\in\mathbb{R}^2 : y = -\frac{1}{bm}x\}.$$

From now on, we assume that all solutions are admissible, that is for any solution $\{(x_n,y_n)\}_{n=-1}^{\infty}$ of the system (1.2), the initial points $(x_{-i},y_{-i})\notin F_1$ if $a\neq 1$ or $(x_{-i},y_{-i})\notin F_2$ if $a=1$, $i=0,1$.

Theorem 2.2. Assume that $|a|<1$. Then the equilibrium point $(\frac{b}{1-a},1)$ of the system (1.2) is globally asymptotically stable.

Proof.
Using formulas (2.3) and (2.4), we have

$$(x_n,y_n)\to(\frac{b}{1-a},1),\ \text{as } n\to\infty. \quad (2.7)$$

That is, the equilibrium point $(\frac{b}{1-a},1)$ of the system (1.2) is a global attractor.
Using Theorem (2.1)(1), the proof follows.

We give the following result without proof as a consequence of the solution form of the system (1.2).

Theorem 2.3. Assume that $a\neq 1$. The following statements are true:

1. If $a>1$, then the solution $\{(x_n,y_n)\}_{n=-1}^{\infty}$ is unbounded, namely:

$$\{(x_{2n+1},y_{2n+1})\}_{n=-1}^{\infty}\to(-\infty\ .\ sgn(\alpha_1),\frac{1}{a}),\ \text{as } n\to\infty,$$

and

$$\{(x_{2n+2}, y_{2n+2})\}_{n=-1}^{\infty} \to (-\infty \,.\, sgn(\alpha_2), \frac{1}{a}), \text{ as } n \to \infty.$$

2. If $a = 1$, then the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is unbounded, namely:
   $\{(x_n, y_n)\}_{n=-1}^{\infty} \to (\infty \,.\, sgn(b), 1)$ as $n \to \infty$.

Theorem 2.4. Assume that $a \neq 1$. Then the set $I = \{(x, y) \in \mathbb{R}^2 : (a-1)x + by = 0\}$ is an invariant set for the system (1.2).

Proof.
Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a solution of the system (1.2) such that $(x_{-i}, y_{-i}) \in I$, $i = 0, 1$. Then

$$x_1 = \frac{x_{-1}}{y_{-1}} = \frac{b}{1-a} \text{ and } y_1 = \frac{x_{-1}}{ax_{-1} + by_{-1}} = 1.$$

This implies that $(x_1, y_1) \in I$. Similarly, we can show that $(x_2, y_2) \in I$. Assume that $(x_t, y_t) \in I$, $-1 \le t \le n_0 - 1$ for a certain $n_0 \in \mathbb{N}$. Then

$$x_{n_0} = \frac{x_{n_0-1}}{y_{n_0-1}} = \frac{b}{1-a} \text{ and } y_{n_0} = \frac{x_{n_0-1}}{ax_{n_0-1} + by_{n_0-1}} = 1.$$

This implies that $(x_{n_0}, y_{n_0}) \in I$ and the proof is completed.

3. Behaviors of Solutions of the System (1.2)

This section is devoted to study the behaviors of the admissible solutions of the system (1.2). During this section, assume that $a \in \mathbb{R}_+ - \{1\}$ and consider the real-valued functions

$$f(x) = a^x \alpha + b, \; g(x) = \frac{a^x \alpha + b}{a^{x+1}\alpha + b}.$$

For $\alpha b < 0$, denote $l_1 = \frac{\ln(-\frac{b}{\alpha})}{\ln a}$.
We shall introduce the following two Lemmas to be used in the subsequent results.

Lemma 3.1. For the function $f(x)$, the following statements are true:

1. When $\alpha b > 0$, then $f(x) > 0$ ($f(x) < 0$) if $\alpha > 0$ ($\alpha < 0$).
2. When $\alpha b < 0$, we have the following:
   (a) If $\alpha > 0$, then we have the following:
      i. If $0 < a < 1$, then $f(x) < 0$ for all $x > 0$ ($x > l_1$) when $-\frac{b}{\alpha} \in]1, \infty[$ ($-\frac{b}{\alpha} \in]0, 1[$).
      ii. If $a > 1$, then $f(x) > 0$ for all $x > 0$ ($x > l_1$) when $-\frac{b}{\alpha} \in]0, 1[$ ($-\frac{b}{\alpha} \in]1, \infty[$).
   (b) If $\alpha < 0$, then we have the following:
      i. If $0 < a < 1$, then $f(x) > 0$ for all $x > 0$ ($x > l_1$) when $-\frac{b}{\alpha} \in]1, \infty[$ ($-\frac{b}{\alpha} \in]0, 1[$).
      ii. If $a > 1$, then $f(x) < 0$ for all $x > 0$ ($x > l_1$) when $-\frac{b}{\alpha} \in]0, 1[$ ($-\frac{b}{\alpha} \in]1, \infty[$).

Proof.

1. The proof is clear and will be omitted.

2. Assume that $\alpha b < 0$.

   (a) When $\alpha > 0$, we have the following:

   i. If $0 < a < 1$, then for $-\frac{b}{\alpha} \in]1,\infty[$ we have $f(x) < \alpha + b < 0$ for all $x > 0$. Otherwise, if $-\frac{b}{\alpha} \in]0,1[$, then
   $f(x) < f(\frac{\ln(\frac{-b}{\alpha})}{\ln a}) = a^{\frac{\ln(\frac{-b}{\alpha})}{\ln a}}\alpha + b = 0$ for all $x > l_1$.

   ii. If $a > 1$, then for $-\frac{b}{\alpha} \in]0,1[$ we have $f(x) > \alpha + b > 0$ for all $x > 0$. Otherwise, if $-\frac{b}{\alpha} \in]1,\infty[$, then
   $f(x) > f(\frac{\ln(\frac{-b}{\alpha})}{\ln a}) = a^{\frac{\ln(\frac{-b}{\alpha})}{\ln a}}\alpha + b = 0$ for all $x > l_1$.

   (b) When $\alpha < 0$, we have the following:

   i. If $0 < a < 1$, then for $-\frac{b}{\alpha} \in]1,\infty[$ we have $f(x) > 0$ for all $x > 0$. Otherwise, if $-\frac{b}{\alpha} \in]0,1[$, then
   $f(x) > f(\frac{\ln(\frac{-b}{\alpha})}{\ln a}) = a^{\frac{\ln(\frac{-b}{\alpha})}{\ln a}}\alpha + b = 0$.

   ii. If $a > 1$, then for $-\frac{b}{\alpha} \in]0,1[$ we have $f(x) < 0$ for all $x > 0$. Otherwise, if $-\frac{b}{\alpha} \in]1,\infty[$, then
   $f(x) < f(\frac{\ln(\frac{-b}{\alpha})}{\ln a}) = a^{\frac{\ln(\frac{-b}{\alpha})}{\ln a}}\alpha + b = 0$ for all $x > l_1$.

Lemma 3.2. For the function $g(x)$, the following statements are true:

1. When $\alpha b > 0$, then $g(x) > 0$ for all $x > 0$.

2. When $\alpha b < 0$, we have the following:

   (a) If $0 < a < 1$, then either $g(x) > 0$ for all $x > 0$ when $-\frac{b}{\alpha} \in]1,\infty[$ or $g(x) > 0$ for all $x > l_1$ when $-\frac{b}{\alpha} \in]0,1[$.

   (b) If $a > 1$, then either $g(x) > 0$ for all $x > 0$ when $-\frac{b}{\alpha} \in]0,1[$ or $g(x) > 0$ for all $x > l_1$ when $-\frac{b}{\alpha} \in]1,\infty[$.

Proof.

1. The proof is clear and will be omitted.

2. Assume that $\alpha b < 0$.

   (a) When $\alpha > 0$ and $b < 0$, we get $\alpha a + b < \alpha + b$.
   If $-\frac{b}{\alpha} \in]1,\infty[$, then $g(x) > \frac{\alpha+b}{\alpha a+b} = g(0)$ for all $x > 0$.
   Otherwise, there exists $l_1 = \frac{\ln(\frac{-b}{\alpha})}{\ln a} > 0$, $g(x) > g(l_1) = \frac{a_1^l\alpha+b}{a^{l_1+1}\alpha+b} = 0$, for all $x > l_1$.
   Now, when $\alpha < 0$ and $b > 0$, we get $\alpha + b < \alpha a + b$.
   If $-\frac{b}{\alpha} \in]1,\infty[$, then $g(x) > \frac{\alpha+b}{\alpha a+b} = g(0)$ for all $x > 0$.
   Otherwise, there exists $l_1 = \frac{\ln(\frac{-b}{\alpha})}{\ln a} > 0$, $g(x) > g(l_1) = \frac{a_1^l\alpha+b}{a^{l_1+1}\alpha+b} = 0$, for all $x > l_1$.

   (b) The proof is similar to that of (2a) and is omitted.

Consider the sets:

$$D_+ = \{(x,y) \in \mathbb{R}^2 : \frac{x}{y} > \frac{b}{1-a}\},$$

$$D_- = \{(x,y) \in \mathbb{R}^2 : \frac{x}{y} < \frac{b}{1-a},\}$$

where $a \neq 1$, b is a nonzero real number and let $(\lceil . \rceil$ denote the ceiling function.

3.1. Case $0 < a < 1$.

Theorem 3.3. Assume for $i = 1,2$ that either $(x_{-2+i}, y_{-2+i}) \in D_+$ with $b > 0$ or $(x_{-2+i}, y_{-2+i}) \in D_-$ with $b < 0$ (respectively). Then except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located either in the 1^{st} quadrant or the 2^{nd} quadrant (respectively).

Proof.
When $(x_{-2+i}, y_{-2+i}) \in D_+$, $i = 1,2$, we get $\alpha_1 > 0$ and $\alpha_2 > 0$.
Using formulas (2.3) and (2.4), we get

$$sgn(x_{2m+i}) = sgn(\frac{a^m\alpha_i + b}{1-a}) = 1, i = 1,2.$$

Similarly,

$$sgn(y_{2m+i}) = sgn\left(\frac{a^m\alpha_i + b}{a^{m+1}\alpha_i + b}\right) = 1,\ i = 1,2.$$

Then we conclude (using Lemma (3.1) (1) and Lemma (3.2) (1)) that, except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 1^{st} quadrant.
When $(x_{-i}, y_{-i}) \in D_-$ with $b < 0$ for $i = 1,2$, the proof is similar and is omitted.

Theorem 3.4. Assume for $i = 1,2$ that either $(x_{-2+i}, y_{-2+i}) \in D_-$ with $b > 0$ or $(x_{-2+i}, y_{-2+i}) \in D_+$ with $b < 0$ (respectively). Then the following statements are true:

1. If $-\frac{b}{\alpha_1}, -\frac{b}{\alpha_2} \in]1,\infty[$, then except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located either in the 1^{st} quadrant or the 2^{nd} quadrant (respectively).

2. If $-\frac{b}{\alpha_1}, -\frac{b}{\alpha_2} \in]0,1[$, then there exists a positive integer n_0 such that $\{(x_n, y_n)\}_{n=n_0}^{\infty}$ is located either in the 1^{st} quadrant or the 2^{nd} quadrant (respectively).

Proof.
We shall prove only when $(x_{-2+i}, y_{-2+i}) \in D_-$ with $b > 0$, $i = 1,2$. For the other case, the proof is similar and will be omitted.
Assume that $(x_{-2+i}, y_{-2+i}) \in D_-$, $i = 1,2$. Then $\alpha_i < 0$, $i = 1,2$.

1. If $-\frac{b}{\alpha_1}, -\frac{b}{\alpha_2} \in]1,\infty[$, then using Lemma (3.1) (2b) and Lemma (3.2) (2a), we get

$$sgn(x_{2m+i}) = sgn(\frac{a^m\alpha_i + b}{1-a}) = 1,\ i = 1,2,\ m = 0,1,...,$$

and

$$sgn(y_{2m+i}) = sgn\left(\frac{a^m\alpha_i + b}{a^{m+1}\alpha_i + b}\right) = 1,\ i = 1,2\ m = 0,1,....$$

Therefore, except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 1^{st} quadrant.

2. If $-\frac{b}{\alpha_1}, -\frac{b}{\alpha_2} \in]0,1[$, then using Lemma (3.1) (2b) and Lemma (3.2) (2a), we conclude that there exists a positive integer $\lceil \frac{\ln(-\frac{b}{\alpha_i})}{\ln a} \rceil$ such that

$$sgn(x_{2m+i}) = sgn(\frac{a^m\alpha_i + b}{1-a}) = 1,\ m \geq \lceil \frac{\ln(-\frac{b}{\alpha_i})}{\ln a} \rceil,\ i = 1,2,$$

and

$$sgn(y_{2m+i}) = sgn\left(\frac{a^m\alpha_i + b}{a^{m+1}\alpha_i + b}\right) = 1,\ m \geq \lceil \frac{\ln(-\frac{b}{\alpha_i})}{\ln a} \rceil,\ i = 1,2.$$

Now, we claim that

$$sgn(x_n) = 1,\ n \geq n_0,$$

where

$$n_0 = max\{2\lceil \frac{\ln(-\frac{b}{\alpha_1})}{\ln a} \rceil + 1, 2\lceil \frac{\ln(-\frac{b}{\alpha_2})}{\ln a} \rceil + 2\} - 1.$$

To prove the claim, let $n'_0 := max\{2\lceil \frac{\ln(-\frac{b}{\alpha_1})}{\ln a} \rceil + 1, 2\lceil \frac{\ln(-\frac{b}{\alpha_2})}{\ln a} \rceil + 2\}$. We have three cases to consider:

- If $\alpha_1 = \alpha_2 := \alpha$, then $n'_0 = 2\lceil \frac{\ln(-\frac{b}{\alpha})}{\ln a} \rceil + 2$.
  But
  $$sgn(x_n) = 1, n = 2\lceil \frac{\ln(-\frac{b}{\alpha})}{\ln a} \rceil + 1.$$
  Then
  $$sgn(x_n) = 1,\ n \geq n'_0 - 1 = n_0.$$
- If $\alpha_1 < \alpha_2$, then $\lceil \frac{\ln(-\frac{b}{\alpha_1})}{\ln a} \rceil > \lceil \frac{\ln(-\frac{b}{\alpha_2})}{\ln a} \rceil$.
  It follows that
  $$n'_0 - 1 = 2\lceil \frac{\ln(-\frac{b}{\alpha_1})}{\ln a} \rceil \geq 2\lceil \frac{\ln(-\frac{b}{\alpha_2})}{\ln a} \rceil + 2.$$
  Therefore,
  $$sgn(x_n) = 1,\ n \geq n'_0 - 1 = n_0.$$
- If $\alpha_1 > \alpha_2$, then $\lceil \frac{\ln(-\frac{b}{\alpha_1})}{\ln a} \rceil < \lceil \frac{\ln(-\frac{b}{\alpha_2})}{\ln a} \rceil$.
  It follows that
  $$n'_0 - 1 = 2\lceil \frac{\ln(-\frac{b}{\alpha_2})}{\ln a} \rceil + 1 \geq 2\lceil \frac{\ln(-\frac{b}{\alpha_1})}{\ln a} \rceil + 3.$$
  Therefore,
  $$sgn(x_n) = 1,\ n \geq n'_0 - 1 = n_0.$$

The claim is proved.
Therefore, for $n \geq n_0 = max\{2\lceil \frac{\ln(-\frac{b}{\alpha_1})}{\ln a} \rceil + 1, 2\lceil \frac{\ln(-\frac{b}{\alpha_2})}{\ln a} \rceil + 2\} - 1$, (x_n, y_n) is located in the 1^{st} quadrant.

Theorem 3.5. Assume that $(x_{-1}, y_{-1}) \in D_+$ and $(x_0, y_0) \in D_-$. Then the following statements are true:

1. If $b > 0$, then either the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ (except (possibly) for the initial conditions) is located in the 1^{st} quadrant when $-\frac{b}{\alpha_2} \in]1, \infty[$ or there exists a positive integer n_2 such that $\{(x_n, y_n)\}_{n=n_2}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_2} \in]0, 1[$.

2. If $b < 0$, then either the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ (except (possibly) for the initial conditions) is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_1} \in]1, \infty[$ or there exists a positive integer n_1 such that $\{(x_n, y_n)\}_{n=n_1}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_1} \in]0, 1[$.

Proof.

1. Assume that $b > 0$ and let $-\frac{b}{\alpha_2} \in]1,\infty[$. Then $\alpha_1 b > 0$ and $\alpha_2 b < 0$. Using Lemma (3.1) and Lemma (3.2), we get
$$sgn(x_{2m+i}) = sgn(\frac{a^m\alpha_i + b}{1-a}) = 1,\ i = 1,2,\ m = 0,1,...,$$
and
$$sgn(y_{2m+i}) = sgn\left(\frac{a^m\alpha_i + b}{a^{m+1}\alpha_i + b}\right) = 1,\ i = 1,2\ m = 0,1,....$$
Therefore, we have except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 1^{st} quadrant.
Otherwise, if $-\frac{b}{\alpha_2} \in]0,1[$, then there exists a positive integer $m_2 := \lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil$ such that
$$sgn(x_{2m+2}) = sgn(\frac{a^m\alpha_2 + b}{1-a}) = 1,\ m \geq m_2,$$
and
$$sgn(y_{2m+2}) = sgn\left(\frac{a^m\alpha_2 + b}{a^{m+1}\alpha_2 + b}\right) = 1,\ m \geq m_2.$$
Then $\{(x_n, y_n)\}_{n=n_2}^{\infty}$ is located in the 1^{st} quadrant, where $n_2 = 2m_2 + 1$.
Note that:
$$sgn(x_{2m_2+1}) = 1 \text{ and } sgn(y_{2m_2+1}) = 1.$$
2. When $b < 0$, the proof is similar and is omitted.

Note: In Theorem (3.5), we have $n_1 = \lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil$ and $n_2 = \lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil + 1$.

Theorem 3.6. Assume that $(x_{-1}, y_{-1}) \in D_-$ and $(x_0, y_0) \in D_+$.

1. If $b > 0$, then either the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ (except (possibly) for the initial conditions) is located in the 1^{st} quadrant when $-\frac{b}{\alpha_1} \in]1,\infty[$ or $\{(x_n, y_n)\}_{n=n_1}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_1} \in]0,1[$.
2. If $b < 0$, then either the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ (except (possibly) for the initial conditions) is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_2} \in]1,\infty[$ or $\{(x_n, y_n)\}_{n=n_2}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_2} \in]0,1[$.

Proof.
The proof is similar to that of Theorem (3.5) and is omitted. Note: In Theorem (3.6), the values of n_1 and n_2 are:
$$n_1 = 2\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil \text{ and } n_2 = \lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil + 1.$$
To illustrate Theorem (3.3) and Theorem (3.5), we give the following numerical examples:

Example (1) Assume that $a = 0.8, b = -0.4$ and the initial values are $(x_{-1}, y_{-1}) = (3,-1)$, $(x_0, y_0) = (-7,0.2)$ $((x_{-2+i}, y_{-2+i}) \in D_-,\ i = 1,2)$. Then except (possibly) for the initial values, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 2^{nd} quadrant.
Here $\alpha_1 = -0.2$, $\alpha_2 = -6.6$.

The values of the first 30 terms (including the initial values) of the solution are:

$(3,-1)$, $(-7,0.2)$, $(-3,1.07143)$, $(-35,1.23239)$, $(-2.8,1.06061)$, $(-28.4,1.22837)$, $(-2.64,1.05096)$, $(-23.12,1.22354)$, $(-2.512,1.0425)$, $(-18.896,1.21778)$, $(-2.4096, 1.03519)$, $(-15.5168,1.21098)$, $(-2.32768,1.02897)$, $(-12.8134,1.20305)$, $(-2.26214, 1.02373)$, $(-10.6508,1.19395)$, $(-2.20972,1.01935)$, $(-8.9206,1.18366)$, $(-2.16777, 1.01572)$, $(-7.53648,1.17223)$, $(-2.13422,1.01274)$, $(-6.42919,1.1598)$, $(-2.10737, 1.0103)$, $(-5.54335,1.14658)$, $(-2.0859,1.0083)$, $(-4.83468,1.13284)$, $(-2.06872,1.00669)$, $(-4.26774,1.11891)$, $(-2.05498,1.00538)$, $(-3.81419, 1.10513)$, $(-2.04398,1.00432)$, $(-3.45136,1.09183)$.

(See figure 1).

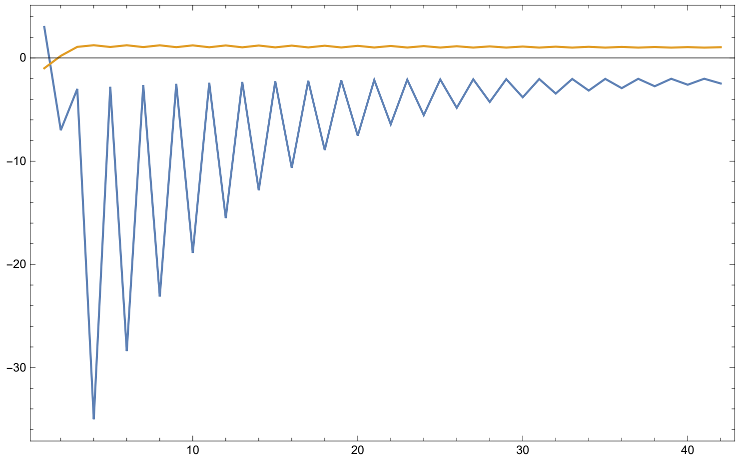

Figure 1. $x_{n+1} = \frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{0.8x_{n-1}-0.4y_{n-1}}$

Example (2) Assume that $a = 0.5, b = 2$ and the initial values are $(x_{-1}, y_{-1}) = (3,0.5)$, $(x_0, y_0) = (26,-0.2)$ $((x_{-1}, y_{-1}) \in D_+, (x_0, y_0) \in D_-)$. Then for $n \geq 13$, (x_n, y_n) is located in the 1^{st} quadrant.
Here $n_2 = 2\lceil \frac{\ln(-\frac{b}{\alpha_2})}{\ln a} \rceil + 1 = 13$, where $\frac{\ln(-\frac{b}{\alpha_2})}{\ln a} = \frac{2}{67} \in]0,1]$, $\alpha_2 = -67$.
The values of the first 30 terms (including the initial values) of the solution are:

$(3,0.5)$, $(26,-0.2)$, $(6,1.2)$, $(-130,2.06349)$, $(5,1.11111)$, $(-63,2.13559)$, $(4.5,1.05882)$, $(-29.5,2.31373)$, $(4.25,1.0303)$, $(-12.75,2.91429)$, $(4.125,1.01538)$, $(-4.375,23.3333)$, $(4.0625,1.00775)$, $(-0.1875,-0.0983607)$, $(4.03125,1.00389)$, $(1.90625,0.645503)$, $(4.01563,1.00195)$, $(2.95312,0.849438)$, $(4.00781,1.00098)$, $(3.47656,0.92999)$, $(4.00391,1.00049)$, $(3.73828,0.966179)$, $(4.00195,1.00024)$, $(3.86914,0.983371)$, $(4.00098,1.00012)$, $(3.93457,0.991754)$, $(4.00049,1.00006)$, $(3.96729,0.995894)$, $(4.00024,1.00003)$, $(3.98364,0.997951)$.

Clear that $x_n > 0$ and $y_n > 0$, $n \geq 13$. (See figure 2).

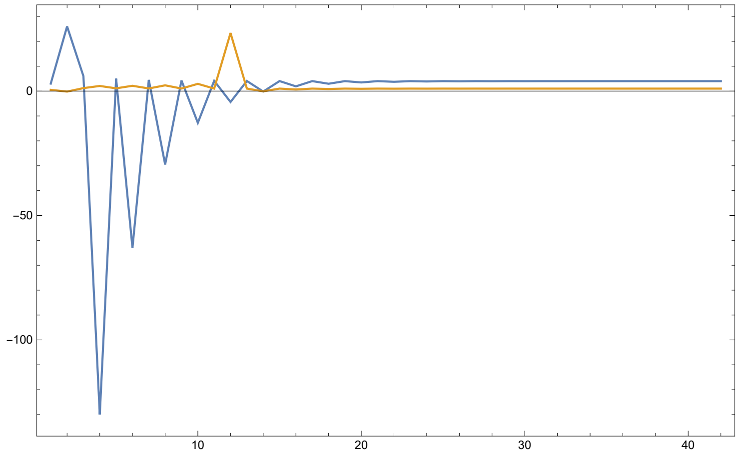

Figure 2. $x_{n+1} = \frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{0.5x_{n-1}+2y_{n-1}}$

3.2. Case $a > 1$.

Theorem 3.7. Assume for $i = 1,2$ that $(x_{-2+i}, y_{-2+i}) \in D_+$. Then we have the following:

1. If $b > 0$, then either the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ (except (possibly) for the initial conditions) is located in the 1^{st} quadrant when $-\frac{b}{\alpha_i} \in]0,1[$, $i = 1,2$ or there exists a positive integer n_0 such that $\{(x_n, y_n)\}_{n=n_0}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_i} \in]1,\infty[$, $i = 1,2$.
2. If $b < 0$, then except for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 1^{st} quadrant.

Proof.
When $(x_{-2+i}, y_{-2+i}) \in D_+$ for $i = 1,2$, we get $\alpha_i < 0$ for $i = 1,2$.

1. If $b > 0$, then $\alpha_i b < 0$, $i = 1,2$. Using Lemma (3.1) and Lemma (3.2), we conclude that except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 1^{st} quadrant when $\max\{-\frac{b}{\alpha_1}, -\frac{b}{\alpha_2}\} < 1$. When $\min\{-\frac{b}{\alpha_1}, -\frac{b}{\alpha_2}\} > 1$, there exist two positive integers m_1 and m_2 such that the subsequences $\{(x_{2m+1}, y_{2m+1})\}_{n=m_1}^{\infty}$ and $\{(x_{2m+2}, y_{2m+2})\}_{n=m_2}^{\infty}$ are located in the 1^{st} quadrant, where $m_1 = \lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil$ and $m_2 = \lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil$. Therefore, we conclude that $\{(x_n, y_n)\}_{n=n_0}^{\infty}$ is located in the 1^{st} quadrant, where $n_0 = \max\{2\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil + 1, 2\lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil + 2\} - 1$.
2. When $b < 0$, the proof is a direct consequence of applying Lemma (3.1) (1) and Lemma (3.2) (1).

Theorem 3.8. Assume for $i = 1,2$ that $(x_{-2+i}, y_{-2+i}) \in D_-$. Then we have the following:

1. If $b > 0$, then except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 2^{nd} quadrant.
2. If $b < 0$, then either the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ (except (possibly) for the initial conditions) is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_i} \in]0,1[$, $i = 1,2$ or $\{(x_n, y_n)\}_{n=n_0}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_i} \in]1,\infty[$, $i = 1,2$.

Proof.
The proof is similar to that of Theorem (3.7) and is omitted. To illustrate Theorem (3.8), we give the follow-

ing numerical example:

Example (3) Assume that $a = 1.2, b = -3$ and the initial values are $(x_{-1}, y_{-1}) = (-2.7, -0.3)$, $(x_0, y_0) = (12.1, 1.1)$ $((x_{-2+i}, y_{-2+i}) \in D_-,\ i = 1,2)$. Then for $n \geq 17$, (x_n, y_n) is located in the 2^{nd} quadrant.
Here $n_0 = \max\{2\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil + 1, 2\lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil + 2\} - 1 = 17$, where $\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil = 6$ and $\lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil = 8$.
The values of the first 30 terms (including the initial values) of the solution are:

$$(-2.7, -0.3),\ (12.1, 1.1),\ (9, 1.15385),\ (11, 1.07843),\ (7.8, 1.22642),\ (10.2, 1.1039),$$
$$(6.36, 1.37306),\ (9.24, 1.14243),\ (4.632, 1.81051),\ (8.088, 1.20616),\ (2.5584, 36.5068),$$
$$(6.7056, 1.3287),\ (0.07008, -0.0240337),\ (5.04672, 1.65138),\ (-2.9159, 0.448664),$$
$$(3.05606, 4.5799),\ (-6.49908, 0.601828),\ (0.667277, -0.303409),\ (-10.7989, 0.676679),$$
$$(-2.19927, 0.390002),\ (-15.9587, 0.720469),\ (-5.63912, 0.577368),\ (-22.1504, 0.748818),$$
$$(-9.76695, 0.6635),\ (-29.5805, 0.768393),\ (-14.7203, 0.712352),\ (-38.4966, 0.782516),$$
$$(-20.6644, 0.743396),\ (-49.1959, 0.793034),\ (-27.7973, 0.76457),\ (-62.0351, 0.801051),$$
$$(-36.3567, 0.779718).$$

Clear that $x_n < 0$ and $y_n > 0$, $n \geq 17$. (See figure 3).

Theorem 3.9. Assume that $(x_{-1}, y_{-1}) \in D_-$ and $(x_0, y_0) \in D_+$. Then we have the following:

1. If $b > 0$, then except for the initial conditions we have, the subsequence $\{(x_{2m+1}, y_{2m+1})\}_{m=-1}^{\infty}$ is located in the 2^{nd} quadrant and either the subsequence $\{(x_{2m+2}, y_{2m+2})\}_{m=-1}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_2} \in]0,1[$, or the subsequence $\{(x_{2m+2}, y_{2m+2})\}_{m=m_2}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_2} \in]1,\infty[$.
2. If $b < 0$, then except for the initial conditions we have, the subsequence $\{(x_{2m+2}, y_{2m+2})\}_{m=-1}^{\infty}$ is located in the 1^{st} quadrant and either the subsequence $\{(x_{2m+1}, y_{2m+1})\}_{m=-1}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_1} \in]0,1[$, or the subsequence $\{(x_{2m+1}, y_{2m+1})\}_{m=m_1}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_1} \in]1,\infty[$.

Proof.
Assume that $(x_{-1}, y_{-1}) \in D_-$ and $(x_0, y_0) \in D_+$. Then $\alpha_1 > 0$ and $\alpha_2 < 0$.

1. When $b > 0$, then $\alpha_1 b > 0$ and $\alpha_2 b < 0$. Using Lemmas (3.1) (1) and (3.2) (1), we conclude that except (possibly) for the initial conditions we have, the subsequence $\{(x_{2m+1}, y_{2m+1})\}_{m=-1}^{\infty}$ is located in the 2^{nd} quadrant.
   If $-\frac{b}{\alpha_2} \in]0,1[$, then using Lemmas (3.1) (2b) and (3.2) (2b), we conclude that except (possibly) for the initial conditions we have, the subsequence $\{(x_{2m+2}, y_{2m+2})\}_{m=-1}^{\infty}$ is located in the 1^{st} quadrant.
   Otherwise, if $-\frac{b}{\alpha_2} \in]1,\infty[$, then the subsequence $\{(x_{2m+2}, y_{2m+2})\}_{m=m_2}^{\infty}$ is located in the 1^{st} quadrant.
2. The proof is similar to (1) and is omitted.

Theorem 3.10. Assume that $(x_{-1}, y_{-1}) \in D_+$ and $(x_0, y_0) \in D_-$.

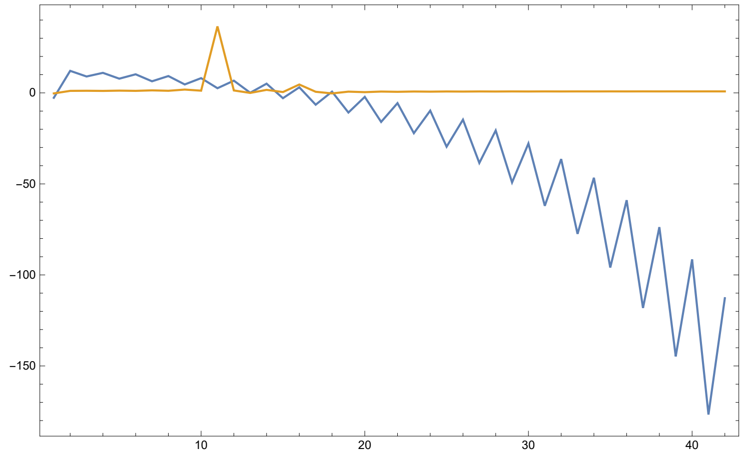

Figure 3. $x_{n+1} = \frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{1.2x_{n-1}-3y_{n-1}}$

1. If $b > 0$, then except (possibly) for the initial conditions we have, the subsequence $\{(x_{2m+2}, y_{2m+2})\}_{m=-1}^{\infty}$ is located in the 2^{nd} quadrant and either the subsequence $\{(x_{2m+1}, y_{2m+1})\}_{m=-1}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_1} \in]0,1[$, or the subsequence $\{(x_{2m+1}, y_{2m+1})\}_{m=m_1}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_1} \in]1,\infty[$.

2. If $b < 0$, then except (possibly) for the initial conditions we have, the subsequence $\{(x_{2m+1}, y_{2m+1})\}_{m=-1}^{\infty}$ is located in the 1^{st} quadrant and either the subsequence $\{(x_{2m+2}, y_{2m+2})\}_{m=-1}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_2} \in]0,1[$, or the subsequence $\{(x_{2m+2}, y_{2m+2})\}_{m=m_2}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_2} \in]1,\infty[$.

Proof.
The proof is similar to that of Theorem (3.9) and is omitted. To illustrate Theorem (3.10), we give the following numerical example:

Example (4) Assume that $a = 1.5, b = 1$ and the initial values are $(x_{-1}, y_{-1}) = (3.9, -2)$, $(x_0, y_0) = (-1.5, 0.5)$ $((x_{-1}, y_{-1}) \in D_+, (x_0, y_0) \in D_-)$. Then the solution $\{(x_n, y_n\}_{n=-1}^{\infty}$ has the property that:
Except (possibly) for the initial values, $\{(x_{2m+2}, y_{2m+2}\}_{m=-1}^{\infty}$ is located in the 2^{nd} quadrant and $\{(x_{2m+1}, y_{2m+1}\}_{m=10}^{\infty}$ is located in the 1^{st} quadrant.
Here $m_1 = \lceil \frac{\ln(-\frac{b}{\alpha_1})}{\ln a} \rceil = 10$, where $\frac{x_{-1}}{y_{-1}} = -1.95$ and $\frac{x_0}{y_0} = -3$, $-\frac{b}{\alpha_1} = 40 \in]1,\infty[$
The values of the first 30 terms (including the initial values) of the solution are:

(3.9, −2), (−1.5, 0.5), (−1.95, 1.01299), (−3., 0.857143), (−1.925, 1.01987), (−3.5, 0.823529), (−1.8875, 1.03072), (−4.25, 0.790698), (−1.83125, 1.0483), (−5.375, 0.761062), (−1.74687, 1.07811), (−7.0625, 0.736156), (−1.62031, 1.13271), (−9.59375, 0.716453), (−1.43047, 1.24855), (−13.3906, 0.701596), (−1.1457, 1.59446), (−19.0859, 0.690796), (−0.718555, 9.23212), (−27.6289, 0.683151), (−0.077832, −0.0881199), (−40.4434, 0.67784), (0.883252, 0.379913), (−59.665, 0.6742), (2.32488, 0.5181), (−88.4976, 0.671727), (4.48732, 0.580433), (−131.746, 0.670057), (7.73098, 0.613742), (−196.62, 0.668935), (12.5965, 0.633157), (−293.929, 0.668182).

(See figure 4).

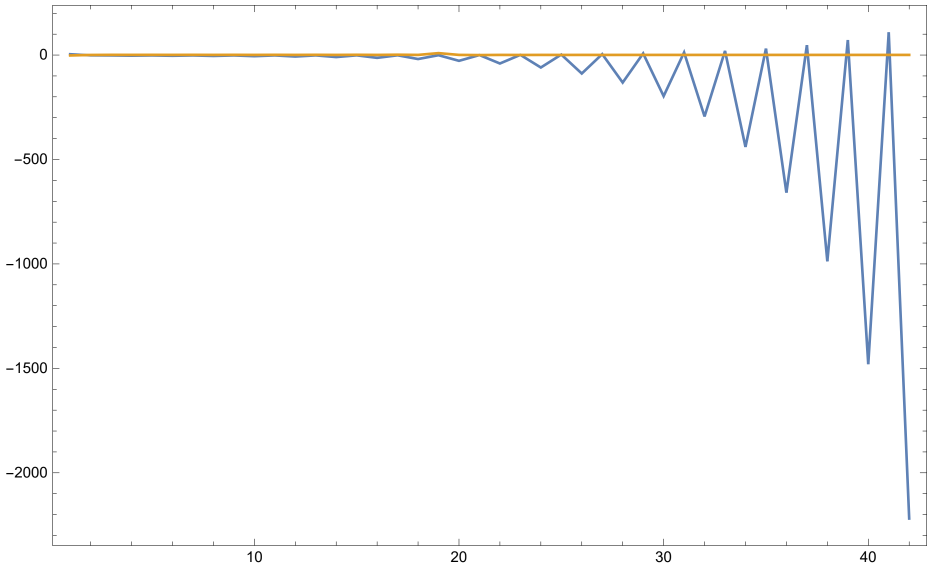

Figure 4. $x_{n+1} = \frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{1.5x_{n-1}+y_{n-1}}$

Discussions and Conclusions

In this paper, we studied the admissible solutions of the non-linear discrete system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{ax_{n-1}+by_{n-1}}, \quad n = 0, 1, \ldots,$$

where a, b and the initial values x_{-1}, x_0, y_{-1}, y_0 are non-zero real numbers. We discussed the linearized and global stability to the steady state $(\frac{b}{1-a}, 1)$ when $a \neq 1$ as well as introducing the forbidden sets. For $a \in \mathbb{R}_+ - \{1\}$, we showed any admissible solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is either entirely located in a certain quadrant of the plane or there exists a natural number $N > 0$ (we calculated its value) such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is located. We conjecture that the same results can be obtained for the discrete system

$$x_{n+1} = \frac{x_{n-k}}{y_{n-k}}, \quad y_{n+1} = \frac{x_{n-k}}{ax_{n-k}+by_{n-k}}, \quad n = 0, 1, \ldots,$$

where a, b are non-zero real numbers and the initial points (x_{-i}, y_{-i}), where $i = 0, 1, ..., k$ are non-zero real numbers.

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Competing Interests

The authors declare that they have no competing interests.

Author Contributions

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Conflicts of Interest

The authors declare no conflict of interest.

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