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Research Article

I -Sequentially Connectedness

Hürmet Fulya AKIZ[1*], Kadriye BAŞAR[2]

[1]Yozgat Bozok University, Faculty of Science and Letters, Department of Mathematics, 66100, Yozgat, Türkiye
[2] Yozgat Bozok University, Vocational School, Department of Computer Programming, 66100, Yozgat, Türkiye

Abstract

This study, first, focuses on understanding two important ideas in ideal topological spaces: I-sequential openness and I-sequential closedness. We start by explaining what these sets are like and how they behave. Then, we talk about their interiors and closures. After that, we look at how these sets relate to the idea of connectedness, which is a key concept in topology. We call this connection I-sequentially connectedness. This helps us understand how sets are connected in ideal topological spaces.

Keywords: I-sequentially open set, I-sequentially topological space, I-sequentially connectedness.

1. INTRODUCTION

In a topological space (X,τ), the largest open set contained in a set $O \subseteq X$ is referred to as the interior of O, denoted by $int(O)$. Similarly, the smallest closed set containing O is known as the closure of O, denoted by $cl(O)$ [1,2]. Various concepts of sets, functions, and spaces in topology have been studied by scientists for many years, contributing to the reexamination and analysis of many concepts with new definitions. Among these, seminal works by Levine [3], Njastad [4], Mashhour et al. [5], and Abd El-Monsef et al. [6] have focused on semi-open sets, α-open sets, pre-open sets, and β-open sets, respectively. These new definitions of open and closed sets have facilitated the redefinition of many concepts in topological spaces. With the definition of open sets, fundamental concepts such as closed sets, closure, interior points, and boundaries have become of great interest and have attracted much attention in topology.

Given a topological space (X,τ), a collection I of subsets of X is called an ideal if it satisfies certain conditions. The triple (X,τ,I) is then referred to as an ideal topological space. The local function $(\cdot)^* : \mathcal{P}(X) \to \mathcal{P}(X)$ associated with O with respect to τ and I was introduced by Kuratowski [7] and Hayashi [8], respectively. The set $O^*(I,\tau)$ is defined as $\{x \in X \mid \text{for all } U \in \mathcal{G}_x, (U \cap O) \notin I\}$, where $\mathcal{G}_x = (U \in \tau | x \in U)$. The Kuratowski closure operator $Cl^*(.)$ consists of the union of O and $O^*(I,\tau)$. If $O \subset intO^*$, then O is called an I-open set [8].

Recently, new definitions of open and closed sets have been obtained using operations defined on a space X. These operations are typically based on obtaining new definitions using the images of open sets in X. For instance, in [9], generalized open sets such as γ_{ij}-pre-I-open and γ_{ij}-b-I-open sets have been introduced in ideal bitopological spaces. Furthermore, there has been a redefinition of topological structures within ideal topological spaces, with the notion of connectedness being defined in [10,11]. Various concepts of continuity in ideal topological spaces have also been recently investigated [12–15]. Additionally, convergence types in topological spaces, as discussed in [15,16], have been extended to ideal topological spaces [17]. In [20], α-open, β-open and semi-open sets are examined in ideal topological spaces, and continuity is studied using these sets. Additionally, in [21], another form of continuity in ideal topological spaces is explored using theta open sets.

This study primarily delved into the notions of I-sequential openness and I-sequential closedness within ideal topological spaces. It commenced by elucidating the characteristics of these sets,followed by discussions on their interior and closures. Subsequently, the notion of connectedness, a fundementsl concept in topology, was revisited within the context of ideal topological spaces through the

[1]https://orcid.org/0000-0002-8547-2175 [2]https://orcid.org/0000-0003-2126-8199

[*]**Correspondence: fulya.gencel@bozok.edu.tr**

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utilization of these sets. The study provided insights into the properties and theoretical aspects of this concept, termed *I*-sequentially connectedness.

From now on, in this article, the terms "I-open set", "I-closed set", "I-sequentially open set", "I-sequentially closed set" and "I-sequentially connectedness" will be denoted as "I-ops", "I-cls", "I-sops", "I-scls" and "I-SC", respectively. Also, ℕ denotes the set of natural numbers.

2. MATERIAL AND METHODS

In reference [7], the subsequent definition is introduced.

Definition 2.1 Let X be a nonempty set. A family of sets $I \subseteq 2^X$ is known as an ideal on the condition that,

1. $\emptyset \in I$,

2. $U, V \in I$ leads to $U \cup V \in I$,

3. $U \in I$ and $V \subseteq U$ leads to $V \in I$.

If $I \neq \emptyset$ and $X \notin I$, then I is referred to as nontrivial.

An ideal I is said to be admissible if it includes all singleton sets. For further examples on this topic, one can refer to [18].

Subsequently, (Υ, τ) and I represent a topological space and a nontrivial ideal of N which denotes the set of all positive integers, respectively.

Definition 2.2 [19] A sequence x_n in X is said to be *I*-convergent to $x_0 \in X$ if for any open set U containing x_0, the set $\{\{n\} \in \mathbb{N} ; \{x_n\} \notin U \in I\}$.

We denote *I*-limit of x_n as *I*-$lim\, x_n = x_0$. If I is admissible, then I -convergence and ordinary convergence coincide. Morever, if I does not contain any infinite set, then both concepts coincide as well.

In a topological space X, a set $U \subseteq X$ is considered open if and only if every point $u \in U$ has an open neighborhood contained in U. Now, we introduce the definition of an I-sops in a topological space.

Definition 2.3 [19] A set U is said to a be I-sops if no sequence in $X \setminus U$ has an I-limit in U.

Proposition 2.4 [19] Every open set is an I-sops.

Now, let's provide an example of an I-sops that is not open.

Example 2.5 [19] Consider the countable complement topology $(\mathbb{R}, \tau_c)$ on ℝ. Let x_n be a sequence with an I-limit y. Then, the set $U = \mathbb{R} \setminus \{ x_n : n \in \mathbb{N} \} \cup \{ y \}$ is a neighborhood of y. Therefore, U must contain x_n for infinitely many n. If $x_n = y$ for sufficiently large n, it is possible. Consequently, a sequence in a set O can only I -converge to an element of O. Thus, every subset of ℝ is *I*-sequentially open. However, not every subset is open.

Definition 2.6 [19] A topological space is termed I-sequential if every set U is open if and only if it satisfies the property of being I-sequentially open.

Definition 2.7 [19] A subset C of X is termed a I-scls if and only if no sequence in F I-converges to a point not in F.

3. RESULTS AND DISCUSSION

In this chapter, we will introduce new types of sets defined using the concept of convergence. Additionally, we will provide some properties of these concepts. Furthermore, the notion of connectedness in topological spaces will be reintroduced using the concept of I-convergence.

Theorem 3.1 The union of any collection of I-squentially open subsets in X remains I-sequentially open.

Proof: Consider a collection $I = \{ U_i | i \in I \}$ of I-sequentially open subsets of X, and let x be an element in $\bigcup_{i\in I} U_i$. For each $i \in I$, x belongs to U_i. Since U_i is I-sequentially open, no sequence outside U_i converges to x in I, indicating that $U_i \subseteq \bigcup_{i\in I} U_i$ remains I-sequentially open. Therefore, $\bigcup_{i\in I} U_i$ is I-squentially open.

Definition 3.2 Let X be a topological space, I be a nontrivial ideal of $\mathbb{N}$, $U \subseteq \Upsilon$, and $a \in U$. We define U as an I-sequentially neighborhood of a if there exists an I-sequentially open subset G of X containing a and wholly contained in U.

A subset U of X is considered I-sops if, for each point within U, there exists a neighborhood of that point contained entirely within U.

Theorem 3.3 A subset U of X is I-sops if and only if there exists an I-sequentially open neighborhood U_a for each $a \in X$ such that U_a is entirely contained within X.

Proof: Choosing $X = U_a$ clarifies the proof. Conversely, if there exists an I-sequentially open neighborhood U_a for each $a \in X$ such that U_a is wholly contained within X, then the union $X \subseteq \bigcup_{a\in X} U_a$ is a I-sops, as each U_a is a I-sops.

Now, let's define the notion of an I-sequentially interior and explore its properties.

Definition 3.4 Consider a topological space X and a subset $V \subseteq X$. We define the I-sequentially interior of V abbreviated as $Isint\ (V)$,as the largest I-sops contained within V, represented by the set

$$\bigcup\{U \subseteq V \mid U \text{ is } I - \text{sequentially open}\}.$$

Some properties of I-sequentially interior of a set is as follows:

Theorem 3.5 For $O,\ P \subseteq X$ and $\{O_i | i \in I\}$ a class of subsets of X. Then,

(i) $Isint\ (O)$ is a I-sops,

(ii) $Isint\ (O) \subseteq O$,

(iii) O is a I-sops if and only if $O = Isint\ (O)$,

(iv) If $O \subseteq P$, then $Isint\ (O) \subseteq Isint\ (P)$,

(v) $Isint(\bigcap_{i\in I} O_i) \subseteq \bigcap_{i\in I} Isint\ (O_i)$,

(vi) $\bigcup_{i\in I} Isint\ (O_i) \subseteq Isint\ (\bigcup_{i\in I} O_i)$.

Proof: (i) The proof is obtained from Definition 3.4.

(ii) The proof is obtained from Definition 3.4.

(iii) Since $Isint\ (O)$ is the largest I-sops contained by O, then $O = Isint\ (O)$.

(iv) From Definition 3.4.

$$Isint\ (O) = \bigcup\{U \subseteq O \mid U \text{ is } I - \text{sequentially open}\}$$

and

$$Isint\ (P) = \bigcup\{U \subseteq P \mid U \text{ is } I - \text{sequentially open}\}$$

Since $O \subseteq P$, then $Isint\ (O) \subseteq Isint\ (P)$.

(v) Let $x \in Isint(\bigcap_{i\in I} O_i)$. By Definition 3.4, there is an I-sops set U such that $x \in U \subseteq (\bigcap_{i\in I} O_i)$. For each $i \in I$, since $x \in U \subseteq (\bigcap_{i\in I} O_i) \subseteq O_i$, it follows that $x \in Isint\ (O_i)$ for each $i \in I$. Therefore, $x \in \bigcap_{i\in I} Isint\ (O_i)$, which shows that $Isint(\bigcap_{i\in I} O_i) \subseteq \bigcap_{i\in I} Isint\ (O_i)$.

(vi) It follows from (iv).

Definition 3.6 A function $f : X \to Y$ is termed I-sequentially open if every I-sops O in X maps to an I- sops in Y under f.

Theorem 3.7 A function $f : X \to Y$ is I-sequentially open if and only if $f\ (I\ int(O)\ ) \subseteq I\ int(\ f\ (O))$ for $O \subseteq X$.

Proof: Let f be I- sequentially open and $O \subseteq X$. Since $I\ int(O) \subseteq O$ then $f\ (I\ int(O)\ ) \subseteq f(O)$, implying

$I\ int(f\ (I\ int(O)\ )) \subseteq f(O)$. Since $f\ (I\ int(O))$ is I-sops, then $f\ (I\ int(O)\ ) \subseteq I\ int\ f(O)$. Conversely assuming $f\ (I\ int(O)\ ) \subseteq I\ int\ f(O)$ for every $O \subseteq X$, let $P \subseteq X$ be I- sops. Then $f\ (P) \subseteq I\ int\ f(P)$ and so $f(O)$ is I- sops.

Definition 3.8 For a subset $O \subset X$ and $x_0 \in X$, if there exists a sequence $\{x_n\}$ of points in U such that $I - lim x_n = x_0$, then x_0 is said to be in the I-sequential closure of U. The I-sequential closure of a set O is defined by

$$I\ cl(O) = \cap \{K \mid O \subseteq K \text{ and } K^c \text{ is } I - \text{sequentially open}\}$$

Theorem 3.9 If $x \in I\ cl(O)$, then for every I-sequentially open neighborhood U of x, we have $O \cap U = \emptyset$.

Proof: Let $x \in I\ cl(O)$. According to Definition 3.8,

$$I\ cl(O) = \cap \{K \mid O \subseteq K \text{ and } K^c \text{ is } I - \text{sequentially open}\}$$

Thus, for every I-scls K containing O, we have $x \in K$. If U is a I-sequentially open neighborhood of x, then $O \cap U = \emptyset$.

If instead, $O \cap U = \emptyset$, then $O \subseteq U^c$ where U^c is a I-scls and $x \notin U^c$. This leads to a contradiction. Therefore, $O \cap U = \emptyset$.

The following theorem outlines some properties of the I-closure of a set:

Theorem 3.10 Let X be a topological space, O and P be subsets of X, and $\{O_i \mid i \in I\}$ be a class of subsets of X. The following properties hold:

(i) If $O \subseteq P$, then $I\ cl(O) \subseteq I\ cl(P)$,

(ii) $I\ cl(O)$ is I-scls,

(iii) O *is* I-scls if and only if $O = I\ cl(O)$,

(iv) $\bigcup_{i\in I} I\ cl(O_i) \subseteq I\ cl(\bigcup_{i\in I} O_i)$,

(v) $I\ cl\ (\bigcap_{i\in I} O_i) \subseteq \bigcap_{i\in I} Icl\ (O_i)$.

Proof: (i) From Definition 3.8.

$$I\ cl(O) = \cap \{K \mid A \subseteq K \text{ and } K^c \text{ is } I - \text{ sequentially open}\}$$

and

$$I\ cl(O) = \cap \{K \mid A \subseteq K \text{ and } K^c \text{ is } I - \text{ sequentially open}\}.$$

Since $O \subseteq P$, then $I\ cl(O) \subseteq I\ cl(P)$.

(ii) This can be obtained directly by Definition 3.8.

(iii) The proof is clear from Definition 3.8.

(iv) Since $\subseteq O_i \subseteq \bigcup_{i\in I} O_i$ for each $i \in I$, we have that $I\,cl(O_i) \subseteq I\,cl(\bigcup_{i\in I} O_i)$. This implies that $\bigcup_{i\in I} I\,cl(O_i) \subseteq I\,cl(\bigcup_{i\in I} O_i)$.

(v) Let $x \in Icl(\bigcap_{i\in I} O_i)$. By Theorem 3.9, there exists an I-sequentially open neighborhood U of x such that $U \cap (\bigcap_{i\in I} O_i) \neq \emptyset$. For any $i \in I$, $\bigcap_{i\in I} O_i \subseteq O_i$, so $O_i \cap U \neq \emptyset$. This implies that $x \in I\,cl(O_i$ for each $i \in I$. Thus, $I\,cl\,(\bigcap_{i\in I} O_i) \subseteq \bigcap_{i\in I} Icl\,(O_i)$.

Definition 3.11 For $\emptyset \neq O \subseteq X$, O is termed I-SC if it cannot be expressed as a union of non-empty disjoint I -scls in O, X is called I -SC if there are no non-empty, disjoint I -scls of X whose union is X.

Lemma 3.12 Let $O \subseteq X$. Then,

(i) O is I-SC,

(ii) O can not be expressed as a union of non-empty disjoint I-scls in O,

(iii) O can not be expressed as a union of non-empty disjoint I-sops in O,

(iv) In O, there are no proper subsets that are both I-sops and I-scls.

Theorem 3.13 For any I-SC subset O of X ,its I-sequentially continuous image under a function $f: X \to Y$ reamins I-SC.

Proof: Let $f: X \to Y$ be an I-sequentially continuous function and let $A \subseteq X$ be an I-SC subset. Assume for contradiction that $f(A)$ is not I-SC. Then there exist two disjoint I-scls. O and O' in $f(A)$ such that $O \cup O' = f(A)$ and $O \cap O' = \emptyset$. Since f is I sequentially continuous, the preimages $f^{-1}(O)$ and $f^{-1}(O')$ are I-closed in X. Therefore, $A = f^{-1}(O) \cup f^{-1}(O')$ with $f^{-1}(O) \cap f^{-1}(O') = \emptyset$, contradicting the assumption that A is I-connected. Thus, $f(A)$ is I-SC.

Proposition 3.14 Let $P \subseteq O \subseteq X$. where O is a I-scls in X, and P is a I-scls in O. Then P is a I-scls in X.

Proof: Suppose k is a point in the I-sequential closure of P in X .Then, there exists a sequence $x = (x_n)$ in O such that $I-lim\, x_n = k \in X$. Since $P \subseteq O$, it follows that $Icl(P) \subseteq Icl(O)$, and thus $k \in Icl(O)$. As O is a I-scls in X, we have $k \in O$. Therefore, k belongs to the I-sequential closure of P in O. Since P is a I-scls in O, it implies $k \in O$. This completes the proof.

Lemma 3.15 Let O be an I-SC subset of X. If U and V are nonempty disjoint I-scls of X such that $O \subseteq U \cup V$, then either $O \subseteq U$ or $O \subseteq V$.

Proof: Suppose $O \nsubseteq U$ and $O \nsubseteq V$ .Then there exist $x \in O$ such that $x \notin U$, and similarly, $x \in V$. Thus, $O \cap V$ and $O \cap U$ are both non-empty, contradicting the assumption. Hence, either $O \subseteq U$ or $O \subseteq V$.

Lemma 3.16 Let $O \subseteq \Upsilon$, and let U be an I-sops and I-scls subset of X. If O is I-SC, then either $O \subseteq U$ or $O \subseteq X \setminus U$.

Proof: If $U = \emptyset$ or $U = X$, the proof is trivial. Otherwise, $O \subseteq U \cup (X \setminus U)$, and by Lemma 3.14, either $O \subseteq U$ or $O \subseteq X \setminus U$.

Corollary 3.17 If O is an I-SC subset of X, then so is $Icl(O)$.

4. CONCLUSION

In this study, the focus was on the concepts of I-sops and I-scls in ideal topological spaces. Firstly, the properties of these sets were provided. Discussions were held on their interiors and closures. Subsequently, using these sets, the concept of connectedness, an important notion in topology, was reintroduced for ideal topological spaces. Properties and theoretical information regarding this concept, called I-sequential connectedness, were presented. With this study, concepts and topics in topology can be examined in ideal topological spaces. Further research can explore new topics and make comparisons using the concept of convergence in sequences.

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AUTHOR'S CONTRIBUTIONS

The authors contributed equally.

CONFLICTS OF INTEREST

The authors declare no conflict of interest.

RESEARCH AND PUBLICATION ETHICS

The authors declare that this study complies with Research and Publication Ethics.

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