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NUMERICAL SOLUTIONS TO THE STOCHASTIC SYSTEMS WITH FRACTIONAL OPERATORS

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Abstract

Original scientific paper

Fractional-stochastic differential equations are widely used tools to simulate a wide - range of engineering and scientific phenomena. In this paper, the applicability of the approach of indeterminate coefficients to various fractional-stochastic models is examined. These models have a fractional white noise term and are mostly produced by fractional-order derivative operators. We also investigate applications of a polynomial chaos algorithm to stochastic Lotka-Volterra and Benney systems. Fractional-stochastic equations are entirely novel systems that have the potential to function as models for a wide range of scientific and engineering phenomena. It is noted that fractional-order systems with uncertainty or a noise term can benefit from the effective use of Galerkin-type approaches in this article.

Keywords: *Galerkin methods, numerical solutions, stochastic systems with fractional operators.*

KESİRLİ MERTEBELİ DİFERANSİYEL DENKLEMLERİN YENİ SAYISAL ÇÖZÜMLERİ

Özet

Orijinal bilimsel makale

Kesirli stokastik diferansiyel denklemler, çok çeşitli mühendislik ve bilimsel olguları simüle etmek için yaygın olarak kullanılan araçlardır. Bu makalede, belirsiz katsayılar yaklaşımının çeşitli kesirli stokastik modellere uygulanabilirliği incelenmiştir. Bu modeller kesirli beyaz gürültü terimine sahiptir ve çoğunlukla kesirli dereceli türev operatörleri tarafından üretilir. Ayrıca polinom kaos algoritmasının stokastik Lotka-Volterra ve Benney sistemlerine uygulamalarını da araştırıyoruz. Kesirli stokastik denklemler, çok çeşitli bilimsel ve mühendislik problemleri için model olarak işlev görme potansiyeline sahip tamamen yeni sistemlerdir. Bu makalede Galerkin tipi yaklaşımların etkin kullanımından ve belirsizlik veya gürültü terimi içeren kesirli dereceli sistemlere uygulanabilirliği araştırılmıştır.

Anahtar Kelimeler: *Galerkin yöntemleri, sayısal çözümler, skokastik sistemler, kesirli operatörler.*

1 Introduction

Stochastic differential equations and fractional order derivative operators, {2,4,5,11,12,13,14,15} are invaluable tools for applications in many different areas of science and engineering. For example, they are used in monitoring chemical processes operating under uncertainty, in image processing such as feature extraction and image segmentation, in analyzing, designing, and testing electronic systems, and in many other engineering applications such as big data analysis, robotics, machine learning, and artificial intelligence.

A fractional-stochastic differential equation has some terms of fractional-order derivative/integral operators, certain deterministic operators, noise (expressed as a derivative of the Wiener process), and fractional-Brownian motion. There may be more terms in these kinds of equations, including Levy-type noise or jump. Some of the well-known {1,3} fractional-order operators with nonsingular kernels and integral properties, such as Riemann-Liouville, Caputo, Grünwald–Letnikov, and Atangana-Baleanu, are more efficient than deterministic maps. Ito's formula is primarily used to obtain exact solutions of fractional-stochastic differential equations, but these solutions are only obtained in a few specific instances. Thus, among the most important and helpful numerical solution methods in computational mathematics are those using Euler's type, finite differences, and indeterminate coefficients (Galerkin, Least-square, Collocation, Wavelets, and Finite Elements).

The primary goal of the current research work is to examine how some finite-element, polynomial chaos and Galerkin methods can be applied to specific and original differential equations. Deterministic derivative operators, fractional-order derivative operators in the Caputo and

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Riemann-Liouville (RL) sense, fractional-Brownian motion, standard Brownian motion, and fractional white noise are the fractional-stochastic systems of equations that are examined in this work. Not a single equation or system taken into consideration in this study has ever been researched in the past. According to this perspective, the current work represents a fresh and innovative addition to science. Investigating the suitability of various indeterminate coefficient approaches for fractional-stochastic models renders this work novel and highly valuable for computer science researchers.

The novelty of this research study can be presented as follows: First of all, both research topics and tools, fractional-order stochastic differential equations and fractional Brownian motion are the tools that can be used in the modeling of many different engineering and scientific phenomena in physics and biology in a highly efficient manner. In this paper, we employ both tools and apply them to the models not considered in the literature in the way we study. Fractional Brownian motion is a very helpful tool for researchers because of its long-range dependence, correlated time increments, and Hurst parameter features. We address the conjunction of fractional operators with white noise and fractional Brownian motion; these tools' combination and application to scientific and engineering problems are still in their infancy. From these points of view, the present study is going to be useful for the researchers in these areas.

2 Fundamentals of Fractional Operators

In recent years, fractional-order calculus, often known as differential equations, has become increasingly important in applied and computational mathematics. These equations are produced by fractional-order operators, such as derivative and integral operators. They worked in various scientific fields, including engineering, physics, economics, and mathematics. Because fractional-order operators may compute non-locally and account for historical impacts, they are more effective and preferred in applications.

Suppose that $g:(0,\infty)\mapsto\mathbb{R}$ is a function and fractional-order operator with the order β for g is defined as

$$J^{\beta}g(s)=\frac{1}{\Gamma(\beta)}\int_0^t(s-\sigma)^{\beta-1}\,g(\sigma)d\sigma,$$

inhere $\Gamma(\cdot)$ denotes Gamma function.

For $(m\in\mathbb{N})$, Caputo-type derivative operator of order β, $m-1<\beta<m$, of $g(s)$ is given by

$$D^{\beta}g(s)=J^{m-\beta}\,\frac{d^m}{dt^m}g(s).$$

The Riemann–Liouville (RL) type integral, $I_x^{\alpha}f(x)$ and derivatives $R_x^{\alpha}f(x)$ of a function, say $f(x)$, of order α is:

$$I_x^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-\rho)^{\alpha-1}f(\rho)d\rho,\quad x\in[a,b].$$

It is obvious that:

$$I_x^{\beta}\left(I_x^{\alpha}f(x)\right)=I_x^{\beta+\alpha}f(x)\text{ and }\frac{d}{dx}I_x^{1+\alpha}=I_x^{\alpha}f(x).$$

$$R_x^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_a^x\frac{f(\rho)d\rho}{(x-\rho)^{\alpha+1-n}}\quad\text{for}\quad n-1<\alpha\le n.$$

For $n=2$, i.e. for $1<\alpha\le 2$, one has

$$R_x^{\alpha}f(x)=\frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_a^x\frac{f(\rho)d\rho}{(x-\rho)^{\alpha-1}}.$$

3 The Galerkin Method

One of the unique varieties of procedures for unknown coefficients is the Galerkin approach. Next, we go over the basic concepts of the Galerkin technique in brief.

Let us consider the differential equation (which can be hybrid, fractional, deterministic, or stochastic).

$$Fu(x)=k(x)$$

where F is a derivative operator, functions $u(x)$ and $k(x)$ are given functions. Write $u(x)$ as

$$u(x)=\sum_{i=1}^{n}d_i\,\varphi_i(x)$$

in which $\varphi_i(x)$ is known as the coordinate function and the coefficients, d_i are constants or functions to be determined. Let us define the so-called residual term $R(x)$ as

$$R(x):=Fu(x)-k(x)$$

and define the following operator

$$I_j(d_1,d_2,\dots,d_n):=\int_a^b W_j(x)R(x)dx=0.$$

Choosing weight function $W_j(x)$ in this equation as $W_j(x):=\varphi_i(x)$, the Galerkin method (one of the undetermined coefficients methods) becomes the name of the resulting technique.

4 Brownian Motion

Brownian motion (or Wiener process) is a stochastic process $(W_x)_{x\ge0}$ defined on a probability space $(\mathcal{X},\mathcal{F},\mathcal{P})$ with:

- $W_0=0$,
- the function $x\mapsto W_x$ is a continuous function with probability 1,
- the increments $W_{x+n}-W_x$ have a normal distribution, $\mathcal{N}(0,n)$.

Independent increments mean $W_x - W_l$ and $W_k - W_m$ are independent random variables for $0 \le l \le x \le m \le k$. $W(x)$ satisfies that:

$$\frac{dW(x)}{dx} \approx \frac{d\widehat{W(x)}}{dx} = \sum_{k=1}^{n} \rho_k \, \delta_k(x),$$

$$\rho_k = \frac{1}{\sigma}\int_I \delta_k(x) dW(x)$$

$$\in \mathcal{N}\left(0, \frac{1}{\sigma}\right), \text{ that is a standard normal random variable}$$

$$dW(x) \coloneqq dW(x_{i+1}) - dW(x_i), \text{ where}$$

$I = \{I_k | I_k := [x_{k-1}, x_k),\ x_k - x_{k-1} = \sigma,\ x_0 = 0,\ x_n = 1\}$

If f is a bounded function, it is not hard to see that

$$E\left(\int_I f\left(dW(x) - d\widehat{W(x)}\right)\right) = 0.$$

Another important and highly helpful method in science, engineering, and finance modeling a wide range of events is fractional-Brownian motion or fBm. It is a very useful tool for scientists because of its long-range dependence, correlated time increments, and Hurst parameter features. The definition and intriguing characteristics of fBm, for instance, can be found in {10} for interested readers. The literature has a wide variety of fBm kinds. We think about

$$B^H(s,x) = \sum_{k=1}^{\infty} \sqrt{\mu_k}\, e_k(x)\alpha_k(s), \quad \sum_i \mu_i < \infty,$$
$$e_k(x) = \sqrt{2}sin(k\pi x),\ k = 1,2,\ldots,$$
$$B^H(s_n,x) - B^H(s_{n-1},x) = \sum_{k=1}^{\infty} \sqrt{\mu_k}\, e_k(x)\big(\alpha_k(s_n) - \alpha_k(s_{n-1})\big),$$
$$\alpha_k(s_n) - \alpha_k(s_{n-1}) = \sqrt{\Delta_t}\,\zeta_{k,n}, \quad \zeta \in \mathcal{N}(0,1).$$

In the following part, we present and analyze the first fractional-stochastic differential model. We deal with numerical solutions of a partial differential equation (PDE) that is fractionally stochastic and contains various derivatives of an unknown function of s and x, as well as a Levy noise term, fractional white noise, and a second-order time derivative. We examine a unique differential problem that has not been approached using finite element methods in previous research. The goal of analyzing such an equation is to demonstrate how original the equation is, as well as how helpful these kinds of equations are in representing a wide range of phenomena in science and engineering. Numerical solutions of the following equation are of interest to us:

$$u_{ss}(s,x) - u_{xx}(s,x) - u(s,x) + dB^H(s) - \frac{\partial}{\partial x}\left[f\big(u(s,x)\big)\frac{\partial u(s,x)}{\partial x}\right] = g(s,x) \quad (1)$$

where $(s,x) \in \Omega := [0,1] \times [0,1]$ and $u(s,x) = 0$ in $\partial\Omega$. Weak form of ([maineq2]):

$$\langle u_{ss}(s,x), w\rangle - \langle u_{xx}(s,x), w\rangle - \langle u(s,x), w\rangle + \langle dB^H(s,x), w\rangle \left\langle -\frac{\partial}{\partial x}\left[f\big(u(s,x)\big)\frac{\partial u(s,x)}{\partial x}\right], w\right\rangle = \langle g(s,x), w\rangle \quad (2)$$

After using a finite element approach to solve (1), we finally arrive at an element matrix equation that can be solved using both an implicit numerical method and a finite difference. Let us now write (2) in variational form first. To begin doing this, let's first express the unknown function $u(s,x)$as

$$u(s,x) := \sum_{i=1}^{M} a_i(s)\varphi_i(x)$$

where $\varphi_i(x)$ is a coordinate function.
We write $u(s,x)$ in the equation (2) using the variational formulation technique. We then multiply each term in (2) by $\varphi_j(x)$, and integrate the resulting function on the boundaries of x. Consequently, we get that

$$\sum_{i=1}^{M} \ddot{a}_i(s)\int_0^1 \varphi_i(x)\varphi_j(x)dx - \sum_{i=1}^{M} a_i(s)\int_0^1 \frac{d^2\varphi_i(x)}{dx^2}\varphi_j(x)dx - \sum_{i=1}^{M} a_i(s)\int_0^1 \varphi_i(x)\varphi_j(x)dx + \int_0^1 \varphi_j(x)dB^H(s,x)dx$$
$$\int_0^1 \frac{\partial}{\partial x}\left[f\left(\sum_{i=1}^{M} a_i(s)\varphi_i(x)\right)\sum_{i=1}^{M} a_i(s)\frac{d\varphi_i(x)}{dx}\right]\varphi_j(x)dx = \int_0^1 g(s,x)\varphi_j(x)dx \quad (3)$$

Our objective is to formulate the equation (3) using a few matrices. To that end, allow us to define the following matrices:
Define $K_{M\times M}$ with entries K_{ij} as

$$K_{ij} = \int_0^1 \varphi_i(x)\varphi_j(x)dx.$$

Hence, the first integral in (3) can be written

$$K^T\ddot{a}$$

where a is a $M \times 1$ vector with entries a_i, $i = 1,2,\ldots,N$. Using the boundary conditions and integration by parts method, the second integral is written as

$$N^T a$$

where $N_{M\times M}$ is a matrix.
Now, let us write the last two terms in (3) as follows:

$$\sum_{i=1}^{M} a_i\,(s)\int_0^1 f\left(\sum_{i=1}^{M} a_i\,(s)\varphi_i(x)\right)\frac{d\varphi_i(x)}{dx}\frac{d\varphi_j(x)}{dx}dx = \langle g(s,x), w\rangle \quad (4)$$

Concerning the unknown $a_i(s)$ in equation ([togethers]), it is evident that we have a nonlinear system of equations. Consequently, (4) can be expressed in terms of a few matrices as

$$L(a)a = c$$

where $L_{M\times M}$ is a matrix with:

$$L_{ij} = \int_0^1 \left[f\left(\sum_{i=1}^{M} a_i\,(s)\varphi_i(x)\right)\frac{d\varphi_i(x)}{dx}\frac{d\varphi_j(x)}{dx}\right]dx$$

and c is a $M\times 1$ matrix with entries $c_i = \int_0^1 g\,(s,x)\varphi_j(x)dx$.

We express the fourth term, using the fBm notations that were discussed in the preceding section. The equality

$$\int_0^1 d\,B^H(s,x)\varphi_j(x)dx = \sum_{k=1}^{M}\sqrt{\mu_k}\,\alpha_k(s)\int_0^1 e_k\,(x)\varphi_j(x)dx,$$

can be written as a vector, say d, with the size of $M\times 1$. As a result, after correctly organizing these matrices in the equation (4), we get the matrix equation:

$$K^T\ddot{a} - (N^T + K^T)a - L(a)a = c - d \quad (5)$$

Now, we may express the temporal derivative using the finite difference approximation. Therefore, we write

$$K^T\left[\frac{a_{k+1} - 2a_k + a_{k-1}}{\Delta t^2}\right] - N^T a_k - K^T a_k - L(a_k)a_k = c - d$$

from which we obtain that

$$a_{k+1} = [(K^T)^{-1}(N^T a_k + K^T a_k + L(a_k)a_k + c - d)]\Delta t^2 + 2a_k - a_{k-1}$$

In the following section, we investigate numerical solutions to fractional-stochastic differential equations using the Riemann-Liouville interpretation of the fractional derivative operator.

5 Fractional-Stochastic Equation

Consider

$R_x^{1+\alpha}u(x) - u_{xxx}(x) + dW(x) = g(x),\ x\in[0,1]$ (6)

with $u(0) := 0,$ and $u(1) := k \in \mathbf{R} - 0,$ where $R_x^{1+\alpha}u(x)$ is the Riemann-Liouville type derivative of $u(x)$. Let's use the above-described Galerkin technique methodology to apply the Galerkin method to Eq. (6):

$$\int_0^1 R_x^{1+\alpha}\,u(x)\varphi_j(x)dx - \int_0^1 u_{xxx}\,(x)\varphi_j(x)dx + \int_0^1 d\,W(x)\varphi_j(x)dx = \int_0^1 g\,(x)\varphi_j(x)dx \quad (7)$$

where $\varphi_j(x)$ is nodal based function defined on Ω. Defining

$$u = \sum_{k=0}^{M} c_k\,\varphi_k(x)$$

Let

$$\Delta := \frac{1}{n}$$

$\varphi_j(x)$ is an appropriate base function. Using integration by part in the second term in (7), we get:

$$\sum_{k=0}^{M} c_k\int_0^1 R_x^{\alpha}\,\varphi_k(x)\varphi_j(x)dx + \sum_{k=0}^{M} c_k\int_0^1 \frac{d^2\varphi_k(x)}{dx^2}\frac{d\varphi_j(x)}{dx}dx + \sum_{k=}^{M}\int_0^1 \rho_k\,\delta_k(x)\varphi_k(x)dx = \int_0^1 g\,(x)\varphi_j(x)dx$$

This equation's discretization is provided as follows:

$$\frac{1}{\Delta}\left[\sum_{k=0}^{M} c_k\int_{x_{l-1}}^{x_l} R_x^{\alpha}\,\varphi_k(x)dx - \sum_{k=0}^{M} c_k\int_{x_l}^{x_{l+1}} R_x^{\alpha}\,\varphi_j(x)dx\right] + \sum_{k=0}^{M} c_k\int_0^1 \frac{d^2\varphi_k(x)}{dx^2}\frac{d\varphi_j(x)}{dx}dx + \sum_{k=1}^{M}\int_0^1 \rho_k\,\delta_k(x)\varphi_k(x)dx + \sum_{k=}^{M}\int_0^1 \rho_k\,\delta_k(x)\varphi_k(x)dx = \int_0^1 g\,(x)\varphi_j(x)dx$$

This equation can be expressed as a matrix equation or in terms of matrices as follows:

$$(F^T - L^T)c + M - G = 0,$$

in which,

$$M = \sum_{k=1}^{M}\int_0^1 \rho_k\,\delta_k(x)\varphi_k(x)dx$$

$$L_{ij} = \int_0^1 \frac{d^2\varphi_k(x)}{dx^2}\frac{d\varphi_j(x)}{dx}dx$$

$$G_j = \int_0^1 g\,(x)\varphi_j(x)dx$$

The application of the Galerkin method to a system of fractional-order differential equations will be examined in the following section. Since we had investigated how noise behaves in fractional-order differential equations in the earlier instances, we do not add a noise term, therefore these equations are somewhat deterministic fractional differential equations.

6 A Fractional-Stochastic System of PDEs

Consider

$$\begin{aligned} D_s^\alpha u(s,x) &= \beta u_{xx}(s,x) + v_x(s,x) + h(s,x) \\ D_s^\alpha v(s,x) &= v_{xx}(s,x) + \tau u_x(s,x) \end{aligned} \quad (8)$$

where α, β, τ are positive parameters, $x \in [0, A]$, $A > 0$, $0 \le t$, with

$$u(0,x) = u(s,0) = v(s,0)$$

$$v(0,x) = u(s,A) = v(s,A) = 0, \;\; x \in [0,A]$$

$$\frac{\partial u(s,x)}{\partial x} = \frac{\partial v(s,x)}{\partial x} = 0, \;\; \text{at } x = 0,$$

and $x = \alpha, \; t \ge 0$.

These kinds of systems of partial differential equations can simulate a variety of physics, chemical, and engineering problems. We chose to explore this system of equations since it is an original differential equation, and systems of differential equations are used to model many scientific phenomena.

The system is transformed into a linear system of finite element matrix equations using the FEM, a kind of weighted residual approach. The solutions to these matrix equations are found by highly efficient computing methods.

This system of fractional order differential equations is solved using a Galerkin type me method as follows:

$$\begin{aligned} u(s,x) &\approx \hat{u}(s,x) = N(x)U^e(s) \\ v(s,x) &\approx \hat{v}(s,x) = N(x)V^e(s) \end{aligned}$$

where $N(x)$ is a row vector made up of what are known as basis functions on a L-length space element. Note that in the weighted residual method, the basis functions we use in this study serve as the weight functions. As a result, we employ a finite element method—a kind of weighted residual technique—in the following section.

Taking note of them, we can see that the approximate system of equations (8) can be expressed as follows:

$$\begin{aligned} D_s^\alpha \hat{u}(s,x) &= \beta\hat{u}_{xx}(s,x) + \hat{v}_x(s,x) + h(s,x) \\ D_t s^\alpha \hat{v}(s,x) &= \hat{v}_{xx}(s,x) + \tau\hat{u}_x(s,x) \end{aligned} \quad (9)$$

We investigate the one-space element Γ_e, which is an interval since we study on an interval, which is a one-dimensional space. Considering this, the first equation in the system (9) can be written as

$$D_s^\alpha \hat{u}(s,x) = \beta\hat{u}_{xx}(s,x) + \hat{v}_x(s,x) + h(s,x)$$

Thus, we have

$$\int_{\Gamma_e} N^T(x)\begin{bmatrix} N(x)D_s^\alpha U^e(s) \\ -\beta\dfrac{d}{dx}\left(\dfrac{dN(x)}{dx}U^e(s)\right) \\ +\dfrac{dN(x)}{dx}V^e(s) \\ +h(s,x) \end{bmatrix} d\Gamma_e = \{0\},$$

Using the integration-by-parts method, we have

$$\begin{aligned} &\int_{\Gamma_e} [\,N^T(x)N(x)D_s^\alpha U^e(s) \\ &-\beta\frac{dN(x)}{dx}\frac{dN^T(x)}{dx}U^e(s)N^T(x)\frac{dN(x)}{dx}V^e(s) \\ &+N^T(x)h(s,x)]d\Gamma_e - \left[\beta N^T(x)\frac{N(x)}{dx}U^e(s)\right]_{\partial\Gamma} = \{0\}, \end{aligned}$$

by $N(x) \equiv 0$ on $\partial\Gamma$, we get

$$\int_{\Gamma_e}\begin{bmatrix} N^T(x)N(x)D_s^\alpha U^e(s) \\ -\beta\dfrac{dN(x)}{dx}\dfrac{dN^T(x)}{dx}U^e(s) \\ +N^T(x)\dfrac{dN(x)}{dx}V^e(s) - N^T(x)h(s,x) \end{bmatrix} d\Gamma_e = \{0\},$$

(10)

We write the equation (10) via matrices as

$${}^eD_s^\alpha U^e(s) + ([B(x)]^e)U^e(s) + ([C(x)]^e)V^e(s) = H^e(s,x) \quad (11)$$

where

$$\begin{aligned} [A(x)]^e &= \int_{\Gamma_e} N^T(x)N(x)d\Gamma_e \\ [B(x)]^e &= -\int_{\Gamma_e}\beta\frac{dN^T(x)}{dx}\frac{dN(x)}{dx}d\Gamma_e \\ [C(x)]^e &= \int_{\Gamma_e} N^T(x)\frac{dN(x)}{dx}d\Gamma_e \\ H^e(s,x) &= N^T(x)h(s,x) \end{aligned}$$

Now, we express the second equation in the system (9)

$$D_s^\alpha v(s,x) = v_{xx}(s,x) + \tau u_x(s,x)$$

as:

$$\int_{\Gamma_e} N^T(x)\begin{bmatrix} N(x)D_s^{\alpha}V^e(s) \\ -\frac{d}{dx}\left(\frac{dN(x)}{dx}V^e(s)\right) \\ +\tau\frac{dN(x)}{dx}U^e(s)\end{bmatrix}d\Gamma_e = \{0\} \quad (12)$$

The equation (12) can be restated as

$$\int_{\Gamma_e}[\,N^T(x)N(x)D_s^{\alpha}V^e(s) - \frac{dN^T(x)}{dx}\frac{dN(x)}{dx}V^e(s) +$$

$$\tau N^T(x)\frac{dN(x)}{dx}U^e(s)]d\Gamma_e - \left[N^T(x)\frac{dN(x)}{dx}V^e(s)\right]_{\partial\Gamma_e} = \{0\}$$

by the boundary conditions, we get

$$\int_{\Gamma_e}[\,N^T(x)N(x)D_s^{\alpha}V^e(s) - \frac{dN^T(x)}{dx}\frac{dN(x)}{dx}V^e(s) +$$

$$\tau N^T(x)\frac{dN(x)}{dx}U^e(s)]d\Gamma_e = \{0\}$$

Likewise, we have:

$${}^eD_s^{\alpha}V^e(s) + \left(\frac{1}{\beta}[B(x)]^eV^e(s) = [C(x)]^eU^e(s)\right) \quad (13)$$

Now, we can rewrite the equations (11) and (13) by localizing the element matrices as:

$${}^eD_s^{\alpha}U^e(s) + ([B(x)]^e)U^e(s) + ([C(x)]^e)V^e(s) = H^e(s,x)$$

$${}^eD_s^{\alpha}V^e(s) + \left(\frac{1}{\beta}[B(x)]^eV^e(s) = [C(x)]^eU^e(s)\right)$$

In the following part, we apply the polynomial chaos method to two nonlinear systems of stochastic ordinary differential equations: the Lotka-Volterra system and the Banney equation, a nonlinear stochastic differential equation.

7 Polynomial Chaos Method

Because they can accurately simulate systems with uncertainty, randomness, and noise, stochastic differential equations are important in both engineering and economics. This section examines the suitability of the polynomial chaos (PC) approach for several partial differential equations and stochastic nonlinear predator-prey scenarios. Specifically, we are interested in the approximate solutions of the Benney and stochastic Lotka-Volterra equations. A particular instance of indeterminate coefficients or the Galerkin method is the polynomial chaos approach. We provide approximate solutions for stochastic systems and show simulations to illustrate the impact of uncertainties on the systems. This is the first study in the history of research to demonstrate how the polynomial chaos approach may be used to the Lotka-Volterra system and the Benney equation.

The Lotka-Volterra (LV) equations are a couple of first-order, non-linear differential equation systems that are mostly used in mathematical biology, namely in the modeling of predator-prey model dynamics (see, for example, 6,7,8,9). The Lotka-Volterra equations are regarded as random or stochastic processes. We present a novel method for numerical solutions of Lotka-Volterra (LV) equations with some uncertainty present in the system, based on polynomial chaos (PC) expansions. One of the most effective methods for perturbing a stochastic differential equation into a deterministic system of equations is the PC technique.

Numerous methods have been proposed in the literature to solve stochastic linear partial differential equations. As an example, consider the stochastic LV model employing the semi-martingale technique at Klebaner et al. They establish a huge deviation principle and derive a bound for the asymptotic time to the prey population's extinction.

The polynomial chaos expansion approach is then briefly reviewed, and stochastic Lotka-Volterra models and Benney equations are introduced. We then address the numerical solutions of these systems by using the PC method on them.

8 Stochastic Galerkin Method

Given a probability space denoted by $(\Omega, \mathcal{A}, \mathcal{P})$, we can investigate the following general time-dependent stochastic partial differential equation with random coefficients:

$$\begin{aligned} \frac{\partial u}{\partial s} + \Gamma(s,x,\nu;u) &= f,\ (s,x) \in \mathcal{D}\times[0,T_f] \\ \mathcal{C}(s,x,\nu;u) &= g,(s,x) \in \partial\mathcal{D}\times[0,T_f] \\ \mathcal{K}(0,x,\nu;u) &= h,\quad x \in \mathcal{D}, \end{aligned}$$

where the coordinates are ν −random or uncertain, s −time, and x −space. This system's solution could be expressed as

$$u(s,x,\nu): \mathcal{D}\times[0,T_f]\times\Omega \to \mathbb{R}$$
$$\mathcal{D}\subset\mathbb{R}^D,\ D = 1,2,3.$$

We need to compute $u(s,x,\nu)$ to solve this system. First, let's presume that:

$$a(x,y) \equiv a\big(x,y_1(\nu),\dots,y_d(\nu)\big).$$

Then,

$$u\big(s,x,y_1(\nu),\dots,y_d(\nu)\big): \mathcal{D}\times[0,T]\times\prod_{i=1}^{d}\Gamma_i \to \mathbb{R}$$

$$y_i: \Omega \to \Gamma_i$$

Some common techniques in the literature to calculate $u(s,x,\nu)$ include the Galerkin, finite elements, wavelets, collocation, and least-squares approaches for indeterminate coefficients. We employ the stochastic Galerkin approach, which is best explained as follows: Given the random space

$$\mathcal{S} \equiv L_2\big(\mathcal{L},\mathcal{P}_y\big),$$

i. Construct an approximate space.

$$\mathcal{S}_p \equiv Span\{\psi_0(y), \psi_1(y), \dots, \psi_p(y)\} \subset \mathcal{S}$$

ii. Show the ambiguities as

$$\hat{a}(x,y) = \sum_i a_i\,(x)\psi_i(y)$$

iii. Approximate the solution

$$\hat{u}(x,t,y) = \sum_i u_i\,(x,t)\psi_i(y)$$

- Calculate the coefficients, $u_i(x,t)$, by Galerkin projection.

It is possible to describe any random variable $u(v) \in L_2(\Omega,\mathcal{P})$ as

$$\begin{aligned} u(v) &= u_0 H_0 \\ &+ \sum_{i=1}^{\infty} u_i\, H_1\big(y_i(v)\big) \\ &+ \sum_{i=1}^{\infty}\sum_{k=1}^{i} u_{ik}\, H_2\big(y_i(v), y_k(v)\big) \\ &+ \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sum_{j=1}^{k} u_{ikj}\, H_3\left(y_i(v), y_k(v), y_j(v)\right) + \cdots \end{aligned}$$

The multi-dimensional Hermite polynomial is denoted by H_n. Alternatively expressed as

$$u(y) \equiv u\big(y_1(v), \dots, y_d(v)\big),$$

the approximation can alternatively be expressed as finite-dimensional uncertainty. In plainer language, we say that

$$u\big(y(v)\big) = \sum_{i=0}^{\infty} u_i\,\psi_i\big(y(v)\big) = \sum_{i=0}^{\infty} u_i\,\psi_i(y)$$

where $\psi_i(y)$ is a multi-dimensional Hermite polynomial. The basis $\{\psi_i(y)\}_{i=0}^{\infty}$ is a complete basis in $L_2\big(\mathbb{R}^d, \mathcal{P}_y\big)$, i.e.,

$$\mathcal{S} \;=\; Span\{\psi_0(y), \psi_1(y), \dots\} = L_2\big(\mathbb{R}^d, \mathcal{P}_y\big)$$

A finite order approximation

$$\hat{u}(y) \;=\; \sum_{i=0}^{P} u_i\,\psi_i(y)$$

corresponds to a p-th (total) order approximation of $u(y)$ in

$$\mathcal{S}_p \;=\; Span\{\psi_0(y), \psi_1(y), \dots, \psi_p(y)\} \subset \mathcal{S}$$

where the generalized Fourier coefficients u_i are obtained by

$$u_i = \langle u\psi_i\rangle / \langle\psi_i^2\rangle$$

The expansion converges in a mean-square sense

$$\lim_{p\to\infty}\langle (u-\hat{u})^2\rangle = 0$$

In terms of computation, the representation

$$\langle f\rangle \;=\; \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty} f\,(\xi)e^{-\frac{|\xi|^2}{2}}d\xi,$$

By illustrating the resolution $u(s,x,w)$ as

$$u(s,x,v) \;=\; \sum_{i=0}^{P} u_i(s,x)\psi_i(\xi)$$

Extension of the solution

$$\tilde{u}(x,t,y) \;=\; \sum_{j=0}^{P} u_j(x,t)\psi_j(y)$$

Galerkin projection on $\mathcal{S}_p$:

$$\begin{aligned} \langle\frac{\partial\tilde{u}}{\partial s} + \Gamma(x,t,\cdot;\tilde{u}) - f, \psi_i\rangle &= 0 \quad i = 0,1,\dots,P \\ \langle\mathcal{C}(x,t,\cdot;\hat{u}) - g, \psi_i\rangle &= 0 \quad i = 0,1,\dots,P \\ \langle\mathcal{K}(x,0,\cdot;\hat{u}) - h, \psi_i\rangle &= 0 \quad i = 0,1,\dots,P. \end{aligned}$$

9 Solutions of stochastic Lotka-Volterra system

The Lotka-Volterra system:

$$\begin{aligned} u'^{(s,x,v)} &= u(s,x,v)\big(a - bv(s,x,v)\big) \\ &= u(s,x,v)a - bu(s,x,v)v(s,x,v), \end{aligned}$$

$$\begin{aligned} v'(s,x,v) = \; & v(s,x,v)\big(-c + du(s,x,v)\big) = \\ & -cv(s,x,v) + du(s,x,v)v(s,x,v), \end{aligned} \qquad (14)$$

where $a,\ b,\ c,\ d$ are constants.

We have that:

$$\begin{aligned} u(s,x,v) &= \sum_{i=0}^{M} u_i(s,x)\psi_i\big(\xi(v)\big), v(s,x,v) \\ &= \sum_{j=0}^{M} v_j(s,x)\psi_j\big(\xi(v)\big) \end{aligned}$$

$$a(v) \;=\; \sum_{i=0}^{M} a_i\psi_i\big(\xi(v)\big), \quad b(v) \;=\; \sum_{i=0}^{M} b_i\psi_i\big(\xi(v)\big),$$

$$c(v) \;=\; \sum_{i=0}^{M} c_i\psi_i\big(\xi(v)\big), \quad d(v) \;=\; \sum_{i=0}^{M} d_i\psi_i\big(\xi(v)\big)$$

Writing these in (14), we obtain the following system:

$$\sum_{i=0}^{M} \frac{\partial u_i}{\partial s}\psi_i = \sum_{i=0}^{M}\sum_{j=0}^{M} u_i(s,x)\psi_i a_j\psi_j - \sum_{j=0}^{M}\sum_{i=0}^{M}\sum_{k=0}^{M} b_j\psi_j u_i(s,x)\psi_i v_k(s,x)\psi_k$$

$$\sum_{j=0}^{M} \frac{\partial v_j}{\partial s}\psi_j = -\sum_{i=0}^{M}\sum_{j=0}^{M} c_i\psi_i v_j(s,x)\psi_j + \sum_{j=0}^{M}\sum_{i=0}^{M}\sum_{k=0}^{M} d_j\psi_j u_i(s,x)\psi_i v_k(s,x)\psi_k$$

$$\frac{\partial u_m}{\partial s} = \sum_{i=0}^{M}\sum_{j=0}^{M} \frac{<\psi_m\psi_i\psi_j>}{<\psi_m^2>} u_i(s,x)a_j - \sum_{j=0}^{M}\sum_{i=0}^{M}\sum_{k=0}^{M} \frac{<\psi_m\psi_j\psi_i\psi_k>}{<\psi_m^2>} b_j u_i(s,x)v_k(s,x)$$

$$\frac{\partial v_m}{\partial s} = -\sum_{i=0}^{M}\sum_{j=0}^{M} \frac{<\psi_m\psi_i\psi_j>}{<\psi_m^2>} c_i v_j(s,x) + \sum_{j=0}^{M}\sum_{i=0}^{M}\sum_{k=0}^{M} \frac{<\psi_m\psi_j\psi_i\psi_k>}{<\psi_m^2>} d_j u_i(s,x)v_k(s,x)$$

$$\frac{\partial u_m}{\partial s} = \frac{1}{<\psi_m^2>}\sum_{i=0}^{M}\sum_{j=0}^{M} u_i(s,x)a_j e_{mij} - \frac{1}{<\psi_m^2>}\sum_{j=0}^{M}\sum_{i=0}^{M}\sum_{k=0}^{M} b_j u_i(s,x)v_k(s,x)e_{mjik}$$

$$\frac{\partial v_m}{\partial s} = -\frac{1}{<\psi_m^2>}\sum_{i=0}^{M}\sum_{j=0}^{M} c_i v_j(s,x)e_{mij} + \frac{1}{<\psi_m^2>}\sum_{j=0}^{M}\sum_{i=0}^{M}\sum_{k=0}^{M} d_j u_i(s,x)v_k(s,x)e_{mjik}$$

where $m = 0,1,2,\dots,N$, $e_{mij} = <\psi_m\psi_i\psi_j>$, $e_{mjik} = <\psi_m\psi_j\psi_i\psi_k>$.

10 Numerical Solution of Stochastic Benney Equation

The Benney equation is defined as follows:

$$u_s(s,x) + (u^n)_x(s,x) + u_{xx}(s,x) + \mu u_{xxx}(s,x) + u_{xxxx}(s,x) = 0, \qquad (15)$$

where μ is a constant term. We will take $n = 2$ and consider the stochastic Benney equation. Stochastic Benney equation (15) can be expressed as:

$$u_s(s,x,v) + 2u(s,x,v)u_x(s,x,v) + u_{xx}(s,x,v) + \mu(s,v)u_{xxx}(s,x,v) + u_{xxxx}(s,x,v) = 0,$$

with $u(0,x,v) = u^0(x,v)$. By writing

$$u(s,x,v) = \sum_{i=0}^{M} u_i(s,x)\psi_i\big(\xi(v)\big),\mu(v) = \sum_{i=0}^{M} \mu_i\psi_i\big(\xi(v)\big),$$

$$u(s,x,v) = \sum_{i=0}^{M} u_i(s,x)\psi_i\big(\xi(v)\big),$$

$$u^0(x,v) = \sum_{i=0}^{M} u_i^0(x)\psi_i\big(\xi(v)\big).$$

Thus, we write the equation (15)

$$\sum_{i=0}^{M} \frac{\partial u_i}{\partial s}\psi_i + 2\sum_{i=0}^{M}\sum_{j=0}^{M} u_j\psi_j \frac{\partial u_i}{\partial x}\psi_i + \sum_{i=0}^{M} \frac{\partial^2 u_i}{\partial x^2}\psi_i + \sum_{i=0}^{M}\sum_{j=0}^{M} \mu_j\psi_j \frac{\partial^3 u_i}{\partial x^3}\psi_i + \sum_{i=0}^{M} \frac{\partial^4 u_i}{\partial x^4}\psi_i = 0$$

with the initial condition

$$\sum_{i=0}^{M} u_i(0,x)\psi_i = \sum_{i=0}^{M} u_i^0(x)\psi_i.$$

Therefore,

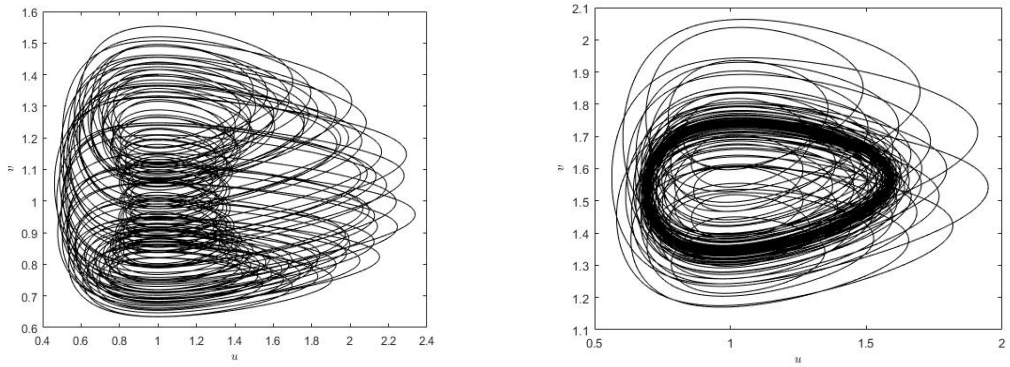
$$\frac{\partial u_k}{\partial s} + 2\sum_{i=0}^{M}\sum_{j=0}^{M} \frac{<\psi_k\psi_j\psi_i>}{<\psi_k^2>} u_j \frac{\partial u_i}{\partial x} + \frac{\partial^2 u_k}{\partial x^2} + \sum_{i=0}^{M}\sum_{j=0}^{M} \frac{<\psi_k\psi_j\psi_i>}{<\psi_k^2>} \mu_j \frac{\partial^3 u_i}{\partial x^3} + \frac{\partial^4 u_k}{\partial x^4} = 0.$$

$$\frac{\partial u_k}{\partial s} + \frac{2}{\psi_k^2}\sum_{i=0}^{M}\sum_{j=0}^{M} u_j \frac{\partial u_i}{\partial x} e_{kji} + \frac{\partial^2 u_k}{\partial x^2} + \frac{1}{<\psi_k^2>}\sum_{i=0}^{M}\sum_{j=0}^{M} \mu_j \frac{\partial^3 u_i}{\partial x^3} e_{kji} + \frac{\partial^4 u_k}{\partial x^4} = 0.$$

11 Numerical Simulations

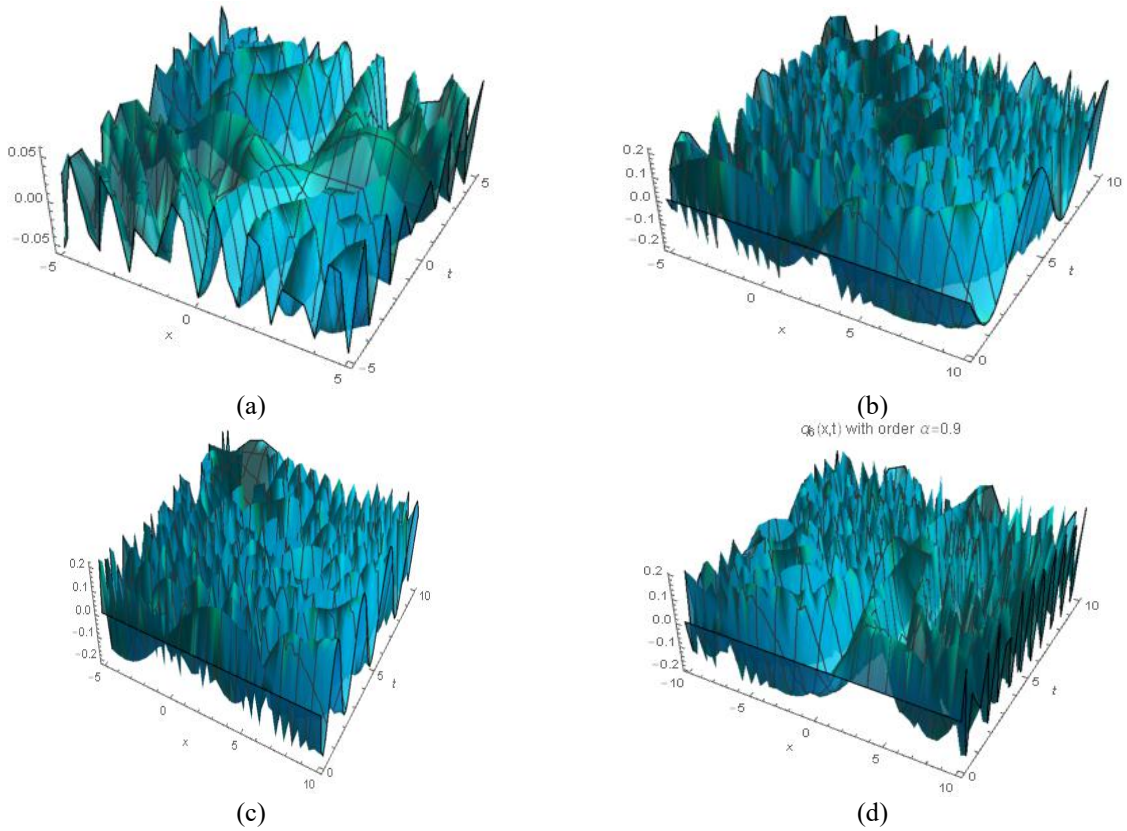
We have demonstrated the suitability of the polynomial chaos approach for both Benney equations and Lotka-Volterra systems.

Figures 1a-1b show numerical simulations for Eq. (14). Figures 2a-2d show numerical simulations for Eq. (15).



(a) (b)

Figure 1. Numerical simulation for Eq. (14).



(a) (b)

(c) (d)

Figure 2. Numerical simulation for Eq. (15).

12 Conclusion and Outlook

The stochastic Galerkin and polynomial chaos methods are the type of computational techniques that can be considered as some of the types of undetermined coefficients methods. The underlying and main idea of these methods is to express the unknown function as a series including some unknown coefficients format and determine those coefficients eventually. In the stochastic Galerkin methods, the resulting system is transformed into a linear system of finite element matrix equations, a kind of weighted residual approach. The solutions to these matrix equations are found by highly efficient computing methods. These matrix equations consist of a row vector made up of what are known as basis functions on a L-length space element (interval). Note that in the weighted residual method, the basis functions we use in this study serve as the weight functions. Polynomial chaos methods work at some probability spaces and are quite like Galerkin methods and can be considered as special cases of Galerkin methods. Polynomial chaos expansions, the series is written in terms of some special series such as the Hermitian series as a different case from the Galerkin method.

In this paper, we investigated if numerical solution techniques of the Galerkin type could be used for the

approximations of fractional-stochastic models. One of the main advantages of the Galerkin-type undetermined coefficients method is that this method is a highly efficient numerical method besides its ease of use. The downside of using these methods is that these methods require working in probability space, and in high-dimensional spaces, this can be a bit of a problem in terms of CPU costs.

The effects of fractional-order operators, fractional Brownian motion, and Gaussian white noise were studied. We also investigated how well the Polynomial Chaos approach worked for approximating solutions for the Benney and stochastic LV systems. Based on computational studies, it is possible to develop a strong and highly efficient use of Galerkin-type approaches. We will be dealing with fractional-stochastic equations involving Levy and Poisson jumps uncertainty in the future development of our study endeavor. We will look into the use of numerical techniques based on random walks in conjunction with Galerkin methods in future studies as an extension of the present work.

Declaration

Ethics committee approval is not required.

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