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$\mathcal{H}_\infty$-Norm Evaluation of Transfer Matrices of Dynamical Systems via Extended Balanced Singular Perturbation Method

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Highlights

- This paper focuses on computation of H_∞-norm of a transfer matrix of a dynamical system.
- A hybrid method is proposed for the higher order models.
- An efficient result was obtained within a satisfactory margin of error.

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Abstract

In this paper, we use a hybrid approach known as the extended balanced singular perturbation technique to compute the H_∞-norm of a transfer matrix of a dynamical system. The transfer matrix's order is first reduced using the balanced truncation approach, and its H_∞-norm is then found using the singular perturbation method. Both the singular perturbation technique and the balanced truncation approach methods are provided with computer algebraic instructions. The method is then applied to a decentralized interconnected system, and the error analysis of the solution is investigated.

1. INTRODUCTION

H_∞-control was first introduced to the literature by Zames in 1981 [1]. H_∞-control, a robust and precise methodology in control theory, finds diverse applications across multiple industries, significantly affecting the reliability and performance of complex systems. In aerospace engineering, it ensures stable flight conditions for aircraft and spacecraft amidst uncertainties. The automotive sector benefits from H_∞-control strategies, enhancing vehicle stability and performance in dynamic environments. Robotics and automation systems leverage H_∞-control to achieve precision and robustness in manufacturing processes and autonomous vehicles. In power systems and energy infrastructure, H_∞-control contributes to stability and efficiency, particularly in the realm of renewable energy and smart grids. Moreover, biomedical engineering utilizes H-infinity control for the development of precise and resilient medical devices, elevating the quality of healthcare delivery. This versatile control strategy stands as a cornerstone in addressing the challenges posed by disturbances and uncertainties across these diverse industrial domains.

The main motivation of H_∞ control is to create a powerful technique that works efficiently even if come up against undesirable factors or situations such as irregularities, disturbances, modelling errors etc. That is, obtaining measurable optimization for multi-variable cases while shrinking modelling errors and undetermined disturbances, at the same time. H_∞-control represents "the space of all bounded analytic matrix valued functions in the open right-half complex plane". H_∞-control aims to formulate the problem of sensitivity reduction as an optimization problem by an operator norm which is called H_∞-norm. In other

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words, to design "the best" controller when compared to the other controllers which minimizes the H_∞-norm of the transfer function of the system. From 1980s to present, H_∞-control has been studied by many researchers from various disciplines and a number of methods have been developed to compute H_∞-norm of transfer matrices (or functions) of dynamical systems. Since H_∞-norm associated with the largest singular value, these methods focus on singular value evaluation and have been used in many works. Bisection method [2-4], two step algorithm [5], two-sided Jacobi's method [6], householder bidiagonalization technique [7], singular perturbation method [8], extended balanced singular perturbation method [9, 10], etc. can be given as examples.

In high-level control problems such as control of huge space structures and power systems, researchers encounter excessive number of components and parameters and this excessiveness naturally, causes to spend much more time and effort. To overcome this undesirable circumstance, control theorists seek out some alternative operations to transform the high-order models to lower order and more convenient models which are easier to design in practice. These operations are called model order reduction. The governing idea of model order reduction is to convert a high-level model to a smaller size that is easier to solve with preserving structural features of the original model [11-14]. In other words, model order reduction means to find a suitable balanced realization and to truncate this realization without compromising the structural integrity of the original system. This operation is known as balanced truncation approach.

Let $\mu > 0$ be a parameter, a dynamical system which contains some state component derivatives with μ-coefficients is called a singular perturbation model. Singular perturbation models are represented by following set of equations,

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u \quad (1a)$$

$$\mu\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u \quad (1b)$$

$$y = C_1x_1 + C_2x_2 + Du \quad (1c)$$

here x_1, x_2 are called slow and fast variables, respectively, (1a), (1b) are called slow(powerful) and fast(weak) subsystems, respectively and μ is called perturbation parameter.

Analysis of these system types is done by singular perturbation theory. Singular perturbation theory means to investigate behavior of solutions of the system (1) for an interval $0 \leq t \leq T$ (or $0 \leq t < +\infty$). The basic idea of singular perturbation method is to protect the slow (low-frequency) part(1a) while neglecting the fast(high-frequency) part(1b) of the above equation. When considered from this point of view the method can be associated with a dominant mode state. In other words, it is process of examining solutions of the given system for $\mu = 0$ [15, 16]. μ-parameter may correspond to different concepts depending on the structure of the system. For example, it represents machine reactance or transients in voltage regulators in power systems, actuators in industrial control, enzymes in biochemical models and fast neutrons in nuclear reactor models.

The hybrid strategy known as the extended balanced singular perturbation method combines the balanced truncation approach with the singular perturbation method. First, the balanced truncation strategy is used to decrease the model order. Next, the singular perturbation method is used to derive the norm of the reduced model's transfer function.

This paper is organized in 7 sections. Some fundamental definitions and notations which will be used the next sections are given in section 2. In section 3, algorithm of balanced truncation approach, with the computer algebraic commands, is given. In section 4, algorithm of singular perturbation method that is applied to the balanced system obtained in the previous section, and its computer algebraic commands are given. Section 5 is about extended balanced singular perturbation method and error analysis that is based on comparing of the solutions of the original system and reduced-order model. In section 6, H_∞-norm of the transfer matrix of a numerical example (a decentralized interconnected system) is computed by extended balanced singular perturbation method for order of 3. Finally, section 7 is about efficiency of the algorithm and convincing error tolerance according to given error bound criterion.

2. PRELIMINARIES

Consider the linear dynamic system;

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du \tag{2}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$. Transfer matrix (or function) of the system (2) is defined as;

$$G(s) = C(sI - A)^{-1}B + D \tag{3}$$

We know that, transfer matrix of a system is defined in frequency domain while the state-space notation in time domain.

Let $\lambda_j(M)$, $\sigma_j(M)$ denote the j^{th} eigenvalue and j^{th} singular value of a matrix M respectively, where $\sigma_j(M) = \sqrt{\lambda_j(MM^T)}$. M is stable if $Re(\lambda_j(M)) < 0$ for all j.

The set of all analytic and bounded matrix valued function defined on complex open right-half plane $\mathbb{C}^+ = \{s \in \mathbb{C} \mid Re(s) > 0\}$ is called H_∞-space. In other words, if a matrix valued function $G: \mathbb{C}^+ \longrightarrow \mathbb{C}^{n,m}$ satisfies the conditions;

- $G(s)$ is analytic on complex open right-half plane
- $\lim_{\sigma\to 0^+} G(\sigma + j\omega) = G(j\omega)$
- $\sup_{s\in\mathbb{C}^+} \bar{\sigma}(G(s)) < \infty$, where $\bar{\sigma}$ is greatest singular value of the system (2) then it is an element of H_∞.

H_∞-norm of the transfer matrix of $G(s)$ of a stable system (2) is given as follows;

$$\|G\|_\infty = \sup_{Res>0} \sigma_{max}\left(G(s)\right) = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(G(j\omega)\right) \tag{4}$$

where $\sup_{\omega\in\mathbb{R}}$ denotes least upper bound for all frequencies ω which are real.

For the system (2) the matrices; $W_C(t)$ and $W_O(t)$ are called controllable and observable Grammians, respectively, which are defined as follows;

$$W_C(t) = \int_0^t e^{A\tau}BB^Te^{A^T\tau}\,d\tau$$
$$W_O(t) = \int_0^t e^{A^T\tau}C^TCe^{A\tau}\,d\tau \tag{5}$$

and singular values of $W_C(t)W_O(t)$ are called Hankel singular values of the system (2) which describes the energy of each state of the system (2) and are denoted as σH_j for $j = 1,2, \ldots$.

Any positive definite matrix M can be expressed in the form of

$$M = LL^T \tag{6}$$

where L is a lower triangular matrix. The expression (6) and the matrix L are called Cholesky factorization and Cholesky factor of M, respectively.

Let $M \in \mathbb{R}^{m\times n}$ and $rank(M) = r = min\,(m, n)$, the expression

$$M = U\Sigma V^T \tag{7}$$

is called singular value decomposition of the matrix M. Here U and V are orthogonal matrices of type of $m \times m$ and $n \times n$, respectively, that is, $U^T U = I_m$, $V^T V = I_n$ and Σ is a half-diagonal matrix which contains singular values $(\sigma_1, \ldots, \sigma_r)$ of the matrix M. Singular value decomposition can be formulated clearly as follows for a matrix M,

$$M = U\Sigma V^T = \underbrace{\left[\, u_1 \;\middle|\; u_2 \;\middle|\; \cdots \;\middle|\; u_m \,\right]}_{u(m\times m)} \underbrace{\begin{bmatrix} \sigma_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ddots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & 0 & \sigma_r & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \ldots & 0 & \ldots & 0 \end{bmatrix}}_{\Sigma(m\times n)} \underbrace{\begin{bmatrix} v_1^T \\ \hline v_2^T \\ \hline \vdots \\ \hline v_n^T \end{bmatrix}}_{V^T(n\times n)} . \tag{8}$$

3. BALANCED TRUNCATION AND MODEL ORDER REDUCTION

Let (2) be a minimal, asymptotically stable system, the algorithm of balanced truncation approach with the computer algebraic commands as follows;

Step 1. Find controllable and observable Grammians W_C and W_O of the given system through the Lyapunov equations with the computer algebraic commands

Wc=gram(sys,'c')
Wo=gram(sys,'o').

Step 2. Find the Cholesky factors L_C and L_O of W_C and W_O, respectively, such that

$$W_C = L_C L_C^T$$
$$W_O = L_O L_O^T$$

with the computer algebraic commands

Lc=chol(Wc,'lower')
Lo=chol(Wo,'lower').

Step 3. Find the singular value decomposition of $L_O^T L_C$ such that

$$L_O^T L_C = U\Sigma V^T$$

with the computer algebraic command

[U, S, V] =svd(Lo'*Lc).

Step 4. Make the transformation $T = L_C V \Sigma^{-1/2}$ and obtain coefficient matrices of balanced system by similarity transformations as follows,

$$\tilde{A} = T^{-1}AT, \qquad \tilde{B} = T^{-1}B, \qquad \tilde{C} = CT, \qquad \tilde{D} = D$$

where $\tilde{G}(s) = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$ and find controllable and observable Grammians of the balanced system $\widetilde{W}_C$ and $\widetilde{W}_O$ respectively which are given as below,

$$\widetilde{W}_C = T^{-1} W_C T^{-T}$$

$$\widetilde{W}_O = T^T W_O T$$

where $\widetilde{W} = \widetilde{W}_O = \Sigma = diag(\sigma_1, \sigma_1, \ldots, \sigma_n)$.

4. SINGULAR PERTURBATION METHOD

Let $\tilde{G}(s) = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$ be the balanced system obtained by balanced truncation approach, the algorithm of singular perturbation method is given as follows;

Step 1. Separate the balanced system $\tilde{G}(s) = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] \Leftrightarrow \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$ into two subsystems as slow (powerful) and fast(weak). Choose the A_{11} as coefficient matrix of the slow (powerful) part where A_{11}, $\Sigma_1 \in \mathbb{R}^{r\times r}$, for $r \ll n$. Rearrange the matrices $\tilde{A},\ \tilde{B},\ \tilde{C},\ \tilde{D}$ in block matrix form as seen below,

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad \tilde{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \qquad \tilde{D} = D.$$

Add perturbation parameter μ and rewrite $\tilde{G}(s)$ as the followings,

$$\begin{bmatrix} \dot{x}_1 \\ \mu\dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$
$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du.$$

Step 2. Eliminate the fast(weak) part taking $\mu = 0$ and find the system as,

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u$$
$$0 = A_{21}x_1 + A_{22}x_2 + B_2u$$
$$y = C_1x_1 + C_2x_2 + Du$$

and weak variable as, $x_2 = -{A_{22}}^{-1}A_{21}x_1 - {A_{22}}^{-1}B_2u$.

Step 3. Substitute x_2 to other equations to get the final version of the system that denoted by $G_f(s)$
As is below,

$$G_f(s) = \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right] = \left[\begin{array}{c|c} A_{11} - A_{12}{A_{22}}^{-1}A_{21} & B_1 - A_{12}{A_{22}}^{-1}B_2 \\ \hline C_1 - C_2{A_{22}}^{-1}A_{21} & D - C_2{A_{22}}^{-1}B_2 \end{array}\right].$$

Step 4. Obtain the H_∞-norm of $\|G_f(s)\|_\infty$ via computer algebraic computation.

5. EXTENDED BALANCED SINGULAR PERTURBATION METHOD AND ERROR ANALYSIS

The algorithm of extended balanced singular perturbation method consists of 8 steps, the first four being balanced truncation approach and the last four being singular perturbation method which are pointed out in the sections 3 and 4, in detail.

Now, to analyze the error tolerance, first we define modelling error transfer function as follows

$$E_r = [G(s) - G_f(s)].$$

Then, we have a criterion about sufficiency of error tolerance which is based on comparison of two error bounds called actual infinity error bound and theoretical infinity error bound defined in [17, 18] given as below

- actual infinity error bound: $\|E_r\|_\infty = \left\|[G(s) - G_f(s)]\right\|_\infty$
- theoretical infinity error bound: $2\sum_{i=r+1}^{n} \sigma_i$
- the criterion: $\|E_r\|_\infty \leq 2\sum_{i=r+1}^{n} \sigma_i$.

We can summarize algorithm of extended balanced singular perturbation method step by step by constructing the following Table 1.

Table 1. *Algorithm of extended balanced singular perturbation method step by step*

	Extended Balanced Singular Perturbation Method	
	Balanced Truncation Approach	**Singular Perturbation Method**
Step1.	Find Grammians of the original system (W_C, W_O).	Seperate the balanced system $\tilde{G}(s)$ into two parts as; strong and weak.
Step2.	Find Cholesky factors of Grammians (L_C, L_O).	Eliminate the weak part taking $\mu = 0$ and find weak variable x_2.
Step3.	Find singular value decomposition of $L_O^T L_C = U\Sigma V^T$.	Substitute x_2 in other equations, get the final version of the system $G_f(s)$.
Step4.	Make the transformation $T = L_C V \Sigma^{-1/2}$ and find the balanced system $\tilde{G}(s)$.	Obtain the H_∞-norm of $\left\|G_f(s)\right\|_\infty$.
Error Analysis	Compute actual and theoretical infinity error bounds and apply the error tolerance criterion which says actual bound must be less than or equal to theoretical bound.	

6. APPLICATION TO A NUMERICAL EXAMPLE

In this section, we apply the extended balanced singular perturbation method to a numerical example for the case order of 3.

Example 4.1. (A decentralized interconnected system)

For additional details, see [19].

Consider the system (2) with coefficient matrices given as follows:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -8 & 1 & -1 & -1 & -2 & 0 \\ 4 & -0.5 & 0.5 & 0 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = 0\ .$$

We have H_∞-norm of the transfer matrix of the given system computed via computer algebraic as

$\|G(s)\|_{\infty} = 29.6784$. Now, if we apply balanced truncation approach algorithm step by step, we finally get;

$$\tilde{G}(s) = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$$

where

$$\tilde{A} = \begin{bmatrix} -0.1139 & 0.3032 & -0.1793 & -0.1238 & -0.0600 & 0.0158 \\ -0.2941 & -0.2900 & 0.7198 & 0.1681 & 0.2357 & -0.0281 \\ 0.0271 & -0.5608 & -0.5796 & -1.0328 & -0.2514 & 0.0865 \\ 0.0948 & 0.3066 & 1.0322 & -1.8693 & 0.0091 & 0.1389 \\ -0.0222 & -0.1791 & -0.0595 & 0.1382 & -1.3774 & -0.7626 \\ -0.0562 & -0.2046 & -0.2118 & 0.9211 & -1.2014 & -3.7698 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 1.5096 & -1.1876 & 0.2814 & 0.7819 \\ 0.6344 & -1.3681 & 0.3046 & 1.2281 \\ -1.2063 & -1.5068 & 0.0591 & 0.0910 \\ 0.1128 & 1.6074 & 0.4628 & -0.1543 \\ -0.2150 & 0.0400 & 0.0270 & 1.0440 \\ 0.0454 & -0.4450 & 0.1872 & 0.5491 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 1.1887 & -1.0593 & 0.0281 & -0.0249 & 0.3844 & -0.0660 \\ -1.0856 & 1.4815 & -1.2716 & -0.0962 & -0.8222 & -0.0206 \\ 1.2977 & -0.7147 & 1.4044 & 1.6654 & -0.0043 & -0.0061 \\ -0.3228 & 0.2170 & -0.3834 & -0.2253 & 0.5609 & 0.7293 \end{bmatrix},$$

$\tilde{D} = 0$

and Hankel singular values of the original system as,

$\sigma(G) = (19.2254 \quad 6.6806 \quad 3.2240 \quad 0.7581 \quad 0.4133 \quad 0.0712)$.

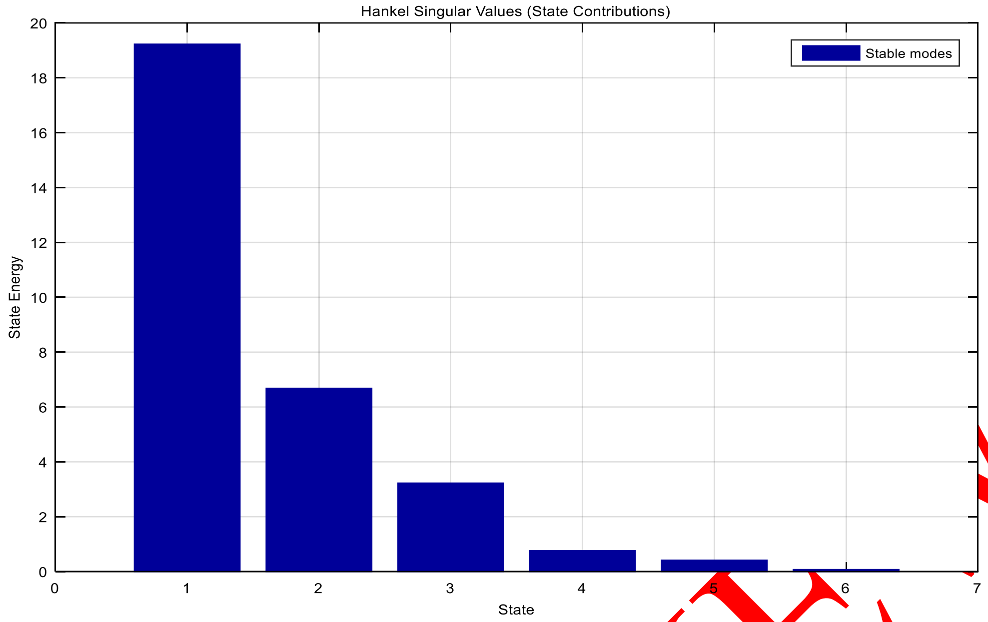

Figure 1. *Hankel singular values of the original system*

It is seen clearly in the Figure 1 the first three Hankel singular values are much greater than the others so we choose $r = 3$ and apply extended balanced singular perturbation method. First separate the balanced system $\tilde{G}(s)$ into two parts as slow (powerful) and fast (weak) and rewrite the system for perturbation parameter $\mu = 0$ as is given below;

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1u \\ 0 &= A_{21}x_1 + A_{22}x_2 + B_2u \\ y &= C_1x_1 + C_2x_2 + Du \end{aligned}$$

where

$$A_{11} = \begin{bmatrix} -0.1139 & 0.3032 & -0.1793 \\ -0.2941 & -0.2900 & 0.7198 \\ 0.0271 & -0.5608 & -0.5796 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -0.1238 & -0.0600 & 0.0158 \\ 0.1681 & 0.2357 & -0.0281 \\ -1.0328 & -0.2514 & 0.0865 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0.0948 & 0.3066 & 1.0322 \\ -0.0222 & -0.1791 & -0.0595 \\ -0.0562 & -0.2046 & -0.2118 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -1.8693 & 0.0091 & 0.1389 \\ 0.1382 & -1.3774 & -0.7626 \\ 0.9211 & -1.2014 & -3.7698 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1.5096 & -1.1876 & 0.2814 & 0.7819 \\ 0.6344 & -1.3681 & 0.3046 & 1.2281 \\ -1.2063 & -1.5068 & 0.0591 & -0.1543 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1128 & 1.6074 & 0.4628 & -0.1543 \\ -0.2150 & 0.0400 & 0.0270 & 1.0440 \\ 0.0454 & -0.4450 & 0.1872 & 0.5491 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1.1887 & -1.0593 & 0.0281 \\ -1.0856 & 1.4815 & -1.2716 \\ 1.2977 & -0.7147 & 1.4044 \\ -0.3228 & 0.2170 & -0.3834 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -0.0249 & 0.3844 & -0.0660 \\ -0.0962 & -0.8222 & -0.0206 \\ 1.6654 & -0.0043 & -0.0061 \\ -0.2253 & 0.5609 & 0.7293 \end{bmatrix},$$

$D = 0$

and from the second equation find weak variable as, $x_2 = -A_{22}{}^{-1}A_{21}x_1 - A_{22}{}^{-1}B_2u$. Continue from

Step 3. Make necessary algebraic matrix operations and finally get, $G_f(s) = \begin{bmatrix} A_f & | & B_f \\ - - - & - & - - - \\ C_f & | & D_f \end{bmatrix}$ where

$$A_f = \begin{bmatrix} -0.1195 & 0.2909 & -0.2447 \\ -0.2884 & -0.2932 & 0.8021 \\ -0.0223 & -0.6970 & -1.1391 \end{bmatrix}, \quad B_f = \begin{bmatrix} 1.5149 & -1.2983 & 0.2528 & 0.7411 \\ 0.5957 & -1.2062 & 0.3395 & 1.4118 \\ -1.2163 & -2.4142 & -0.1902 & -0.0370 \end{bmatrix}$$

$$C_f = \begin{bmatrix} 1.1829 & -1.1146 & -0.0067 \\ -1.0809 & 1.5706 & -1.2957 \\ 1.3823 & -0.4388 & 2.3348 \\ -0.3399 & 0.1274 & -0.4633 \end{bmatrix}, \quad D_f = \begin{bmatrix} -0.0848 & 0.0039 & -0.0221 & 0.3290 \\ 0.1565 & -0.1487 & -0.0103 & -0.6679 \\ 0.1106 & 1.4404 & 0.4262 & -0.1507 \\ -0.0602 & -0.1013 & 0.0174 & 0.3818 \end{bmatrix}$$

Obtain the H_∞-norm via computer algebraic as $\left\|G_f(s)\right\|_\infty = 29.6799$ which is so close to the H_∞-norm of the original system $\|G(s)\|_\infty = 29.6784$.

Now analyze the error tolerance between the original system and balanced-reduced order model via actual and theoretical infinity error bounds as follows,

$$\|E_r\|_\infty = \left\|[G(s) - G_f(s)]\right\|_\infty = 1.5218$$

and for $r = 3$ and $n = 6$,

$$2\sum_{i=r+1}^{n} \sigma_i = 0.7581 + 0.4133 + 0.0712 = 2.4852.$$

It is obvious that $\|E_r\|_\infty \leq 2\sum_{i=r+1}^{n} \sigma_i$ thus we can say that error tolerance is in a satisfied level.

7. RESULTS

In this study, the H_∞- norm of the transfer function of a linear dynamic system for instance $D = 0$ has been computed using the extended-balanced singular perturbation approach. The balanced-reduced order model and original system have almost identical H_∞- norm values, as demonstrated by the solution of a numerical case. Furthermore, based on the reduced order models' error investigation requirement of sufficiently tiny error tolerance, we can state with certainty that the procedure functions successfully.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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