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More Efficient Solutions for Numerical Analysis of the Nonlinear Generalized Regularized Long Wave (Grlw) Using the Operator Splitting Method

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Abstract

Through the use of two numerical techniques, the purpose of this study is to examine the approximate outcomes of the (GRLW) equation. The utilized methods are the collocation method with quintic B-spline, which is based on finite elements and yields good results for nonlinear evolution equations, and the strang splitting technique, which is simple to apply, practical, and quick. In order to provide approximate solutions for the main problem, the collocation method is combined with the Strang splitting method for this study. Three examples—the formation of the Maxwellian initial condition, the interaction of two solitary waves, and a single solitary wave—are taken into consideration in order to assess the accuracy of these algorithms. To demonstrate how closely the exact solutions close to numerical results and to contrast them with other solutions in the literature, error norms, and conservation quantities are computed. Tables and graphs are used to illustrate the solutions that have generated. Based on the results obtained and the practical, easy-to-use, and current features of the methodologies, this article stands out from the rest.

1. Introduction

Analytical solutions of nonlinear evolution equations, especially those containing nonlinear terms, which play an important role in various fields of science such as physics, applied mathematics and engineering problems, may not generally be obtained. Therefore, due to the existence of limited boundary and initial conditions in obtaining analytical solutions, approximate solutions of such equations have become quite suitable for the study of physical phenomena. The regularized long wave (RLW) equation, which was first proposed by Peregrine [1] and forms the basis of the generalized regular long wave (GRLW) equation discussed in this study, is one of the significant patterns in the physics environment due to the fact that it describes phenomena with weak nonlinearity and dispersion waves. Later, the RLW equation was investigated by Benjamin *et al* [2], who disputed it as an improved pattern of the KdV equation, which describes long waves by presuming a small wave amplitude and a large wave length in nonlinear dispersion and great number of physical systems. GRLW equation is connected with the generalized Korteweg-de Vries (GKdV) equation presented as

$$U_t + \varepsilon U^p U_x + \mu U_{xxx} = 0. \tag{1.1}$$

These generally expressed equations are non-linear wave equations with $(p+1)$th non-linearity and they have solitary wave solutions with pulse-like properties. The GRLW equation designed to obtain approximate solutions in this study is described by the form

$$U_t + U_x + p(p+1)U^p U_x - \mu U_{xxt} = 0 \tag{1.2}$$

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with the initial-boundary conditions presented as follows

$$\begin{aligned} U(x,0) &= g(x), \quad x_L \leq x \leq x_R, \\ U(x_L,t) &= U(x_R,t) = 0, \\ U_x(x_L,t) &= U_x(x_R,t) = 0. \end{aligned} \tag{1.3}$$

where physical boundary conditions of this equation are expressed as $U \to 0$ when $x \to \pm\infty$ and here t and x are subscripts that indicate variations in time and space and p is a non-negative integer and μ is a positive constant. $f(x)$ refers to a localized disturbance within the range $[x_L, x_R]$, while U refers to the vertical displacement of the water surface or similar physical quantity. Many scientists have tried to obtain solutions of the (GRLW) equation numerically and analytically. Zhang [3] considered a finite difference method for the (GRLW) equation. Both Karakoç and Bhowmik [4] and Roshan [5] approximated the solutions of the equation using the Petrov–Galerkin method. The Galerkin approximation with cubic B-splines was constructed to acquire the approximate solution of the (GRLW) equation by Zeybek and Karakoç [6]. Zeybek and Karakoç [7] and Karakoç and Zeybek [8] used collocation method with the help of quintic and septic B-splines, respectively, for solitary-wave solutions of the equation. A new compact finite difference method (CFDM) was proposed by [9] for equation. Mokhtari and Mohammadi [10] utilized Sinc-collocation method to the equation. Recently, Karakoç *et al* [11] applied to the equation an exact method named Riccati–Bernoulli sub-ODE method and a numerical method named Subdomain finite element method. By taking $p = 1$ in the (GRLW) equation, the (RLW) equation, which is a special case of this equation, is obtained. Solutions to this equation have been obtained by many methods. One can easily refer to refs. [2], [12]–[26]. If $p = 2$ is taken into account in the (GRLW) equation, the (MRLW) equation, which is a special case of this equation, is gotten. The reader can examine refs. [27, 28] for the solutions of this equation, which have been obtained by many methods.

The aim of this study is to investigate approximate solutions of the equation (1.2). The GRLW equation has been previously solved by the Collocation method. However, in this article, the solutions have been obtained by combining the collocation method with the Strang splitting technique. This method is simple, practical and fast to implement, so it can be preferred more in the literature.The Strang splitting technique, which is one of the Operator splitting techniques that is very practical and produces accurate results, is used to obtain solutions. Two numerical schemes are created for the main equation via the splitting technique. These schemes are applied the collocation method with the help of quintic B-spline. The results obtained are illustrated with tables and graphs.

2. Operator Splitting Method

Operator splitting is an effective technique for solving coupled systems of partial differential equations. Because one obtains a series of equations by dividing a complex equation into simpler and easier parts. Operator splitting means that the spatial differential operator contained in the equations is divided into the sum of different sub-operators with simpler forms, so that the corresponding equations be able to solve more easily. Then, as per the procedure of the splitting technique, a series of sub-equations are solved instead of the main equation. There are operator splitting techniques that include different algorithms such as Lie-Trotter, strang and higher order splitting techniques. In this study, the second order Strang splitting technique, which is one of the easy and convenient splitting techniques used to obtain faster results, will be used. Let's consider a complex problem that has the following form.

$$\frac{dU(t)}{dt} = (\omega_1 + \omega_2)U(t), \;\; U(0) = U_0, t \in [0,T]. \tag{2.1}$$

The problem (2.1) can be split into the following subequations in one dimensional form

$$\begin{aligned} \frac{dU^*(t)}{dt} &= \omega_1 U^*(t), \;\; U^*(t_n) = U^n_{sp} = U_0 \quad , \quad t \in [t_n, t_{n+1}], \\ \frac{dU^{**}(t)}{dt} &= \omega_2 U^{**}(t), \;\; U^{**}(t_n) = U^*(t_{n+1}) \quad , \quad t \in [t_n, t_{n+1}] \end{aligned}$$

in which $U^n_{sp} = U_0$ is known and $(U_{sp})_{t_{n+1}} = U^{**}_{t_{n+1}}$ is the approximate solution at $t_n = t_{n+1}$. Here, ω_1 and ω_2 differential operators. $[0,T]$ is a time interval for arbitrary $T \geq 0$, and this interval can be divided into M subintervals $[t_n, t_{n+1}]$, $(n = 0,1,2,...,M-1)$ that satisfy the condition $0 =\leq t_0 \leq t_1 \leq t_2 ... \leq t_M = T$ and each interval is of length $\Delta t = t_{n+1} - t_n$.
Second order strang splitting technique can be presented with the following algorithm

$$\begin{aligned} \frac{dU^*(t)}{dt} &= \omega_1 U^*(t), \;\; U^*(t_n) = U^{***}(t_n) \quad , \quad t \in [t_n, t_{n+1/2}], \\ \frac{dU^{**}(t)}{dt} &= \omega_2 U^{**}(t), \;\; U^{**}(t_n) = U^*(t_{n+1/2}) \quad , \quad t \in [t_n, t_{n+1}] \\ \frac{dU^{***}(t)}{dt} &= \omega_1 U^{***}(t), \;\; U^{***}(t_{n+1/2}) = U^{**}(t_{n+1}) \quad , \quad t \in [t_{n+1/2}, t_{n+1}] \end{aligned} \tag{2.2}$$

in which $(U^*)_0 = U_0$ and $U^{***}(t_{n+1})$ are the approximate solution at $t_n = t_{n+1}$ [29, 30]. As it is known, solutions of equation (2.1) can be found over the entire time interval. However, instead of doing this, according to the procedure of this algorithm, the first equation of (2.2) is solved with half the time step, then the second equation of (2.2) is solved with the whole time step, and then the first equation of (2.2) is solved again with half the time step. Thus, the process is completed. For the solutions of (2.1) in [31], Taylor series expansion up to the first order and the second order have been used. It has been obtained that the approach has a first-order accuracy of $(O(\Delta t))$ for Lie-Trotter splitting technique and a second-order accuracy $(O(\Delta t^2))$ for Strang splitting technique.

3. The Construction of the Collocation Method

Let the solution range of the main problem $[x_L, x_R]$ be divided into N finite elements of equal length $h = x_{j+1} - x_j$ for the nodes x_j, $j = 0(1)$ such that $x_L = x_0 \leq x_1 \leq \ldots \leq x_N = x_R$. Quintic B-splines $\varphi_{-2}(x), \varphi_{-1}(x), \ldots \varphi_{N+2}(x)$ for nodes x_j can be defined on the interval $[x_L, x_R]$ as follows by [33]

$$\varphi_j(x) = \frac{1}{h^5}\begin{cases} p_0 = (x-x_{j-3})^5, & x \in [x_{j-3}, x_{j-2}] \\ p_1 = p_0 - 6(x-x_{j-2})^5, & x \in [x_{j-2}, x_{j-1}] \\ p_2 = p_1 - 6(x-x_{j-2})^5 + 15(x-x_{j-1})^5, & x \in [x_{j-1}, x_j] \\ p_3 = p_2 - 6(x-x_{j-2})^5 - 20(x-x_j)^5, & x \in [x_j, x_{j+1}] \\ p_4 = p_3 - 6(x-x_{j-2})^5 + 15(x-x_{j+1})^5, & x \in [x_{j+1}, x_{j+2}] \\ p_5 = p_4 - 6(x-x_{j-2})^5 - 6(x-x_{j+2})^5, & x \in [x_{j+2}, x_{mj3}] \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

The numerical solution, $U_N(x,t)$, is defined in terms of quintic B-spline functions with form:

$$U_N(x,t) = \sum_{j=-2}^{N+2} \varphi_j(x)\delta_j(t) \tag{3.2}$$

in which $\delta_j(t)$ is the unknown time-dependent quantity and it is found from the boundary and quintic B-spline collocation conditions. When written instead of B-spline functions (3.1) in the approximate function (3.2), the nodal values U_j, U_j', U_j'' are written as follows depending on $\delta_j(t)$

$$\begin{aligned} U_j &= \delta_{j-2} + 26\delta_{j-1} + 66\delta_j + 26\delta_{j+1} + \delta_{j+2}, \\ U_j' &= \frac{5}{h}(-\delta_{j-2} - 10\delta_{j-1} + 10\delta_{j+1} + \delta_{j+2}), \\ U_j'' &= \frac{20}{h^2}(\delta_{j-2} + 2\delta_{j-1} - 6\delta_j + 2\delta_{j+1} + \delta_{j+2}), \end{aligned} \tag{3.3}$$

and the variation of U with the interval $[x_j, x_{j+1}]$ can be obtained with form

$$U = \sum_{j=-2}^{N+2} \varphi_j \delta_j. \tag{3.4}$$

Now, let's split the GRLW equation as follows:

$$U_t - \mu U_{xxt} = 0, \tag{3.5}$$

$$\tag{3.6}$$

$$U_t - \mu U_{xxt} + U_x + p(p+1)U^p U_x = 0. \tag{3.7}$$

When the nodal values and space derivatives of U_j in (3.3) are used in the (3.5) and (3.7) equations, two ordinary differential equations are obtained as follows

$$\begin{aligned} \dot{\delta}_{j-2} + 26\dot{\delta}_{j-1} + 66\dot{\delta}_j + 26\dot{\delta}_{j+1} + \dot{\delta}_{j+2} - \frac{20\mu}{h^2}(\dot{\delta}_{j-2} + 2\dot{\delta}_{j-1} - 6\dot{\delta}_j + 2\dot{\delta}_{j+1} + \dot{\delta}_{j+2}) \\ + \frac{5}{h}(-\delta_{j-2} - 10\delta_{j-1} + 10\delta_{j+1} + \delta_{j+2}) = 0, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \dot{\delta}_{j-2} + 26\dot{\delta}_{j-1} + 66\dot{\delta}_j + 26\dot{\delta}_{j+1} + \dot{\delta}_{j+2} - \frac{20\mu}{h^2}(\dot{\delta}_{j-2} + 2\dot{\delta}_{j-1} - 6\dot{\delta}_j + 2\dot{\delta}_{j+1} + \dot{\delta}_{j+2}) \\ + \frac{5z_j}{h}(-\delta_{j-2} - 10\delta_{j-1} + 10\delta_{j+1} + \delta_{j+2}) = 0, \end{aligned} \tag{3.9}$$

in which symbol $''.''$ is derivative according to time t and z_j is linearization operation by

$$z_j = p(p+1)(\delta_{j-2}+26\delta_{j-1}+66\delta_j+26\delta_{j+1}+\delta_{j+2})^p.$$

If it is written instead of $\dfrac{\delta_j^{n+1}+\delta_j^n}{2}$ for the quantity δ_j and $\dfrac{\delta_j^{n+1}-\delta_j^n}{\Delta t}$ for the quantity $\dot{\delta}_j$ in Eqs.(3.8) and (3.9), two numerical system presented in the following are acquired ,

$$k_1\delta_{j-2}^{n+1}+k_2\delta_{j-1}^{n+1}+k_3\delta_j^{n+1}+k_4\delta_{j+1}^{n+1}+k_5\delta_{j+2}^{n+1}=k_5\delta_{j-2}^{n}+k_4\delta_{j-1}^{n}+k_3\delta_j^{n}+k_2\delta_{j+1}^{n}+k_1\delta_{j+2}^{n} \tag{3.10}$$

$$l_1\delta_{j-2}^{n+1}+l_2\delta_{j-1}^{n+1}+l_3\delta_j^{n+1}+l_4\delta_{j+1}^{n+1}+l_5\delta_{j+2}^{n+1}=l_5\delta_{j-2}^{n}+l_4\delta_{j-1}^{n}+l_3\delta_j^{n}+l_2\delta_{j+1}^{n}+l_1\delta_{j+2}^{n} \tag{3.11}$$

in which $k_j, l_j (j=1(1)5)$,and z_j are $z_j = p(p+1)U^p$

$$k_1 = 1-\frac{20\mu}{h^2}-\frac{5\Delta t}{2h}, k_2 = 26-\frac{40\mu}{h^2}-\frac{25\Delta t}{h}, k_3 = 66+\frac{120\mu}{h^2},$$
$$k_4 = 26-\frac{40\mu}{h^2}+\frac{25\Delta t}{h}, k_5 = 1-\frac{20\mu}{h^2}+\frac{5\Delta t}{h}$$

$$l_1 = 1-\frac{20\mu}{h^2}-\frac{5z_j\Delta t}{2h}, l_2 = 26-\frac{40\mu}{h^2}-\frac{25z_j\Delta t}{h}, l_3 = 66+\frac{120\mu}{h^2},$$
$$l_4 = 26-\frac{40\mu}{h^2}+\frac{25z_j\Delta t}{h}, l_5 = 1-\frac{20\mu}{h^2}+\frac{5z_j\Delta t}{2h}.$$

Systems (3.10) and (3.11) contain unknown quantities $(N+5)$, while $(N+1)$ consist of linear equations. However, only one solution for these systems must be obtained. While doing this, since the virtual parameters are not in the solution region, these parameters are eliminated by using U and U' in Equation(3.3) and the boundary conditions $U(x_L,t) = U(x_R,t) = 0$ and $U_x(x_L,t) = U_x(x_R,t) = 0$. In this way, the matrix system $(N+1)$ x $(N+1)$ for the $(N+1)$ unknowns quantities is obtained for the systems (3.10) and (3.11).

The closed form of the matrix systems (3.10) and (3.11) above can be expressed as

$$A_1\delta^{n+1} = A_1^T\delta^n$$
$$B_1\lambda^{n+1} = B_1^T\lambda^n$$

for the unknown time dependent quantities $\delta^T = [\delta_0\delta_1...\delta_N]$ and $\lambda^T = [\lambda_0\lambda_1...\lambda_N]$ to be calculated and A_1 and B_1 are coefficient matrices with the form

$$A_1 = \begin{bmatrix}
\bar{k}_3 & \bar{k}_4 & \bar{k}_5 & & & & & & & & \\
k_2 & k_3 & k_4 & k_5 & & & & & & & \\
k_1 & k_2 & k_3 & k_4 & k_5 & & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & & & & & \\
 & & & k_1 & k_2 & k_3 & k_4 & & & & \\
 & & & & k_5 & 1 & & & & & \\
 & & & & \bar{k}_4 & & \bar{k}_1 & & & & \\
 & & & & \bar{k}_3 & & k_1 & \bar{k}_2 & \bar{k}_3 & & \\
 & & & & & & & k_1 & k_2 & k_3 & k_4 \\
 & & & & & & & & k_1 & \bar{k}_2 & \bar{k}_3 \\
 & & & & & & & & & \bar{k}_1 & \bar{k}_2
\end{bmatrix}$$

$$B_1 = \begin{bmatrix}
\bar{l}_3 & \bar{l}_4 & \bar{l}_5 & & & & & & & & \\
l_2 & \bar{l}_3 & l_4 & l_5 & & & & & & & \\
l_1 & l_2 & l_3 & l_4 & l_5 & & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & & & & & \\
 & & & l_1 & l_2 & l_3 & l_4 & & & & \\
 & & & & l_5 & 1 & & & & & \\
 & & & & \bar{l}_4 & & \bar{l}_1 & & & & \\
 & & & & \bar{l}_3 & & l_1 & \bar{l}_2 & \bar{l}_3 & & \\
 & & & & & & & & l_1 & l_2 & l_3 \\
 & & & & & & & & & \bar{l}_2 & \bar{l}_3 \\
 & & & & & & & & & & \bar{l}_2
\end{bmatrix}$$

$$\bar{k_3} = \frac{165}{4}k_1 - \frac{33}{8}k_2 + k_3, \bar{k_4} = \frac{65}{2}k_1 - \frac{9}{4}k_2 + k_4, \bar{k_5} = \frac{9}{4}k_1 - \frac{1}{8}k_2 + k_5,$$

$$\bar{k_2} = -\frac{33}{8}k_1 + k_2, \bar{k_3} = -\frac{9}{4}k_1 + k_3, \bar{k_4} = -\frac{1}{8}k_1 + k_4,$$

$$\bar{\bar{k_2}} = -\frac{1}{8}k_5 + k_2, \bar{\bar{k_3}} = -\frac{9}{4}k_5 + k_3, \bar{\bar{k_4}} = -\frac{33}{8}k_5 + k_4,$$

$$\bar{\bar{k_1}} = \frac{9}{4}k_5 - \frac{1}{8}k_4 + k_1, \bar{\bar{k_2}} = \frac{65}{2}k_5 - \frac{9}{4}k_4 + k_2, \bar{\bar{k_3}} = \frac{165}{4}k_5 - \frac{33}{8}k_4 + k_3,$$

$$\bar{l_3} = \frac{165}{4}l_1 - \frac{33}{8}l_2 + l_3, \bar{l_4} = \frac{65}{2}l_1 - \frac{9}{4}l_2 + l_4, \bar{l_5} = \frac{9}{4}l_1 - \frac{1}{8}l_2 + l_5,$$

$$\bar{l_2} = -\frac{33}{8}l_1 + l_2, \bar{l_3} = -\frac{9}{4}l_1 + l_3, \bar{l_4} = -\frac{1}{8}l_1 + l_4,$$

$$\bar{\bar{l_2}} = -\frac{1}{8}l_5 + l_2, \bar{\bar{l_3}} = -\frac{9}{4}l_5 + l_3, \bar{\bar{l_4}} = -\frac{33}{8}l_5 + l_4,$$

$$\bar{\bar{l_1}} = \frac{9}{4}l_5 - \frac{1}{8}l_4 + l_1, \bar{\bar{l_2}} = \frac{65}{2}l_5 - \frac{9}{4}l_4 + l_2, \bar{\bar{l_3}} = \frac{165}{4}l_5 - \frac{33}{8}l_4 + l_3.$$

In order to produce more attractive, effective and accurate results for each time step, the internal iteration formula presented as follows is applied 3 or 5 times to z_j in Eq.(3.11)

$$(\delta^*)^n = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1}).$$

4. The Initial Vector δ_j^0

To start the iteration process for the systems (3.10) and (3.11), it is necessary to determine the initial vector δ_j^0. For this, initial parameters are computeded utilizing initial condition $U(x_j,0) = U_N(x_j,0) = g_0(x_j), j = 0(1)N$ and 1st and 2nd order derivatives on the boundaries presented with the main problem. In other words, these vectors to be calculated are computed from the system of algebraic equations presented as follows

$$\begin{aligned} \delta_{m-2}^0 + 26\delta_{m-1}^0 + 66\delta_m^0 + 26\delta_{m+1}^0 + \delta_{m+2}^0 &= g_0(x_m), m = 0(1)N \\ -\delta_{-2}^0 - 10\delta_{-1}^0 + 10\delta_1^0 + \delta_2^0 &= g_0^{'}(x_L), \\ \delta_{-2}^0 + 2\delta_{-1}^0 - 6\delta_0^0 + 2\delta_1^0 + \delta_2^0 &= g_0^{''}(x_L), \\ \delta_{N-2}^0 + 2\delta_{N-1}^0 - 6\delta_N^0 + 2\delta_{N+1}^0 + \delta_{N+2}^0 &= g_0^{''}(x_R), \\ -\delta_{N-2}^0 - 10\delta_{N-1}^0 + 10\delta_{N+1}^0 + \delta_{N+2}^0 &= g_0^{'}(x_R). \end{aligned} \tag{4.1}$$

In conclusion, the matrix equation for the initial vector δ^0 is acquired by

$$\begin{bmatrix} 54 & 60 & 6 & & & & & \\ 25.25 & 67.5 & 26.25 & 1 & & & & \\ 1 & 26 & 66 & 26 & 1 & & & \\ & & & \ddots & & & & \\ & & 1 & 26 & 66 & 26 & 1 & \\ & & & 1 & 26.25 & 67.5 & 25.25 & \\ & & & & 6 & 60 & 54 & \end{bmatrix} \begin{bmatrix} \delta^0{}_0 \\ \delta^0{}_1 \\ \delta^0{}_2 \\ . \\ . \\ . \\ \delta^0{}_{N-2} \\ \delta^0{}_{N-1} \\ \delta^0{}_N \end{bmatrix} = \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ . \\ . \\ . \\ U_{N-2} \\ U_{N-1} \\ U_N \end{bmatrix}.$$

With the current symbolic programming languages, calculating such matrices is fairly simple and useful. These features of the schemes that are being presented are indicative of their dependability and resilience.

5. Stability Analysis of Numerical Algorithm

Von Neumann theory is used to analyze the stability of the Strang splitting method applied to the GRLW equation. Let the growth factors of a typical Fourier mode be described as follows for stability analysis based on Von Neumann theory of systems (3.10) and (3.11)

$$\delta_j^n = \rho_1^n e^{ij\gamma h}, \tag{5.1}$$

$$\Psi_j^n = \rho_2^n e^{ij\gamma h}. \tag{5.2}$$

Here, γ represents the mode number and h denotes the element size. The Fourier mode (5.1) is substituted for (3.10) and the Fourier mode (5.2) is substituted for (3.11). The Fourier mode method cannot be applied to the system (3.11) because it contains a nonlinear term $p(p+1)U^pU_x$. Instead, the system must first be linearized and then the Von Neumann method is applied, assuming that the amount of $p(p+1)U^p$ in the nonlinear term is taken as a local constant like z_j. One of the most popular methods for analyzing the stability analysis of approximation systems for linear or linearized partial differential equations is Von Neumann analysis. Using the Euler formula $e^{i\Phi} = cos\Phi + isin\Phi$, the following growth factors are obtained: ρ_1 and ρ_2

$$\rho_1 = \frac{A_1 - iB_1}{A_1 + iB_1}, \quad \rho_2 = \frac{A_1 - iC_1}{A_1 + iC_1}, \tag{5.3}$$

$$A_1 = \left(2 - \frac{40\mu}{h^2}\right) cos(2\gamma h) + \left(52 - \frac{80\mu}{h^2}\right) cos(\gamma h) + \left(66 + \frac{120\mu}{h^2}\right),$$

$$B = \frac{5\Delta t}{h} sin(2\gamma h) + \frac{50\Delta t}{h} sin(\gamma h),$$

and

$$C = \frac{5z_m\Delta t}{h} sin(2\gamma h) + \frac{50z_m\Delta t}{h} sin(\gamma h).$$

For $k_1, k_2, ..., k_9, k_{10}$ and $l_1, l_2, ..., l_9, l_{10}$ founded in section 3. It can be written $|\rho_1|.|\rho_2| = 1$. For the entire system with the Strang Splitting algorithm because $|\rho_1| \leq 1$, and $|\rho_2| \leq 1$ according to the von Neumann theory, which are satisfied. This makes it obvious that the systems (3.10) and (3.11) are unconditionally stable. Equation (5.3) yields $|\rho_1| = |\rho_2| = 1$, which explains this.

6. Numerical Experiments and Discussion

The error norms L_2 and L_∞ to demonstrate the perfection of numerical schemes in terms of accuracy and at the same time, invariants I_1, I_2 and I_3 such as mass, momentum and energy are examined to report how well numerical schemes preserve physical quantities. These are given in the following format

$$L_2 = ||U - U_N||_2 = \sqrt{h \sum_{j=0}^{N} (U - U_N)^2},$$

$$L_\infty = ||U - U_N||_\infty = \max_j |U - U_N|,$$

$$I_1 = \int_{x_L}^{x_R} U dx,$$
$$I_2 = \int_{x_L}^{x_R} [U^2 + \mu (U_x)^2] dx,$$
$$I_3 = \int_{x_L}^{x_R} [U^4 - \mu (U_x)^2] dx.$$

The analytical solution of the GRLW equation is presented as follows in [7]

$$U(x,t) = \left(\frac{c(p+2)}{2p} sech^2 \left[\frac{p}{2} \sqrt{\frac{c}{\mu(c+1)}} (x - (c+1)t - x_0) \right] \right)^{1/p}$$

in which $\dfrac{c(p+2)}{2p}$ is the amplitude, $c+1$ is the wave speed in the direction diffusion and x_0 is an arbitrary constant. In this study, it would be good to mention that these calculations are obtained for the problems of single solitary wave and intersection of two solitary waves and the growth of the Maxwellian initial condition.

Table 1: The error norms at $t = 20$ for $\mu = 1$ of the single solitary wave

	$p=2$			$p=3$			$p=4$		
$c \rightarrow$	0.03	0.1	0.3	0.03	0.1	0.3	0.03	0.1	0.3
$amp. \rightarrow$	0.17	0.31	0.54	0.29	0.43	0.62	0.38	0.52	0.68
h Δt									
L_2x10^3									
0.1 0.01	0.99567	0.01186	0.00785	1.33411	0.01308	0.02711	1.57429	0.01429	0.07221
0.2 0.01	0.87463	0.01082	0.00954	1.17194	0.01199	0.03564	1.38292	0.01357	0.10329
0.1 0.025	0.99567	0.01240	0.04850	1.33411	0.01588	0.16665	1.57430	0.02532	0.44136
0.2 0.025	0.87463	0.01143	0.05016	1.17194	0.01521	0.17511	1.38292	0.02592	0.47230
L_∞x10^3									
0.1 0.01	0.41622	0.00668	0.00353	0.55769	0.00732	0.01290	0.65810	0.00782	0.03584
0.2 0.01	0.41622	0.00668	0.00446	0.55769	0.00732	0.01750	0.65810	0.00782	0.05238
0.1 0.025	0.41622	0.00668	0.02178	0.55769	0.00732	0.07910	0.65810	0.00908	0.21863
0.2 0.025	0.41622	0.00668	0.02268	0.55769	0.00732	0.08369	0.65810	0.00980	0.23440
	$p=6$			$p=8$			$p=10$		
$c \rightarrow$	0.03	0.1	0.3	0.03	0.1	0.3	0.03	0.1	0.3
$amp. \rightarrow$	0.17	0.31	0.54	0.29	0.43	0.62	0.38	0.52	0.68
h Δt									
L_2x10^3									
0.1 0.01	1.88672	0.02196	0.33489	2.07926	0.06272	1.21542	2.20933	0.20864	4.16765
0.2 0.01	1.65737	0.03011	0.58326	1.82651	0.11858	2.68428	1.94078	0.46826	1.21572
0.1 0.025	1.88673	0.09767	2.02469	2.07930	0.36196	7.39167	2.20953	1.22335	27.2734
0.2 0.025	1.65738	0.10812	2.27201	1.82656	0.41870	8.84774	1.94105	1.48236	35.0904
L_∞x10^3									
0.1 0.01	0.78870	0.00848	0.17647	0.86919	0.02863	0.66880	0.92356	0.10223	2.38050
0.2 0.01	0.78870	0.01243	0.31129	0.86919	0.05636	1.48646	0.92356	0.23179	6.92021
0.1 0.025	0.78870	0.04350	1.06550	0.86919	0.170238	4.06219	0.92356	0.60052	15.5740
0.2 0.025	0.78870	0.04869	1.19985	0.86919	0.19792	4.87227	0.92356	0.72958	19.9888

6.1. First example: A single solitary wave

In the first example, to compare numerical solutions, the parameters in the studies [5, 8, 27, 28, 11, 4, 32, 3, 6, 7] are taken into consideration. As in these studies, the solution region $[0,100]$],and $x_0 = 40$, $\mu = 1$ are selected. Calculations are performed for different values $h, \Delta t, p$ and c until time $t = 20$. First, for different values of $\Delta t, h$ and p , the situation with solitary waves with amplitudes of $0.17, 0.31$ and 0.54 for speeds $c = 0.03, 0.1$ and 0.3, respectively, is considered and the solutions are found at time $t = 20$. The results of the error norms L_2 and L_∞ that provide the solutions are depicted in Table 1. This table shows that the error norms L_2 and L_∞ produce results that are as small as intended. Secondly, conservation constants and error norms are calculated at $t = 10$ with different values of $\Delta t, h$ and c for $p = 2, 3$ and 4. The data of these calculations are depicted in Tables $2, 3, 5, 7, 9$ and 11 and based on the results, it is concluded that the conservation quantities are well preserved and the error norms are small enough. Thirdly, the datas of conservation quantities I_1, I_2 and I_3 and error norms L_2 and L_∞ in Tables $3, 5, 7, 9$ and 11 are compared with those obtained by different methods in the literature. The results of the comparison are listed in Tables $4, 6, 8, 10$ and 12. It can be seen from these tables that the solutions obtained thanks to the collocation method combined with the Strang splitting algorithm proposed in this study are as perfect as they are promising. Figure 1 shows the motion of single solitary wave at various times t and with different values of p. From this figure, it is possible to see that the solitary wave, traveling at a constant speed, moves towards the right and still maintains its shape and and increases the energy of this wave with increasing p values.

6.2. Second example: The interaction of two solitary waves

In the second example, the GRLW equation with initial condition presented in the following form, written as the linear sum of two well-separated solitary waves traveling in the same direction and having different amplitudes, is targeted. Numerical calculations are performed with conditions $\Delta t = 0.025, h = 0.2, c_1 = 4, c_2 = 1, x_1 = 25, x_2 = 55, \mu = 1$ for $p = 2$ on the region $[0,250]$ at $t = 0(4)20$, $\Delta t = 0.01, h = 0.1, c_1 = 48/5, c_2 = 6/5, x_1 = 20, x_2 = 50, \mu = 1$ for $p = 3$ on the region $[0,120]$ at $t = 0(1)6$, and $\Delta t = 0.01, h = 0.125, c_1 = 64/3, c_2 = 4/3, x_1 = 20, x_2 = 80, \mu = 1$ for $p = 4$ on the region $[0,200]$ at $t = 0(1)6$. For this purpose, conservation quantities are computed. The solutions of all calculations are reported in Tables 13-15, comparing with those in [7]. As can be observed from these tables, the conservation quantities calculated with the collocation

Table 2: Invariants and errors for single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 0.3$ on the region $[0,100]$ for $p = 2$

t	I_1	I_2	I_3	L_2	L_∞
0	3.58196673	1.34507649	0.15372303	0.000000000000	0.000000000000
2	3.58196673	1.34507640	0.15372312	0.000001861902	0.000001007019
4	3.58196673	1.34507629	0.15372323	0.000003499876	0.000001725688
6	3.58196673	1.34507621	0.15372331	0.000005003949	0.000002352101
8	3.58196673	1.34507616	0.15372336	0.000006443838	0.000002949496
10	3.58196673	1.34507612	0.15372340	0.000007851673	0.000003535734

Table 3: Invariants and errors for single solitary wave with $\Delta t = 0.025, h = 0.2$, $\mu = 1$, and $c = 1$ on the region $[0,100]$ from 0 to 10 in increments of 2 for $p = 2$

t	I_1	I_2	I_3	L_2	L_∞
0	4.44288294	3.29983161	1.41421360	0.0000000000	0.0000000000
2	4.44288294	3.29979589	1.41424932	0.0002939377	0.0001776334
4	4.44288294	3.29977191	1.41427330	0.0005531879	0.0003079917
6	4.44288294	3.29976284	1.41428237	0.0007998396	0.0004328585
8	4.44288294	3.29975926	1.41428595	0.00010430543	0.0005568057
10	4.44288294	3.29975778	1.41428743	0.0012853426	0.0006805326

Table 4: The error norms and invariants of the single solitary wave with $\Delta t = 0.025, h = 0.2$, $\mu = 1$, and $c = 1$ on the region $[0,100]$ for $p = 2$ at $t = 10$

method	I_1	I_2	I_3	L_2	L_∞
present	4.4428829	3.2997577	1.41428743	0.0012853426	0.0006805326
[11]	4.4428679	3.2998244	1.4142061	0.009619	0.004971
[5]	4.44288	3.29981	1.41416	0.00300533	0.00168749
[8]first approach	4.442866	3.299822	1.414204	0.002632463	0.001393064
[8]second approach	4.442866	3.299715	1.414312	0.002571481	0.001340210
[6]	4.4431	3.3003	1.4146	0.0024175	0.0010809
[27] B-spline coll-CN	4.442	3.299	1.413	0.01639	0.00924
[27] B-spline coll + PA-CN	4.440	3.296	1.411	0.0203	0.0112
[28]	4.44288	3.29983	1.41420	0.00930196	0.00543718
[7]	4.4428	3.2997	1.4143	0.0025893	0.0013518
[4]	4.443175	3.300302	1.414692	0.002415468	0.001079686
[32]	4.4431	3.3003	1.4146	0.0024155	0.0010797
[10]	4.4428	3.2998	1.4141	0.0030053	0.0016874

Table 5: Invariants and errors for single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 0.3$ on the region $[0,100]$ from 0 to 10 in increments of 2 for $p = 3$

t	I_1	I_2	I_3	L_2	L_∞
0	3.67755181	1.56574088	0.22683850	0.00000000	0.00000000
2	3.67755181	1.56574072	0.22683866	0.00000297	0.00000183
4	3.67755181	1.56574048	0.22683891	0.00000581	0.00000324
6	3.67755181	1.56574027	0.22683912	0.00000852	0.00000449
8	3.67755181	1.56574010	0.22683928	0.00001119	0.00000570
10	3.67755181	1.56573997	0.22683942	0.00001383	0.00000689

Table 6: Comparation of invariants and errors for single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 0.3$ on the region $[0, 100]$ for $p = 3$

method	I_1	I_2	I_3	L_2	L_∞
method	3.67755181	1.56573997	0.22683942	0.0000138	0.00000689
[5]	3.67755000	1.56574000	0.22683700	0.0000719	0.0000377
[8]second approach	3.67760690	1.56576200	0.22684460	0.0000785	0.0000365
[6]	3.6776	1.5657	0.2268	0.0001913	0.0000779

Table 7: Invariants and errors for single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 0.3$ on the region $[0, 100]$ from 0 to 10 in increments of 2 for $p = 4$

t	I_1	I_2	I_3	L_2	L_∞
0	3.7592300	1.7300029	0.2894090	0.0000000	0.0000000
2	3.7592300	1.7300024	0.2894095	0.0000069	0.0000044
4	3.7592300	1.7300017	0.2894101	0.0000138	0.0000078
6	3.75923000	1.7300012	0.2894107	0.0000206	0.0000111
8	3.75923000	1.7300008	0.2894111	0.0000276	0.0000144
10	3.75923000	1.7300004	0.2894114	0.0000347	0.0000178

Table 8: Comparation of invariants and errors for single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 0.3$ on the region $[0, 100]$ for $p = 4$

method	I_1	I_2	I_3	L_2	L_∞
method	3.7592300	1.7300004	0.2894114	0.0000347	0.0000178
[5]	3.7592300	1.7299900	0.2894060	0.0001225	0.0000662
[8]second approach	3.7592863	1.7300259	0.2894169	0.0000980	0.0000480
[6]	3.7592	1.7300	0.2894	0.0003089	0.0001444

Table 9: Invariants and errors for single solitary wave with $\Delta t = 0.025, h = 0.1$, $\mu = 1$, and $c = 6/5$ on the region $[0, 100]$ from 0 to 10 in increments of 2 for $p = 3$

t	I_1	I_2	I_3	L_2	L_∞
0	3.79712709	2.88122489	0.97293454	0.000000000	0.000000000
2	3.79712709	2.88110865	0.97305079	0.000117031	0.000719921
4	3.79712709	2.88105895	0.97310049	0.000227035	0.000133748
6	3.79712709	2.88104403	0.97311540	0.000033573	0.000195366
8	3.79712709	2.88103884	0.973120604	0.000444397	0.000257147
10	3.79712709	2.88103667	0.97312277	0.000553257	0.000319084

Table 10: Comparation of invariants and errors for the single solitary wave with $\Delta t = 0.025, h = 0.1$, $\mu = 1$, and $c = 6/5$ on the region $[0, 100]$ at $t = 10$ for $p = 3$

method	I_1	I_2	I_3	L_2	L_∞
present	3.797127	2.881036	0.973122	0.005532	0.003190
[11]	3.797185	2.881252	0.973157	0.011026	0.006355
[5]	3.79713	2.88123	0.972243	0.007767	0.004708
[8]first approach	3.797185	2.881252	0.973145	0.008972	0.005175
[8]second approach	3.797133	2.881089	0.973128	0.007778	0.004441
[6]	3.801670	2.888066	0.979294	0.013291	0.008478
[4]	3.797282	2.881293	0..973446	0.006128	0.003722

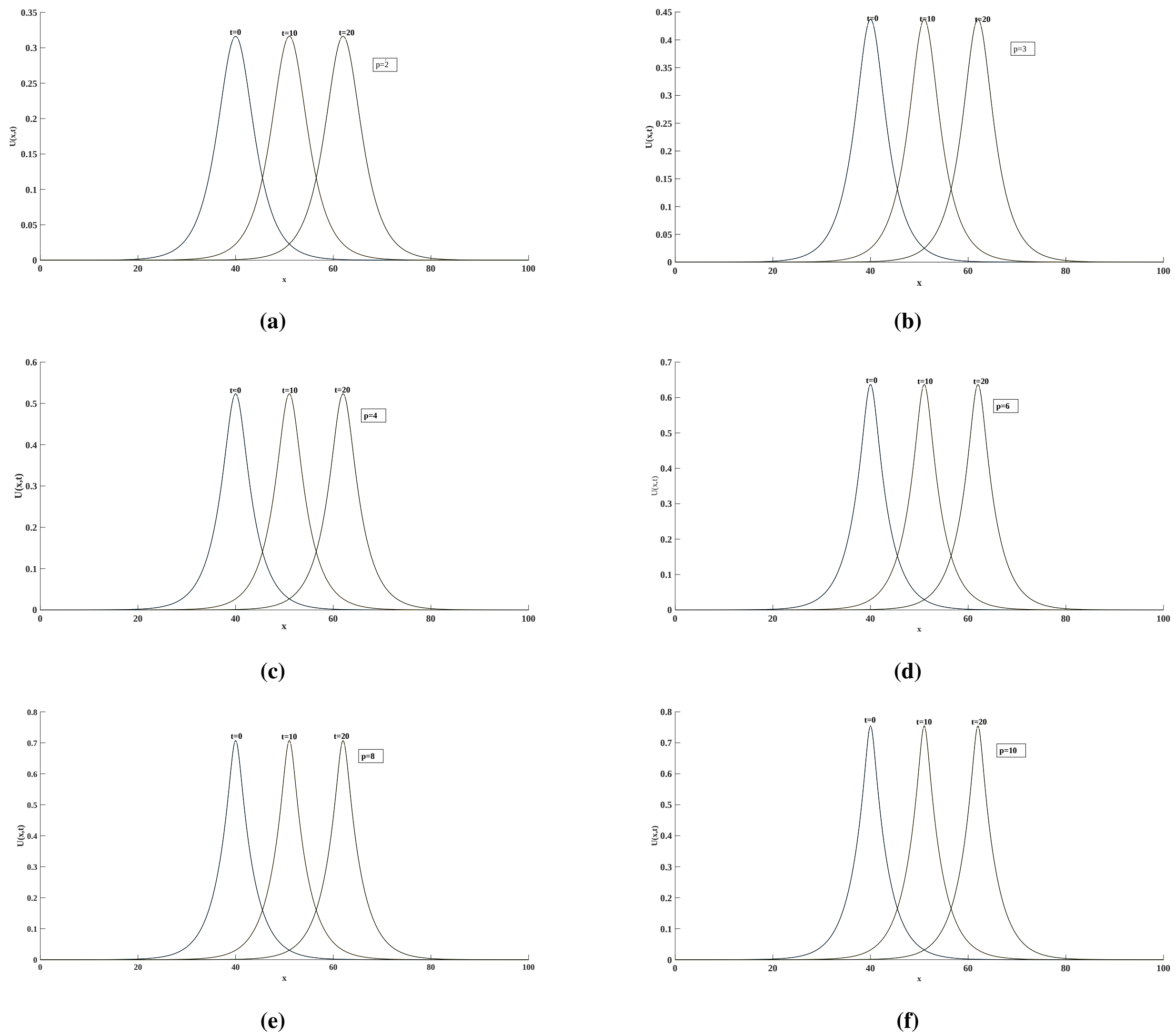


Figure 1: A single solitary wave movement at $[0,100]$ for $c = 0,1$ and $x_0 = 40$

Table 11: Invariants and errors for single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 4/3$ on the region $[0, 100]$ from 0 to 10 in increments of 2 for $p = 4$

t	I_1	I_2	I_3	L_2	L_∞
0	3.46865611	2.67167341	0.72917047	0.000000000	0.000000000
2	3.46865611	2.67163139	0.72921249	0.000481813	0.000307699
4	3.46865611	2.67161752	0.72922636	0.000949370	0.000588519
6	3.46865611	2.67161379	0.72923009	0.000141623	0.000872966
8	3.46865611	2.67161260	0.72923128	0.000188397	0.000115795
10	3.46865611	2.67161212	0.72923176	0.000235269	0.000144089

Table 12: Comparation of invariants and errors for the single solitary wave with $\Delta t = 0.01, h = 0.1$, $\mu = 1$, and $c = 4/3$ on the region $[0, 100]$ at $t = 10$ for $p = 4$

method	I_1	I_2	I_3	L_2	L_∞
present	3.46865611	2.67161212	0.72923176	0.002352	0.001440
[11]	3.468709	2.671696	0.729303	0.008696	0.005314
[5]	3.46866	2.67168	0.728881	0.002460	0.001566
[8]first approach	3.468709	2.671696	0.729258	0.003351	0.002049
[8]second approach	3.468671	2.671658	0.729237	0.002698	0.001656
[6]	3.470439	2.674445	0.731987	0.001511	0.000857
[4]	3.468799	2.671742	0.730001	0.001283	0.000821

method combined with the Strang splitting algorithm are compatible with those in ref.[7] presented with the quintic B-spline collocation method. Figures 2 and 3 depict the action of interaction of two solitary waves for various times. It can be clearly seen from these figures that at $t = 0$, the wave with lower energy is located to the right of the wave with larger energy. Later, the wave with greater energy catches up with the smaller one and leaves it behind.

6.3. Last example: The Maxwellian initial condition

In the last example, the problem of how the Maxwell pulse presented as follows, which appears as the initial condition, turns into a solitary waves is examined.

$$U(x,0) = exp(-(x-40)^2).$$

Here, the value of μ determines how the solution behaves [4]. As a result, for $p = 2, 3, 4$, with values of $\mu = 0.025, 0.05$, and $\mu = 0.1$, numerical calculations are completed until time $t = 0.05$. Table 16 displays the calculated numerical invariants at various t values and this table shows that the invariants are quite compatible among themselves. Figure 4 illustrates how the Maxwellian initial condition developed into solitary waves.

Table 13: Comparison of invariants of two solitary waves with values $\Delta t = 0.025, h = 0.2$, for $x_1 = 25, x_2 = 55, c_1 = 4, c_2 = 1$ on the region $[0, 250]$ at $t = 0(4)20$ for $p = 2$ with those in [7]

	method			[7]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	11.46769767	14.62924187	22.88046714	11.4676	14.6292	22.8803
4	11.46769767	14.62560599	22.88410302	11.4676	14.6277	22.8818
8	11.46769767	14.13410877	23.37560024	11.4676	14.1399	23.3695
12	11.46769767	14.67865616	22.83105285	11.4676	14.6803	22.8292
16	11.46769767	14.64185929	22.86784972	11.4676	14.6442	22.8653
20	11.46769767	14.62835609	22.88135292	11.4676	14.6309	22.8786

Table 14: Comparison of invariants of two solitary waves with values $\Delta t = 0.01, h = 0.1$, for $x_1 = 20, x_2 = 50, c_1 = 48/5, c_2 = 6/5$ on the region $[0,120]$ at $t = 0(1)6$ for $p = 3$ with those in [7]

	method			[7]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	9.69074161	12.94438041	17.01872563	9.6907	12.9443	17.0186
1	9.69074161	12.93790956	17.02519648	9.6894	12.9433	17.0197
2	9.69074161	12.93262436	17.03048168	9.6881	12.9391	17.0239
3	9.69074161	12.31072166	17.65238437	9.6851	12.3044	17.6586
4	9.69074161	12.96109129	17.00201474	9.6860	12.9704	16.9926
5	9.69074161	13.04585327	16.91725276	9.6848	13.0539	16.9091
6	9.69074161	12.99335590	16.96975014	9.6835	13.0028	16.9601

Table 15: Comparison of invariants of two solitary waves with values $\Delta t = 0.01, h = 0.125$, for $x_1 = 20, x_2 = 50, c_1 = 64/3, c_2 = 4/3$ on the region $[0,200]$ at $t = 0(1)6$ for $p = 4$ with those in [7]

	method			[7]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	8.83427261	12.17088582	14.02942463	8.8342	12.1708	14.0294
1	8.83427261	11.47188138	14.72842908	8.6650	11.9332	14.2670
2	8.83427261	11.33376433	14.86654612	8.5662	11.7919	14.4083
3	8.83427261	11.25540256	14.94490789	8.4965	11.6913	14.5090
4	8.83427261	11.20082492	14.99948554	8.4529	11.4644	14.7358
5	8.83427261	11.08672895	14.97358150	8.4089	11.7254	14.4748
6	8.83427261	11.00520465	14.98510581	8.3702	11.5990	14.6012

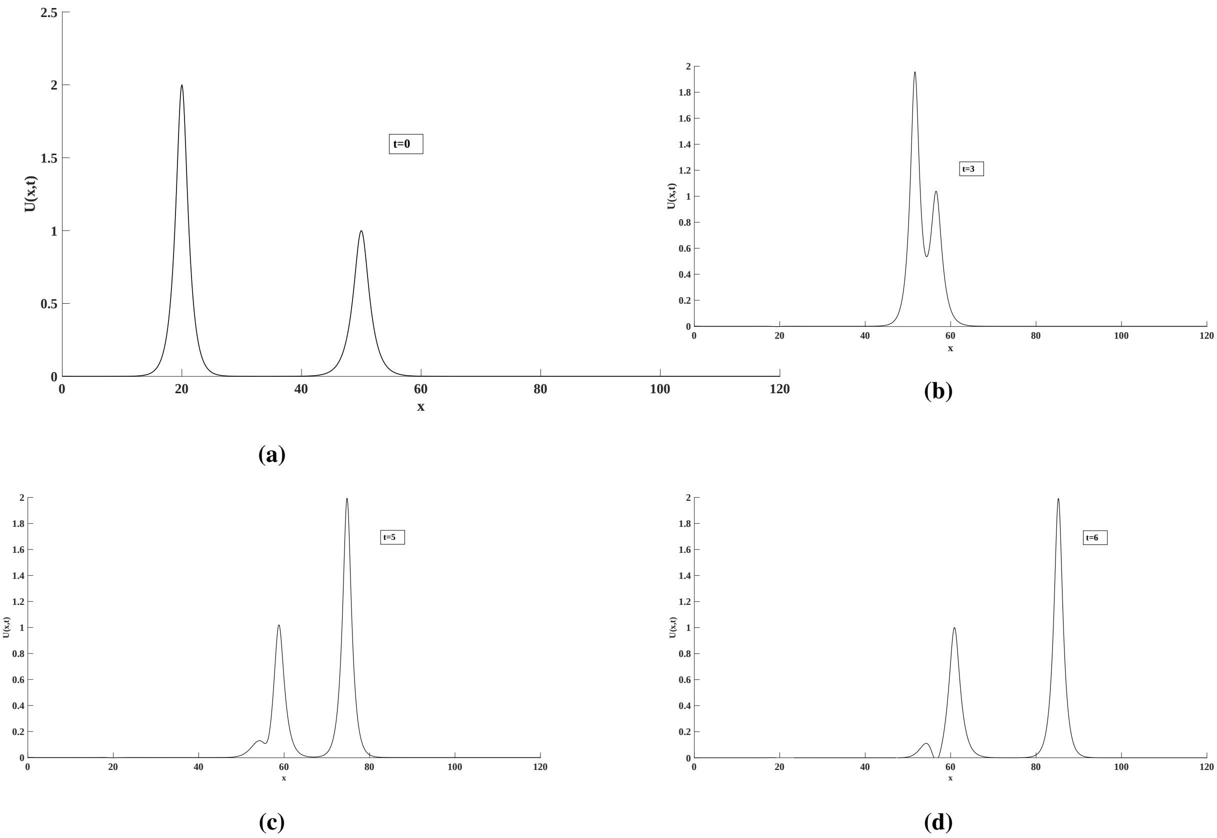


Figure 2: The interactions of two solitary waves at $p = 3$

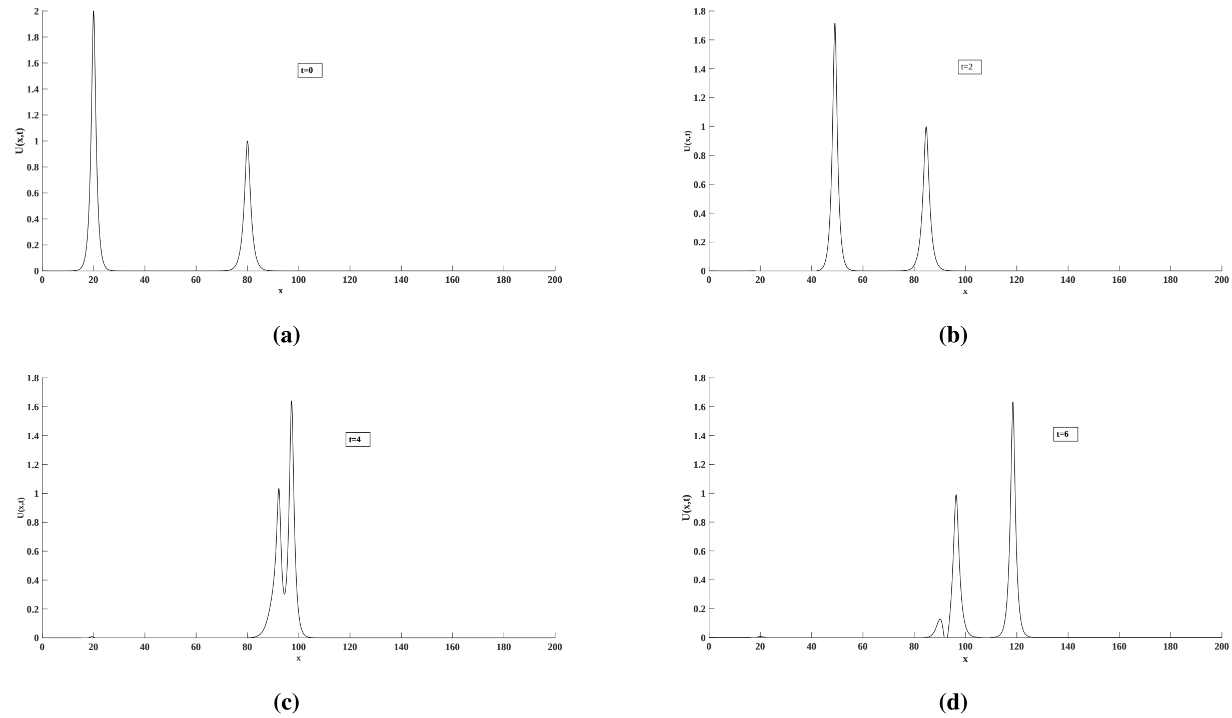
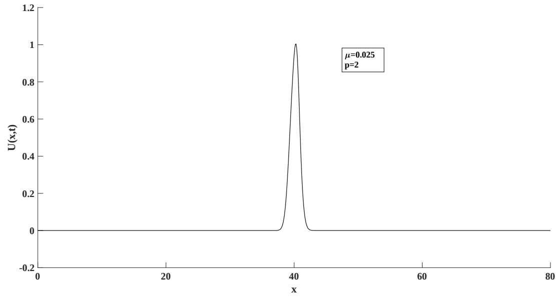


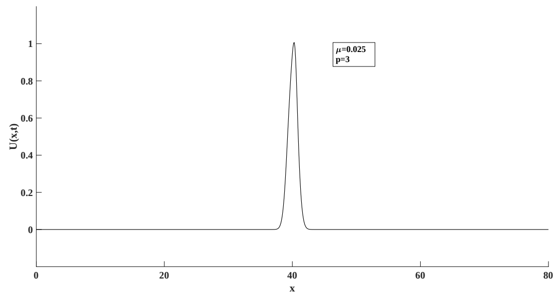
Figure 3: The interactions of two solitary waves at $p = 4$

Table 16: Maxwellian initial condition for various μ values

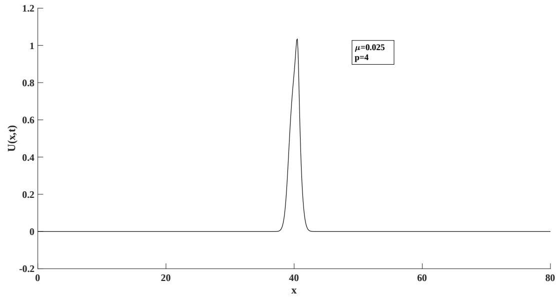
		$p=2$			$p=3$			$p=4$		
μ	t	I_1	I_2	I_3	I_1	I_2	I_3	I_1	I_2	I_3
0.025	0.01	1.77245	1.28464	0.85485	1.77245	1.28464	0.85475	1.77245	1.28464	0.85454
	0.03	1.77245	1.28464	0.85457	1.77245	1.28464	0.85361	1.77245	1.28464	0.85165
	0.05	1.77245	1.28464	0.85399	1.77245	1.28464	0.85125	1.77245	1.28464	0.84541
0.05	0.01	1.77245	1.31597	0.82352	1.77245	1.31597	0.82341	1.77245	1.31597	0.82320
	0.03	1.77245	1.31597	0.82322	1.77245	1.31597	0.82224	1.77245	1.31597	0.82034
	0.05	1.77245	1.31597	0.82261	1.77245	1.31597	0.81988	1.77245	1.31597	0.81455
0.1	0.01	1.77245	1.37864	0.76087	1.77245	1.37864	0.76078	1.77245	1.37864	0.76062
	0.03	1.77245	1.37864	0.76065	1.77245	1.37864	0.75989	1.77245	1.37864	0.75844
	0.05	1.77245	1.37864	0.76021	1.77245	1.37864	0.75809	1.77245	1.37864	0.75410



(a)



(b)



(c)

Figure 4: Graphics of Maxwell initial condition for different p values at $t = 0.05$

7. Conclusion

To obtain the solitary-wave solutions of the GRLW problem, this paper establishes two different linearization techniques, the collocation method and the Strang splitting algorithm. In order to achieve this, the collocation method is combined with the Strang splitting algorithm to perform numerical calculations, and the collocation method is applied to each scheme. In particular, the error norms L_2, L_∞ and the invariants I_1, I_2, and I_3 have been calculated for each of the three examples: A single solitary wave, the interaction of two solitary waves and the Maxwellian initial condition. The results obtained are listed in tables and figures. These tables show how the invariant values agree with other findings and the variations of the invariants are quite small. The figures show that the method applied in the article is compatible with the figures in similar examples in the literature. Compared to previous numerical methods, smaller error norms are obtained. The outcome of the error norms obtained are superior to those from earlier numerical techniques. As a result, based on the results produced in this study, it can be said with certainty that the numerical scheme that has been presented is more preferred and trustworthy for improving the numerical solutions of the physically significant nonlinear partial differential equations.

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