Rings Whose Certain Modules are Dual Self-CS-Baer

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Abstract
In this work, we characterize some rings in terms of dual self-CS-Baer modules (briefly, ds-CS-Baer modules). We prove that any ring \( R \) is a left and right artinian serial ring with \( J^2(R) = 0 \) iff \( R \oplus M \) is ds-CS-Baer for every right \( R \)-module \( M \). If \( R \) is a commutative ring, then we prove that \( R \) is an artinian serial ring iff \( R \) is perfect and every \( R \)-module is a direct sum of ds-CS-Baer \( R \)-modules. Also, we show that \( R \) is a right perfect ring iff all countably generated free right \( R \)-modules are ds-CS-Baer.

Keywords: Dual self-CS-Baer module, Harada ring, Lifting module, Perfect ring, QF-ring, Serial ring

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1. Introduction
Throughout the paper, all rings will have an identity element and all modules will be unitary right modules unless otherwise stated. Let \( M \) be a module and \( N \) a submodule of \( M \). Then \( N \ll M \) means that \( N \) is a small submodule of \( M \) (namely, \( M \) is different from \( N + K \) for every proper submodule \( K \) of \( M \)). \( J(R) \) will denote the Jacobson radical of any ring \( R \) and \( \operatorname{Rad}(M) \) will denote the radical of any module \( M \).

A module \( M \) is called lifting (or satisfies \((D_1)\)), if every submodule \( N \) of \( M \) lies above a direct summand, that is, \( N \) contains a direct summand \( X \) of \( M \) such that \( N/X \ll M/X \) (see [1] and [2]). A module \( M \) is said to be dual self-CS-Baer (briefly, ds-CS-Baer) if for every family \((f_i)_{i \in I}\) of homomorphisms \( f_i : M \to M, \sum_{i \in I} \operatorname{Im}(f_i) \) lies above a direct summand of \( M \) (see [3]). Clearly, every lifting module is ds-CS-Baer. Moreover, if \( R \) is a right Harada ring, then every injective right \( R \)-module is ds-CS-Baer. Because, remember that any ring \( R \) is called a right Harada ring if every injective right \( R \)-module is lifting (see [1]). Recall that any right \( R \)-module \( M \) is called hollow, if every proper submodule of \( M \) is small in \( M \) (see [2, Definition 4.1]) and it is called local, if it is hollow and \( \operatorname{Rad}(M) \neq M \). Note that \( M \) is local iff \( M \) is cyclic and has a unique maximal submodule (see [4, page 357]). It is not hard to see that every hollow module and so every local module is a lifting module.

In recent years, ds-CS-Baer modules and their related topics have been studied by Crivei, Keskin Tütüncü, Radu and Tribak (see for example [3], [5] and [6]). In this paper, we continue the study of ds-CS-Baer modules.

In section 2, we characterize some rings in terms of ds-CS-Baer modules. Among others, we mainly prove the followings:

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(A) Let $R$ be a ring. Then $R$ is an artinian serial ring with $J^2(R) = 0$ iff for every right $R$-module $M$, $R \oplus M$ is ds-CS-Baer (Theorem 2.1).

(B) Let $R$ be a right self-injective ring. Then $R$ is a QF-ring iff every injective right $R$-module is ds-CS-Baer (Theorem 2.3).

(C) Let $R$ be a ring. Then $R$ is a right perfect ring iff every free right $R$-module is ds-CS-Baer (Theorem 2.4).

(D) Let $R$ be a commutative ring. Then $R$ is semiperfect iff every cyclic $R$-module is ds-CS-Baer (Proposition 2.1).

(E) Let $R$ be a commutative ring. Then $R$ is an artinian serial ring iff $R$ is perfect and every 2-f.p. $R$-module is a finite direct sum of ds-CS-Baer modules (Proposition 2.4).

2. Results

We first give the following easy observation.

**Lemma 2.1.** Let $R$ be a ring. Let $M$ be a free right $R$-module. Then $M$ is lifting iff it is ds-CS-Baer.

**Proof.** Let $M$ be a free right $R$-module. Then we can assume that $M = \oplus_{i \in I} R$. Now the result is obvious by the proof of [3, Proposition 9.4].

Let $R$ be a ring and $M$ a module. $M$ is called uniserial if its submodules are linearly ordered by inclusion and is called serial if it is a direct sum of uniserial submodules. The ring $R$ is called right (left) serial if the right (left) $R$-module $R_R$ ($R_L$) is serial. Also $R$ is called artinian serial if it is both right and left artinian serial. By [4, Theorem 32.3], we know that if $R$ is an artinian serial ring, then every right $R$-module and every left $R$-module is a direct sum of uniserial $R$-modules.

Now, we characterize artinian serial rings with $J^2(R) = 0$ via ds-CS-Baer modules.

**Theorem 2.1.** Let $R$ be a ring. Then the following assertions are equivalent:

1. $R$ is an artinian serial ring with $J^2(R) = 0$.
2. Every right $R$-module is lifting.
3. For every right $R$-module $M$, $R \oplus M$ is lifting.
4. For every right $R$-module $M$, $R \oplus M$ is ds-CS-Baer.

**Proof.** (1) $\Leftrightarrow$ (2): It is satisfied by [1, 29.10].

(3) $\Leftrightarrow$ (4): It is proved in [3, Proposition 9.4].

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (2): It is clear since lifting property is preserved by direct summands (see for example [1, Lemma 22.6]).

The next result is a consequence of Theorem 2.1.

**Corollary 2.1.** Let $R$ be a ring. Then $R$ is an artinian serial ring with $J^2(R) = 0$ iff every (finitely generated) right $R$-module is ds-CS-Baer.

**Proof.** This follows from [7, Theorem 3.15], [3, Proposition 9.4] and Theorem 2.1 and the fact that being ds-CS-Baer or lifting is preserved by taking direct summands.

**Remark 2.1.** The left-handed versions of Theorem 2.1 and Corollary 2.1 are equal to being artinian serial ring with $J^2(R) = 0$.  

A finitely generated right \( R \)-module \( M \) is said to be \textit{finitely presented} in case in every exact sequence
\[
0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0
\]
with \( F \) finitely generated and free the kernel \( K \) is also finitely generated. An exact sequence of right \( R \)-modules
\[
P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0
\]
is called a \textit{minimal projective presentation} of \( M \) in case \( P_1 \) and \( P_0 \) are finitely generated projective and \( \text{Ker} f \ll P_1 \) and \( \text{Im} f \ll P_0 \). Let \( M \) a finitely presented right \( R \)-module with no nonzero projective direct summands. Following [4], \( M \) is called a 2-f.p. module if there are primitive idempotents \( e, e_1 \) and \( e_2 \) of \( R \) and there is a minimal projective presentation
\[
eR \rightarrow e_1 R \oplus e_2 R \rightarrow M \rightarrow 0.
\]
Therefore a 2-f.p. module is both 2-primitive generated and finitely presented.

Recall from [8] that a module \( M \) is called \( w \)-local if it has a unique maximal submodule. Clearly, a module \( M \) is local if and only if \( M \) is a cyclic \( w \)-local module.

Next, we can give the following.

\textbf{Theorem 2.2.} Let \( R \) be a ring. Consider the following statements:

1. \( R \) is serial and every direct sum of two ds-CS-Baer right \( R \)-modules and every direct sum of two ds-CS-Baer left \( R \)-modules is ds-CS-Baer.
2. Every finitely presented right \( R \)-module and finitely presented left \( R \)-module is ds-CS-Baer.
3. Every 2-generated finitely presented right \( R \)-module and 2-f.p. left \( R \)-module is ds-CS-Baer.
4. \( R \) is semiperfect and every 2-f.p. right \( R \)-module and 2-f.p. left \( R \)-module is ds-CS-Baer.

Then (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4).

\textit{Proof.} (1) \( \Rightarrow \) (2): Let \( M \) be a finitely presented right \( R \)-module and \( N \) a finitely presented left \( R \)-module. By [9, Corollary 3.4], \( M \) and \( N \) are finite direct sum of cyclic \( w \)-local submodules. In particular, they are finite direct sum of local submodules. Since local modules are lifting, they are also ds-CS-Baer. Therefore \( M \) and \( N \) are ds-CS-Baer by (1).

(2) \( \Rightarrow \) (3) \( \Rightarrow \) (4): These are clear by definitions and [3, Proposition 5.9].

Inspired by Theorem 2.1, we give the following theorem that characterizes QF-rings. First, remember that any ring \( R \) is called a QF-ring, if \( R \) is noetherian and injective as a left (or right) \( R \)-module (see for example [4, page 333]).

\textbf{Theorem 2.3.} Let \( R \) be a right self-injective ring. Then the following assertions are equivalent:

1. \( R \) is a QF-ring.
2. \( R \) is a right Harada ring.
3. For every injective right \( R \)-module \( M \), \( R \oplus M \) is lifting.
4. For every injective right \( R \)-module \( M \), \( R \oplus M \) is ds-CS-Baer.
5. Every injective right \( R \)-module is ds-CS-Baer.

\textit{Proof.} (1) \( \Leftrightarrow \) (2): It is clear by [1, 28.10 and 28.16].

(3) \( \Leftrightarrow \) (4): It is clear by [3, Proposition 9.4].

(2) \( \Rightarrow \) (3): Let \( M \) be an injective right \( R \)-module. By hypothesis, \( R \oplus M \) is an injective right \( R \)-module. Since \( R \) is right Harada, it follows that \( R \oplus M \) is lifting.

(3) \( \Rightarrow \) (2): Let \( M \) be an injective right \( R \)-module. By (3), \( R \oplus M \) is lifting. Therefore, \( M \) is lifting. Hence, \( R \) is a right Harada ring.

(4) \( \Leftrightarrow \) (5): It is clear.
Theorem 2.4. Let $R$ be a ring. Then the following assertions are equivalent:

(1) $R$ is a right perfect ring.

(2) $R^{(1)}$ is a ds-CS-Baer right $R$-module.

(3) Every countably generated free right $R$-module is ds-CS-Baer.

(4) Every free right $R$-module is ds-CS-Baer.

Proof. (1) $\Rightarrow$ (2): Assume that $R$ is a right perfect ring. Consider the right $R$-module $M = R^{(1)}$. By [2, Theorem 4.41], $M$ is lifting, and so it is ds-CS-Baer by definitions.

(2) $\Rightarrow$ (1): Assume that the right $R$-module $R^{(1)}$ is ds-CS-Baer. Since it is free, by Lemma 2.1, it is lifting. Hence it is $\oplus$-supplemented. Therefore, $R$ is a right perfect ring by [7, Theorem 2.10].

(1) $\Rightarrow$ (4): Let $M$ be a free right $R$-module. Then $M$ is projective. So, $M$ is lifting by [2, Theorem 4.41]. Thus, $M$ is ds-CS-Baer by definitions.

(4) $\Rightarrow$ (1): Assume that every free right $R$-module is ds-CS-Baer. Then every free right $R$-module is lifting by Lemma 2.1. By [2, Theorem 4.41], $R$ is a right perfect ring.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2): These are clear. 

Next, we give a characterization of commutative semiperfect rings in terms of cyclic dual self-CS-Baer modules.

Proposition 2.1. Let $R$ be a commutative ring. Then $R$ is semiperfect iff every cyclic $R$-module is ds-CS-Baer.

Proof. Let $R$ be a semiperfect ring. Let $M$ be a cyclic $R$-module. Assume that $M = xR$, where $x \in M$. We know that $M \cong R/I$, for some ideal $I$ of $R$. By [1, 4.9 (1)], since $I$ is fully invariant in $R$, $R/I$ is quasi-projective and hence $M$ is quasi-projective. Then by [2, Theorem 4.41], $M$ is lifting and so $M$ is ds-CS-Baer.

Conversely, assume that every cyclic $R$-module is ds-CS-Baer. Then $R$ is a ds-CS-Baer $R$-module. Therefore by [3, Proposition 5.9], $R$ is semiperfect.

Now, we give a characterization of commutative semiperfect FGC-rings. Let $R$ be a commutative ring. $R$ is called an FGC-ring, if every finitely generated $R$-module is a direct sum of cyclic modules (see [10]).

Proposition 2.2. Let $R$ be a commutative ring. Then the following assertions are equivalent:

(1) Every finitely generated $R$-module is $\oplus$-supplemented.

(2) Every finitely generated $R$-module is a finite direct sum of ds-CS-Baer modules.

(3) $R$ is a semiperfect FGC-ring.

(4) $R$ is a direct sum of almost maximal valuation rings.

Proof. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): These are proved in [7, Proposition 2.8].

(1) $\Rightarrow$ (2): Let $M$ be a finitely generated $R$-module. By (1), $M$ is $\oplus$-supplemented. By [7, Corollary 2.6], $M = \oplus_{i=1}^{n} x_i R$. Note that each $x_i R$ is quasi-projective since $R$ is commutative. Therefore by [2, Theorem 4.41], each $x_i R$ is lifting and so ds-CS-Baer.

(2) $\Rightarrow$ (1): Let $M$ be a finitely generated $R$-module. By (2), $M = \oplus_{i=1}^{n} x_i R$, where each $x_i R$ is ds-CS-Baer. By [3, Proposition 5.12], each $x_i R$ is lifting and hence $\oplus$-supplemented. Therefore by [11, Theorem 1.4], $M$ is $\oplus$-supplemented. 

Corollary 2.2. Let $R$ be a commutative indecomposable ring. Then $R$ is an almost maximal valuation ring iff every finitely generated $R$-module is a direct sum of cyclic ds-CS-Baer $R$-modules.

Next, we characterize commutative serial rings via direct sums of cyclic ds-CS-Baer modules.

Proposition 2.3. Let $R$ be a commutative ring. Then the following assertions are equivalent:

(1) $R$ is serial.
(2) $R$ is semiperfect and every 2.f.p. $R$-module is $\oplus$-supplemented.

(3) $R$ is semiperfect and every finitely presented $R$-module is a finite direct sum of $ds$-CS-Baer modules.

(4) $R$ is semiperfect and every 2-generated finitely presented $R$-module is a finite direct sum of $ds$-CS-Baer modules.

(5) $R$ is semiperfect and every 2.f.p. $R$-module is a finite direct sum of $ds$-CS-Baer modules.

Proof. $(1) \iff (2)$: This follows from [7, Theorem 3.5].

$(1) \Rightarrow (3)$: Clearly, $R$ is semiperfect. Now, let $M$ be a finitely presented $R$-module. Note that $M$ is finitely generated. By [9, Corollary 3.4], $M = \oplus_{i=1}^{n} M_i$, where each $M_i$ is $u$-local and cyclic. Note that each $M_i$ $(1 \leq i \leq n)$ is a local module. Hence each $M_i$ is $ds$-CS-Baer.

$(3) \Rightarrow (4) \Rightarrow (5)$: These are clear.

$(5) \Rightarrow (2)$: Let $M$ be a 2-f.p. $R$-module. By (5), $M = \oplus_{i=1}^{n} M_i$, where each $M_i$ is a cyclic $ds$-CS-Baer $R$-module. By [3, Proposition 5.12], each $M_i$ is lifting and hence $\oplus$-supplemented. Hence $M$ is $\oplus$-supplemented by [11, Theorem 1.4].

Finally, we characterize commutative artinian serial rings as follows.

**Proposition 2.4.** Let $R$ be a commutative ring. Then the following assertions are equivalent:

(1) $R$ is an artinian serial ring.

(2) $R$ is perfect and every 2.f.p. $R$-module is $\oplus$-supplemented.

(3) $R$ is perfect and every $R$-module is a direct sum of $ds$-CS-Baer modules.

(4) $R$ is perfect and every countably generated $R$-module is a direct sum of $ds$-CS-Baer modules.

(5) $R$ is perfect and every finitely presented $R$-module is a finite direct sum of $ds$-CS-Baer modules.

(6) $R$ is perfect and every 2-f.p. $R$-module is a finite direct sum of $ds$-CS-Baer modules.

Proof. $(1) \iff (2)$: It is proved in [7, Corollary 3.13].

$(1) \Rightarrow (3)$: By [4, Corollary 28.8], $R$ is a perfect ring. Now, let $M$ be any $R$-module. By [4, Theorem 32.3], $M = \oplus_{i=1}^{n} M_i$, where each $M_i$ is uniserial. Clearly every uniserial module is hollow. Since $R$ is perfect, then each $M_i$ has small radical (see [4, Remark 28.5]). Therefore, each $M_i$ is local, and so cyclic. Hence $M$ is a direct sum of cyclic $ds$-CS-Baer modules.

$(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$: These are clear.

$(6) \Rightarrow (2)$: Let $M$ be a 2-f.p. $R$-module. By (6), $M = \oplus_{i=1}^{n} M_i$, where each $M_i$ is a cyclic $ds$-CS-Baer $R$-module. By [3, Proposition 5.12], each $M_i$ is lifting and hence $\oplus$-supplemented. Therefore $M$ is $\oplus$-supplemented by [11, Theorem 1.4].

Propositions 2.3 and 2.4 are not true over noncommutative rings as we see in the following example.

**Example 2.1.** (see [7, Example 3.16]) Let $R$ be a local artinian ring with Jacobson radical $J(R)$ such that $J^2(R) = 0$, $Q = R/J(R)$ is commutative, $\dim(QJ(R)) = 1$ and $\dim(J(R)Q) = 2$. Then $R$ is left serial but not right serial. Let $J(R) = uR \oplus vR$. $A_1 = R/J(R)$, $A_2 = R/uR$ and $A_3 = R/vR$ are the only three isomorphism types of indecomposable right $R$-modules. Here each $A_i$ is lifting and hence $ds$-CS-Baer. Note that every right $R$-module is a direct sum of indecomposable modules, and hence a direct sum of cyclic $ds$-CS-Baer modules. However, $R$ is not a serial ring.

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