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On $n-\delta$ −semiprimary Ideals in Commutative Rings

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Abstract – Let R be a commutative ring with identity and n a positive integer. A generalization of prime ideals is introduced in (Anderson and Badawi, 2021). A proper ideal J of R is said to be an n-semiprimary ideal if whenever $a,b\in R$ with $a^nb^n\in J$, then $a^n\in J$ or $b^n\in J$. Let $\delta: Id(R)\longrightarrow Id(R)$ be an expansion function of ideals of R where $Id(R)$ is the set of all ideals of R. The aim of this paper is to introduce the class of $n-\delta$ −semiprimary ideals generalizing the notion of n −semiprimary ideals. We call a proper ideal J of R an $n-\delta$ −semiprimary ideal if whenever $a^nb^n\in J$ for $a,b\in R$, then $a^n\in\delta(J)$ or $b^n\in\delta(J)$. Several properties and characterizations regarding this class of ideals with many supporting examples are presented. Additionally, we call a proper ideal J of R a strongly $n-\delta$ −semiprimary ideal of R if whenever $K^nL^n\subseteq J$ for proper ideals K and L of R, then $K^n\subseteq\delta(J)$ or $L^n\subseteq\delta(J)$. We investigate the relationship between these two concepts. Moreover, the behaviour of $n-\delta$ −semiprimary ideals under homomorphisms, in localization rings, in division rings, in cartesian product of rings and in idealization rings is investigated.

1. Introduction and Preliminaries

Throughout this article, all rings are assumed to be commutative with identity. By $Id(R)$ and $Id(R)^*$, we denote the set of all ideals and particularly, the set of all proper ideals of a ring R, respectively. Recall from Zhao (2001) that a function $\delta: Id(R)\longrightarrow Id(R)$ providing $J\subseteq\delta(J)$, and whenever $J\subseteq K$ implies $\delta(J)\subseteq\delta(K)$ for all $J,K\in I(R)$ is called an expansion of ideals (in briefly e.f.i). For example, the identity function δ_J, where $\delta_J(J)=J$ for all $J\in Id(R)$, is a trivial e.f.i of R. Also, the function $\delta_{\sqrt{J}}(J)=\sqrt{J}$ for each ideal J of R is an e.f.i of R. Generalizing the concept of prime ideals, in 2001, Zhao introduced the concept of δ-primary ideals. According to Zhao (2001), given an e.f.i δ of ideals of R, $J\in Id(R)^*$ is called a δ-primary ideal in R if $a,b\in R$ with $ab\in J$, then $a\in$ J or $b\in\delta(J)$. After that, Badawi et al. (Badawi et al., 2018) defined the class of δ-semiprimary ideals. $J\in Id(R)^*$ is said to be δ-semiprimary in R if $a,b\in R$ with $ab\in J$ implies $a\in\delta(J)$ or $b\in\delta(J)$. As a different generalization of prime ideals, Anderson and Badawi defined n-semiprimary ideals in their recent research (Anderson and Badawi, 2021). Let $n\geq 1$. Then, $J\in Id(R)^*$ is called an n-semiprimary ideal of R if whenever $a^nb^n\in J$ for $a,b\in R$, then $a^n\in J$ or $b^n\in J$. Clearly, 1-semiprimary ideal is a just prime ideal. For the other extentions of prime and primary ideals, the reader may consult for example (Anderson and Badawi, 2011), (Badawi and Fahid, 2018), (Badawi et al., 2018), (Yetkin Celikel, 2021), (Hamoda, 2023) and (Ulucak et al., 2018).

The motivation of writing this article lies to create new concepts that can be used in many branches in commutative algebra and its applications and to develop related results. In section 2, we present the main results concerning $n-\delta$ −semiprimary ideals with supporting examples and counterexamples. Among many results in this paper, the behavior of this class of ideals under homomorphisms, localizations, cartesian products and idealizations are investigated. We proved that if $J\in Id(R)^*$ and $P^n\subseteq\delta(J)$ for a positive integer n where P be a prime ideal of R including J, then J is $k-\delta$ −semiprimary in R for any integer $k\geq n$.

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2. Properties of $n-\delta-$semiprimary ideals

Our starting point is the following definition. Unless otherwise stated, throughout δ is assumed to be an expansion function of ideals (e.f.i) of a ring R and n ∈ ℕ.

Definition 1 Let δ be an e.f.i of R and $J \in Id(R)^*$. Then, J is said to be $n-\delta-$semiprimary in R if whenever $a^n b^n \in J$ for $a, b \in R$, then $a^n \in \delta(J)$ or $b^n \in \delta(J)$.

It is clear to see that a $1-\delta-$semiprimary ideal coincides with $\delta-$semiprimary ideal. Any $\delta-$semiprimary ideal and n-semiprimary ideal is $n-\delta-$semiprimary. However, the converses of these relationships do not hold in general. The following two examples are presented to justify that there are $n-\delta-$semiprimary ideals of a ring R that are not an $n-$semiprimary ideal.

Example 1 Consider $R=\mathbb{Z}$ and $J=16\mathbb{Z}$. Then $\sqrt{J}=2\mathbb{Z}$ and clearly I is $2-\delta_{\sqrt{J}}-$semiprimary in R. However, J is not a $2-$semiprimary in R as $2^2 2^2 \in J$ and $2^2 \notin J$.

Example 2 Consider $R=\mathbb{Z}_2[\{X_n\}_{n=1}^{\infty}]$ and the ideal $K=(\{X_n^n\}_{n=1}^{\infty})$ of R. Then K is $n-\delta_{\sqrt{I}}-$semiprimary in R where $\delta_{\sqrt{K}}(K)=\sqrt{K}=(\{X_n\}_{n=1}^{\infty})$. On the other hand, K is not an $n-$semiprimary ideal for each $n \geq 1$ as $X_{2n}^n X_{2n}^n = X_{2n}^{2n} \in K$, but $X_{2n}^n \notin K$.

Next, we introduce an $n-\delta-$semiprimary ideal of a ring R which is not $\delta-$primary ideal.

Example 3 Let $A=\mathbb{Z}_2[X_1, X_2]$ with indeterminates X_1 and X_2. Take $I=(X_2^2, X_1X_2)A$ and $K=(X_2^2, X_1^2X_2^2)A$ and let $R=A/J$. Then, $\sqrt{I/K}=(X_2, X_1X_2)A/K$. One can check easily that K is $n-\delta_{\sqrt{I}}-$semiprimary in R *for* any $n \geq 1$. However, I/K is not a $\delta_{\sqrt{I}}-$primary in R, since $X_1X_2+K \in I/J$, $X_1+K \notin \delta_{\sqrt{I}}(I/K)=\sqrt{I/K}$ and $X_2+K \notin I/K$.

Proposition 1 For $J \in Id(R)^*$, we have the following statements.

1. If J is $n-\delta-$semiprimary in R, then J is $mn-\delta-$semiprimary in R for all $m \in \mathbb{N}$.

2. J is $n-\delta_I-$semiprimary in R if and only if J is $n-$semiprimary in R.

3. Let δ and γ be two e.f.i of R such that $\delta(J) \subseteq \gamma(J)$. If J is $n-\delta-$semiprimary in R, then J is $\gamma-\delta-$semiprimary in R.

4. If $\delta(\delta(J))=\delta(J)$, then $\delta(J)$ is $n-\delta-$semiprimary in R if and only if $\delta(J)$ is $n-$semiprimary in R.

Proof Straightforward.

Proposition 2 Let J ∈ Id(R)*. Then, J is $n-\delta-$semiprimary in R if and only if either $\delta(J)$ is a prime or $\delta(J)$ is $n-\delta-$semiprimary in R providing $\delta(\delta(J))=\delta(J)$.

Proof Assume that $\delta(J)$ is an $n-\delta-$semiprimary in R and let $a^n b^n \in J$ for $a, b \in R$ and $a^n \notin \delta(J)$. Since $J \subseteq \delta(J)$ and $\delta(J)$ is an $n-\delta-$semiprimary in R, we have $b^n \in \delta(\delta(J))=\delta(J)$, as required. The converse part is clear.

Theorem 1 Let $J \in Id(R)^*$ and $Q^n \subseteq \delta(J)$ for some $n \geq 1$ where Q is prime in of R including J. Then, for all $k \geq n$, J is an $k-\delta-$semiprimary in R.

Proof Let $a^k b^k \in J \subseteq Q$ for $a, b \in R$ and $k \geq n$. Then, $a \in Q$ or $b \in Q$. Hence, $a^k \in Q^k \subseteq \delta(J)$ or $b^k \in Q^k \subseteq \delta(J)$, and therefore J is $k-\delta-$semiprimary in R.

As direct consequences of Theorem 1, we verify the following corollary

Corollary 1 Let R be a Noetherian ring and $J \in Id(R)^*$ such that $\sqrt{J}$ is prime. Then, there is an $n \geq 1$ provided that J is $k-\delta-$semiprimary in R for any $k \geq n$.

Proof Put $Q=\sqrt{J}$. Then there is an $n \geq 1$ satisfying $Q^n \subseteq J \subseteq \delta(J)$ as R is Noetherian. Therefore, from by Theorem 1, J is $k-\delta-$semiprimary for any $k \geq n$.

Corollary 2 For prime ideals $J_1 \subseteq \ldots \subseteq J_k$ of R and positive integers $n_1, \ldots, n_k$, $J = J_1^{n_1} \ldots J_k^{n_k}$ is $m-\delta-$semiprimary where $m \geq n_1 + \cdots + n_k$.

Proof Since $\sqrt{J} = J_1$ is prime and $J_1^n \subseteq J_1^{n_1} \ldots J_k^{n_k} = J$, where $n = n_1 + \cdots + n_k$, Theorem 1 yields that J is $m-\delta-$semiprimary where $m \geq n = n_1 + \cdots + n_k$.

The following example is given to illustrate that the converse of Theorem 1 need not be true.

Example 4 Let $R = \mathbb{Z}_q[X, Y]$, where $q \geq 2$ be a prime integer and let $J = (X^q, Y^q)$. Then $\sqrt{J} = (X, Y)$. $(\sqrt{J})^q \nsubseteq J$ since $YX^{q-1} \notin J$. On the other hand, let $g^q h^q \in \sqrt{J} \subseteq (X, Y)$ for g, h ∈ R. Then, $g \in (X, Y)$ or $h \in (X, Y)$. Thus, $g^q \in J \subseteq \delta(J)$ or $h^q \in J \subseteq \delta(J)$ and hence, J is $q-\delta-$semiprimary in R.

Let $J \in Id(R)^*$. We say that J is $n-\delta-$primary in R if there exists $n \geq 1$ if whenever $ab \in J$ for $a, b \in R$ implies either $a \in \delta(J)$ or $b^n \in \delta(J)$. Next, we show that $n-\delta-$primary ideals are subclass of the class of $n-\delta-$semiprimary ideals.

Proposition 3 Any $n-\delta-$primary ideal is an $n-\delta-$semiprimary ideal.

Proof Let J be $n-\delta-$primary in R. Assume that $a^n b^n \in J$ for $a, b \in R$ and $a^n \notin \delta(J)$. Let k be the minimum positive integer satisfying $a^n b^k \in \delta(J)$. Then, $(a^n b^{k-1})b = a^n b^k \in \delta(J)$. Since $a^n b^{k-1} \notin \delta(J)$ and J is $n-\delta-$primary in R, we have $b^n \in \delta(J)$; so we are done.

Now, we give an example for an $n-\delta-$semiprimary ideal of a ring R which is not $n-\delta-$primary ideal for all n.

Example 5 Let $R = \mathbb{Z}_2[X_1, X_2]$ with indeterminates X_1 and X_2. For all $n \geq 2$, consider $K = (X_1X_2, X_2^n)$. Then, $Q = \sqrt{K} = (X_2)$ is prime in R and $Q^n \subseteq K$. Define $\delta: Id(R) \longrightarrow Id(R)$ by $\delta(J) = J + M$ for each ideal J of R, where (X_1, X_2) is the unique maximal ideal. Thus, δ is an e.f.i of R. By Theorem 1, K is an $n-\delta-$semiprimary in R. However, $X_2X_1 \in K, X_2 \notin \delta(K)$ and $X_1^m \notin \delta(K)$ for any $m \in \mathbb{N}$. Hence, K is not $m-\delta-$primary in R for all $m \in \mathbb{N}$.

Recall from Zhao (2001) that an e.f.i δ of a ring R is said to be intersection preserving if $\delta(I_1 \cap \ldots \cap I_n) = \delta(I_1) \cap \ldots \cap \delta(I_n)$ for any ideals $I_1, \ldots, I_n$ of R.

Proposition 4 Suppose that δ is intersection preserving and $J_1, \ldots, J_t$ are $n-\delta-$semiprimary ideals of R satisfying $\delta(J_i) = \delta(J_k)$ for all $i,\ k \in \{1,2, \ldots, t\}$. Then, $\bigcap_{i=1}^t J_i$ *is* $n-\delta-$semiprimary in R.

Proof Assume that $a^n b^n \in \bigcap_{i=1}^t J_i$ for $a, b \in R$ and $a^n \notin \delta(\bigcap_{i=1}^t J_i)$. Since $\delta(\bigcap_{i=1}^t J_i) = \bigcap_{i=1}^t \delta(J_i) = \delta(J_i)$, we have $a^n \notin \delta(J_i)$. Since $a^n b^n \in J_i$ for all $i \in \{1,2, \ldots, t\}$ and J_i is $n-\delta-$semiprimary, we have $b^n \in \delta(J_i) = \delta(\bigcap_{i=1}^t J_i)$ for all $i \in \{1,2, \ldots, t\}$, so we are done.

Proposition 5 Let $I_1,\ I_2,\ I_3 \in Id(R)^*$ with the order $I_1 \subseteq I_2 \subseteq I_3$. *If* I_3 is an $n-\delta-$semiprimary ideal of R such that $\delta(I_1) = \delta(I_3)$, then I_2 is an $n-\delta-$semiprimary ideal of R.

Proof Let $a^n b^n \in I_2$ for $a, b \in R$ and $a^n \notin \delta(I_2)$. From our assumptions, we have $\delta(I_1) = \delta(I_2) = \delta(I_3)$. From the inclusion $I_1 \subseteq I_2$, we have $a^n b^n \in I_2$. Since I_3 is an $n-\delta-$semiprimary ideal of R and $a^n \notin \delta(I_3)$, we conclude $b^n \in \delta(I_3) = \delta(I_2)$. Thus, I_2 is an $n-\delta-$semiprimary ideal of R.

Recall from (Ulucak et al., 2018) that if $f: R \longrightarrow S$ is a homomorphism or rings, γ and δ are e.f.i of R and S, respectively, then it is said that f is a $\gamma\delta-$ring homomorphism if $\gamma(f^{-1}(J)) = f^{-1}(\delta(J))$ for all $J \in Id(S)$. In this case, we have $f(\gamma(J)) = \delta(f(J))$ for all $J \in Id(R)$.

Proposition 6 Let γ and δ be e.f.i of R and R', respectively, and $f: R \longrightarrow R'$ be a $\gamma\delta-$ring homomorphism.

1. If J' is $n-\delta-$semiprimary in R', then $f^{-1}(J')$ is $n-\gamma-$semiprimary in R.

2. Suppose that f is onto and $J \in Id(R)^*$containing $ker(f)$. Then J is $n-\gamma-$semiprimary in R if and only if $f(J)$ is $n-\delta-$semiprimary in R'.

Proof 1. Let J' be an $n-\delta-$semiprimary in R' and $a^nb^n \in f^{-1}(J')$ for $a,b \in R$. Then, $f(a^nb^n) = f(a^n)f(b^n) \in J'$ which yields that either $f(a^n) \in \delta(J')$ or $f(b^n) \in \delta(J')$. Hence, $a^n \in f^{-1}(\delta(J'))$ or $b^n \in f^{-1}(\delta(J'))$ and we are done as $f^{-1}(\delta(J')) = \gamma(f^{-1}(J'))$.

2. Let $a,b \in R'$ and $a^nb^n \in f(J)$. Say, $a^n = f(c)^n$ and $b^n = f(d)^n$ for some $c,d \in R$. Then, clearly $f(c)^nf(d)^n = f(c^nd^n) \in f(J)$ and $ker(f) \subseteq J$ imply that $c^nd^n \in J$. Since J is an $n-\gamma-$semiprimary in R, we have either $c \in \gamma(J)$ or $d^n \in \gamma(J)$. Thus, $a^n \in f(\gamma(J))$ or $b^n \in f(\gamma(J))$. The claim follows from $f(\gamma(J)) = \delta(f(J))$.

Recall from Ulucak et al. (2018) that if for an e.f.i δ of R and $J \in Id(R)^*$, the function $\delta_q: R/J \longrightarrow R/J$ defined by $\delta_q(K/J) = \delta(K)/J$ for $K \in Id(R)$ with $J \subseteq K$ is also an e.f.i of R/J. Hence, we conclude the next result for quotient rings.

Corollary 3 Let $I,K \in Id(R)^*$ with the order $I \subseteq K$. Then, K is $n-\delta-$semiprimary in R if and only if K/I is $n-\delta_q-$semiprimary in R/I.

Example 6 Consider the polynomial ring $R[X]$ and its e.f.i. δ. Let $\delta_q: R[X]/(X) \longrightarrow R[X]/(X)$ defined by $\delta_q(K/(X)) = \delta(K)/(X)$ for all ideals $(X) \subseteq K$ of $R[X]$. Then, δ_q is an e.f.i of $R[X]/(X) \approx R$. For any $J \in Id(R)^*$, of R, since $(J,X)/(X) \approx J$, from Corollary 3, (J,X) is $n-\delta-$semiprimary in $R[X]$ if and only if J is $n-\delta_q-$semiprimary in R.

Let S be a multiplicatively closed subset (in briefly, m.c.s) of a ring R and δ be an e.f.i of R. Then, a function δ_S defined by $\delta_S(I_S) = (\delta(I))_S$ is an e.f.i of R_S.

Proposition 7 Let S be a m.c.s of R and $J \in Id(R)^*$. Then we have the following statements.

1. Suppose that J is $n-\delta-$semiprimary in R with $J \cap S = \emptyset$. Then, J_S is $n-\delta_S-$semiprimary in R_S.

2. Suppose that J_S is an $n-\delta_S-$semiprimary ideal of R_S satisfying $Z_{\delta(J)}(R) \cap S = \emptyset$. Then J is $n-\delta-$semiprimary in R.

Proof 1. Let $a,b \in R_S$ and $a^nb^n \in J$. Then, $a = \frac{r_1}{s_1}$ and $b = \frac{r_2}{s_2}$ for some $r_1,r_2 \in R$ and $s_1,s_2 \in S$. Hence, $ur_1^nr_2^n \in J$ for some $u \in S$ and so $(ur_1)^nr_2^n \in J$ yields either $(ur_1)^n \in \delta(J)$ or $r_2^n \in \delta(J)$.Thus, we conclude either $a^n = \frac{u^nr_1^n}{u^ns_1^n} \in \delta(J)_S$ or $b^n = \frac{r_2^n}{s_2^n} \in \delta(I)_S$ and we are done as $\delta_S(J_S) = \delta(J)_S$.

2. Let $a,b \in R$ and $a^nb^n \in J$. Then we have $(\frac{a}{1})^n(\frac{b}{1})^n \in J$ which implies either $(\frac{a}{1})^n\ \delta(J_S)$ or $(\frac{b}{1})^n \in \delta(J_S)$. Since $\delta_S(J_S) = (\delta(J))_S$, there are some $u,u' \in S$ satisfying $ua^n \in \delta(J)$ or $u'b^n \in \delta(J)$. Now, $Z_{\delta(I)}(R) \cap S = \emptyset$ implies that we have either $a^n \in \delta(J)$ or $b^n \in \delta(J)$, as required.

Now, we give the following definition.

Definition 2 Let δ be an e.f.i of a ring $R, J \in Id(R)^*$ and $n \geq 1$. J is said to be strongly $n-\delta-$semiprimary in R if whenever $K^nL^n \subseteq I$ for some $K,L \in Id(R)^*$, then $K^n \subseteq \delta(I)$ or $L^n \subseteq \delta(I)$.

Observe that a strongly $1-\delta-$semiprimary ideal is just a $\delta-$semiprimary ideal. Any strongly $n-\delta-$semiprimary ideal is an $n-\delta-$semiprimary ideal. In the following example, we show that those are distinct concepts.

Example 7 Let δ_I be an e.f.i of the ring $\mathbb{Z}_2[X,Y]$, and let $J = (X^2, Y^2)$. By Example 4, I is $2-\delta_J-$semiprimary in $\mathbb{Z}_2[X,Y]$, with prime ideal $K = J = (X,Y)$. It is clear that $K^2K^2 = K^4 \subseteq J$, but $K^2 \nsubseteq J = \delta_J(J)$. Thus, J is not strongly $2-\delta_J-$semiprimary in $\mathbb{Z}_2[X,Y]$.

We recall from Anderson et al. (1994) that for a $J \in Id(R)^*$, the ideal generated by n th powers of elements of J is denoted by $J_n = (a^n : a \in J)$. Note that $J_n \subseteq J^n \subseteq J$ and the equality holds when $n = 1$. Moreover, it is verified that if $n!$ is unit in R, then $J_n = J^n$. Next, we give a characterization for strongly $n-\delta-$semiprimary ideals of R.

Theorem 2 Let δ be an e.f.i of a ring $R, J \in Id(R)^*$ and $n \geq 1$ such that $n!$ is unit. Then we have the following equivalent three conditions.

1. J is strongly $n-\delta-$semiprimary in R.

2. For each element $a \in R$, any $L \in Id(R)$ with $a^n L^n \subseteq J$ and $a^n \not\subseteq \delta(J)$, we have $L^n \subseteq \delta(J)$.

3. J is $n-\delta-$semiprimary in R.

Proof (1) $\Longrightarrow$ (2) Let $a \in R$, $L \in Id(R)$ with $a^n L^n \subseteq J$ and $a^n \not\subseteq \delta(J)$. Put $K = \langle a \rangle$. Then $K^n \nsubseteq J$ and this implies that $L^n \subseteq \delta(J)$, as needed.

(2) $\Longrightarrow$ (3) Suppose that $K^n L^n \subseteq J$ for K , $L \in Id(R)^*$ and $K^n \nsubseteq \delta(J)$. Since $n!$ is a unit, we have $J_n = J^n$, and hence $a^n \not\subseteq \delta(J)$ for some $a \in K$. Thus, $L^n \subseteq \delta(J)$ by (ii).

(3) $\Longrightarrow$ (1) Let $a, b \in R$ and $a^n b^n \in J$. Taking $L = \langle b \rangle$ in (iii), we are done.

Let $R_1, \ldots, R_k$ be commutative rings with identity and $R = R_1 \times \ldots \times R_k$. Recall from Badawi and Fahid (2018) that an ideal of $R = R_1 \times \ldots \times R_k$ has the form $I_1 \times \ldots \times I_k$ for some ideals I_i of R_i for each $i = 1, \ldots, k$. Then, $\delta_\times$ be an e.f.i of R which is defined by $\delta_\times(I_1 \times \ldots \times I_k) = \delta_1(I_1) \times \ldots \times \delta_k(I_k)$ for each ideal I_i of R_i where δ_i is an e.f.i of R_i for each $i \in \{1, \ldots, k\}$. Next, we characterize $n-\delta_\times-$semiprimary ideals of cartesian product of rings.

Theorem 3 Let δ_1 and δ_2 be e.f.i of rings R_1 and R_2, respectively. For $J_1 \times J_2 \in Id(R_1 \times R_2)^*$, the following are equivalent.

1. $J_1 \times J_2$ is an $n-\delta_\times-$semiprimary in $R_1 \times R_2$.

2. J_1 is $n-\delta_1-$semiprimary in R_1 and $\delta_2(J_2) = R_2$ or J_2 is $n-\delta_2-$semiprimary in R_2 and $\delta_1(J_1) = R_1$.

Proof Note that if $\delta_\times(J) = R$, then the claim is clear.

(1) $\Longrightarrow$ (2) Assume that both of $\delta_1(J_1)$ and $\delta_2(J_2)$ are proper. Since $(0,0) = (1,0)^n(0,1)^n \in J$ but neither $(1,0)^n \in \delta_\times(J)$ nor $(0,1)^n \in \delta_\times(J)$, we get a contradiction. Hence, we may assume that $\delta_1(J_1)$ is proper and $\delta_2(J_2) = R_2$. Suppose that $a^n b^n \in J_1$ and $a^n \notin \delta(J_1)$ for some $a, b \in R_1$. Then $(a,0)^n(b,0)^n \in J$ and $(a,0)^n \notin \delta_\times(J)$ which implies $(b,0)^n \in \delta_\times(J)$. Thus, $b^n \in J_1$ and J_1 is $n-\delta_\times-$semiprimary in R_1. In the case of $\delta_1(J_1) = R_1$ and $\delta_2(J_2) = R_2$ is similar.

(2) $\Longrightarrow$ (1) We may suppose that J_1 is $n-\delta_1-$semiprimary in R_1 and $\delta_2(J_2) = R_2$. Let $(a_1,a_2)^n(b_1,b_2)^n \in J = J_1 \times J_2$ such that $(a_1,a_2)^n \not\subseteq \delta_\times(J)$. Then $a_1^n b_1^n \in J_1$ and $a_1^n \notin \delta_1(J_1)$ imlply that $b_1^n \in \delta_1(J_1)$. Hence $(b_1,b_2)^n \in \delta_\times(J)$, so we are done.

In general, we conclude the following result.

Theorem 4 Let $R = R_1 \times \ldots \times R_k$, where $R_1, \ldots, R_k$ are rings for $k \leq 2 < \infty$. Let δ_i be an e.f.i of R_i for each $i \in \{1, \ldots, k\}$. Let $J = J_1 \times \ldots \times J_k \in Id(R)^*$ for some ideals $J_1, \ldots, J_k$ of $R_1, \ldots, R_k$, respectively. Then, we have the following equivalent statements.

1. J is $n-\delta_\times-$semiprimary in R.

2. Either $J = \prod_{r=1}^{k} J_r$ such that for some $t \in \{1, \ldots, k\}$, J_t is an $n-\delta_t-$semiprimary in R_t, and $J_r = R_r$ for every $r \in \{1, \ldots, k\}$ for every $r \in \{1, \ldots, k\}\backslash\{t\}$ or $J = \prod_{r=1}^{k} J_r$ such that for some $t, m \in \{1, \ldots, k\}$.

Proof We use the mathematical induction method. Suppose that $k = 2$. Then the claim holds by Theorem 3. Hence, let $3 \leq k < \infty$. Assume that the claim is true when $A = R_1 \times \ldots \times R_{k-1}$. We verify the claim when $R = A \times R_k$. Since clearly $\delta_A(J_1 \times \ldots \times J_{k-1}) = \delta_1(J_1) \times \ldots \delta_{k-1}(J_{k-1})$, from Theorem 3, J is $n-\delta_\times-$semiprimary in R if and only if either $J = B \times R_k$ for some $n-\delta_A-$semiprimary ideal B of A or $J = A \times B_k$ for some $n-\delta_k-$semiprimary ideal B_k of R_k . It must be clear that for a $P \in Id(A)^*$ is $n-\delta_A-$semiprimary in A if and only if $P = \prod_{r=1}^{k-1} J_r$ such that for some $t \in \{1, \ldots, k-1\}$, J_t is $n-\delta_t-$semiprimary in R_t, and $J_r = R_r$ for every $r \in \{1, \ldots, k-1\}\backslash\{t\}$, we are done.

Let δ be an e.f.i of a ring R. For $I \in Id(R)^*$, we define

$D_R(I) = \{n \in \mathbb{N}: I\ is\ an\ n - \delta - semiprimary\ ideal\ of\ R\}$ and $\mu_R(I) = min\ D_R(I)$.

If $D_R(I) = \emptyset$, we define $\mu_R(I) = \infty$.

Theorem 5 Let δ be an e.f.i of a commutative Noetherian integral domain R. If for any $J \in Id(R)$ with $\mu_R(J) = 2$ implies $J = M^2$ for some maximal ideal M of R, then R is a Dedekind domain.

Proof Assume that J is an ideal of R with $M^2 \subseteq J \subset M$ for a maximal ideal M of R. Then, J is $2 - \delta -$semiprimary in R by Theorem 1. Also, J is not prime (maximal). Thus, $\mu_R(J) = 2$. Thus, $J = M^2$ by assumption. Thus, we have no ideal of R satisfying $M^2 \subset J \subset M$ for every maximal ideal M of R and from Theorem 6.20 in Larson and McCarthy (1971), R is a Dedekind domain.

Recall that an integral domain R is said to be a valuation domain if either $x|y$ or $y|x\ (in\ R)$ for all $0 \neq x, y \in R$. We conclude the following result in valuation domains.

Theorem 6 Let δ be e.f.i of a valuation domain R with $\sqrt{\delta(J)} = \delta(\sqrt{J})$ and $I \in Id(R)^*$ with $K = \sqrt{J}$. *If* K *is* non-idempotent, then J is $n - \delta -$semiprimary in R for some $n \geq 1$.

Proof If $K = \sqrt{J}$ is not idempotent, then $K^n \subseteq J$ for some $n \in \mathbb{N}$, see Theorem 17.1 (5) in Gilmer (1972). Thus, J is $n - \delta -$semiprimary in R by Theorem 1.

Recall from Anderson and Winders (2009) that the idealization ring of an $R -$module M over a ring R defined by $R(+)M = R \times M$ with the following binary operations given by $(x, r) + (y, s) = (x + y, r + s)$ and $(x, r)(y, s) = (xy, yr + xs)$, respectively, and the identity id (1,0). Also, since clearly $(\{0\}(+)M)^2 = \{0\}$, $\{0\}(+)M \subseteq Nil(R(+)M)$.

We define a function $\delta_{(+)}: Id(R(+)M) \longrightarrow Id(R(+)M)$ such that $\delta_{(+)}(J(+)N) = \delta(J)(+)M$ for every ideal $J \in Id(R)$ and every submodule N of M. Then, $\delta_{(+)}$ is an e.f.i of $R(+)M$.

Theorem 7 Let δ be an e.f.i of a ring R, M be an $R -$module, A be a submodule of $M, J \in Id(R)^*$ ideal of R and $n \in \mathbb{N}$. Then, we have the following equivalent conditions.

1. $J(+)A$ is $n - \delta_{(+)} -$semiprimary in $R(+)M$.

2. J is $n - \delta -$semiprimary in R.

Proof (1) ⇒ (2) Let $J(+)A$ be $n - \delta_{(+)} -$semiprimary in $R(+)M$, and let $x^n y^n \in J$ for $x, y \in R$. Then, $(x^n, 0)(y^n, 0) = (x^n y^n, 0) \in J(+)A$. It implies that $x^n \in \delta(J)$ or $y^n \in \delta(J)$ since $J(+)A$ is $n - \delta_{(+)} -$semiprimary in $R(+)M$. Therefore, J is $n - \delta -$semiprimary in R.

(2) ⇒ (1) Assume that J is $n - \delta -$semiprimary in R, and let $(x, r)^n (y, s)^n = (x^n y^n, z) \in J(+)A$ where $z = n(y^n xr + x^n ys)$ and $(x, r), (y, s) \in R(+)M$. Hence, $x^n y^n \in$ J. Since J is $n - \delta -$semiprimary in R, we have $x^n \in \delta(J)$ or $y^n \in \delta(J)$. Since $\delta_{(+)}(J(+)A) = \delta(J)(+)M$, we conclude either $(x, r)^n \in \delta_{(+)}(J(+)A)$ or $(y, s)^n \in \delta_{(+)}(J(+)A)$. Therefore, $J(+)A$ is $n - \delta_{(+)} -$semiprimary in $R(+)M$.

3. Conclusion

In this study, a generalization of both of $n -$semiprimary and $\delta -$semiprimary ideals which is called $n - \delta -$semiprimary ideals is presented. By this way, we found answers to the following questions. What is the location of the algebraic structure of this class of ideals in the literature? Which properties of it is similar to those of $n -$semiprimary and $\delta -$semiprimary ideals? Is this property stable under localizations, homomorphisms, cartesian products and idealizations? (see: Proposition 6 and 7, Corollary 3, Theorems 2, 3 and 7). Consequently, there are many open questions arising from this study. What if one defines weakly $n - \delta -$semiprimary ideals with the following definition, what will be the differences between these two? For example, a proper ideal I of R a weakly $n - \delta -$semiprimary ideal if whenever $0 \neq a^n b^n \in I$ for $a, b \in R$, then $a^n \in \delta(I)$ or $b^n \in \delta(I)$. On the other hand, extending this algebraic structure in rings to submodules, $n - \delta -$semiprimary submodule of an R-module M can be described.

Ethics Permissions

This paper does not require ethics committee approval.

Author Contributions

Both of the authors conceived of the presented idea. Both of the authors discussed the results and contributed to the final manuscript.

Conflict of Interest

There are no conflicts of interests/competing interests.

References

Anderson, D. D., and Winders, M. (2009). Idealization of a module. *Journal of Commutative Algebra*, 1(1), 3-56.

Anderson, D. D., Knopp K. R., and Lewin, R. L. (1994). Ideals generated by powers of elements. *Bulletin of the Australian Mathematical Society*, 49(3), 373-376.

Anderson, D. F., and Badawi, A. (2011). On n −absorbing ideals of commutative rings. *Communications in Algebra*, 39(5), 1646-1672.

Anderson, D. F., and Badawi, A. (2021). On n −semiprimary ideals and n −Pseudo valuation domains. *Communications in Algebra,* 49(2), 500-520.

Badawi, A., and Fahid, B. (2018). On weakly 2 − absorbing δ − primary ideals of commutative rings. *Georgian Mathematical Journal*, 27(4), 1-13.

Badawi A., Sonmez D., and Yesilot, G. (2018). On weakly δ −semiprimary ideals of commutative rings. *Algebra Colloquium*, 25(3), 387-398.

Gilmer, R. (1972). *Multiplicative Ideal Theory*. Marcel Dekker, Inc., New York.

Hamoda, M. (2023). On (m, n) −closed δ −primary ideals of commutative rings. *Palestine Journal of Mathematics*, 12(2), 280–290.

Larson, M. D., and McCarthy, P. J. (1971). *Multiplicative Theory of Ideals*. Academic Press, New York, London.

Ulucak, G., Tekir, Ü., and Koç, S. (2018). On n −absorbing δ −primary ideals. *Turkish Journal of Mathematics*, 42(4), 1833-1844.

Yetkin Celikel, E. (2021). 2 −absorbing δ −semiprimary ideals of commutative rings. *Kyungpook Mathematical Journal*, 61(4), 711-725.

Zhao, D. (2001). δ −primary ideals of commutative rings. *Kyungpook Mathematical Journal*, 41(1), 17-122.