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Weekly Ill-posed integral geometry problems of Volterra type in three-dimensional space

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Abstract

In this paper considers the problem of recovering a function from families of spheres in space. The uniqueness of the solution of the problem is proved by reducing it to the Volterra integral equation of the first and then the second kind. Fourier transform methods are also used. Uniqueness theorems are proved for some new classes of operator equations of Volterra type in three-dimensional space.

Keywords: *Integral geometry problem, Fourier transform, uniqueness theorem, Volterra integral equation.*

Üç boyutlu uzayda Volterra tipindeki, zayıf nokorrekt integral geometri problemi

Öz

Bu makalede küre ailesinden uzaydaki bır fonksiyonnu kurtarılma problemi ele alınmaktadır. Volterranin önce birinci sonra ikinci tür integral denklemine getırmek yoluyla kanıtlanır. Furier değiştirme yöntemleri de kullanılmaktadır. Üç boyutlu uzayda Volterra tipindeki operatör denklemlerinin bazı yeni sınıfları için teklik teoremleri kanıtlanmıştır.

Anahtar kelimeler: *İntegral geometri problemi, Fourier değiştirilmesi, teklik teoremi, Volterra integral denklemi.*

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1. Introduction

Integral geometry problems naturally arise in the study of many mathematical models in seismology, in the interpretation of geophysical and aerospace observations, and in the study of various processes described by kinetic equations and others. The methods developed here are the mathematical foundations of computed tomography. As you know, computed tomography is a rapidly developing field of modern science. One of the central problems of integral geometry is also the problem of restoring a function from its integrals from given manifolds.

Let us give the definition of the problem of integral geometry [1].

Let $x \in R^n$, $x = (x_1,...,x_n)$, $S(y)$ - a family of manifolds in $x \in R^n$, depending on m, $\dim S = p$ the dimension parameter y. Let $u(x)$ be a function defined in some $D \subset R^n$, $\rho(x,y)-$ domain as a function of variables measure x, y, $\omega(y)-$ on a manifold $S(y)$.

Consider the function

$$\int\limits_{S(y)} \rho(x,y)u(x)d\omega = f(y). \qquad (1)$$

Integral geometry is a branch of mathematics that studies various relationships between the elements included in (1).

We will assume that in (1) $S(y)$, $\rho(x,y)$, $f(y)$ are given and consider (1) as a linear operator equation with respect to the function $u(x)$.

Integral geometry problems of Volterra type are those problems that can be reduced to the study of Volterra operator equations in the sense of the definition given by M.M. Lavrentev [1]. We also give definitions of weak and strong ill-posed problems of an integral geometry. The problem of solving equation (1) is called weakly ill-posed if for the given problem and its solution it is possible to choose a pair of function spaces in which a finite number of derivatives are involved in determining the norm such that the inversion operator for this pair of spaces is continuous. If such a pair of spaces does not exist, then the problem is strongly ill-posed [1].

The first results on the uniqueness and stability of integral geometry problems in the case when the manifolds over which the integration is carried out have the form of paraboloids and are invariant under the group of all motions parallel to the $(n-1)$-dimensional hyperplane were obtained by V.G. Romanov [2,3].

In the work of M.M.Lavrentev [6], a very fruitful idea of reducing a wide class of problems of integral geometry to the study of an equation of evolutionary type for some auxiliary function was proposed. In particular, this made it possible to prove the uniqueness theorem for the solution of the original problem. Note that the problem of determining a function from its spherical mean by reducing it to a certain differential equation was studied in the monograph [5]. Mention should also be made of the work [4], in which other classes of Volterra equations in integral geometry were studied.

New classes of problems in integral geometry were developed in the works of A.Kh.Begmatov [7-9]. In his works, problems of integral geometry of Volterra type were studied on the plane and in space.
In [10-12], new classes of problems of integral geometry were studied and new approaches were introduced to the study of problems of recovering a function from weight functions with a singularity.

The paper considers the problem of recovering a function from families of spheres in space. The uniqueness of the solution of the problem is proved by reducing it to the Volterra integral equation of the first and then the second kind. Fourier transform methods are also used. Uniqueness theorems are proved for some new classes of operator equations of Volterra type in three-dimensional space.

2. Results and discussion

Formulation of the problem. Consider the problem of integral geometry for a family of surfaces in half-space $z \geq 0$. The surface over which the integration is carried out is a sphere

$$z^2 - \zeta^2 = (x-\xi)^2 + (y-\eta)^2 .$$

We denote $L_D = \{(x,y,z): x \in R,\ y \in R,\ 0 \leq z \leq D\}$.

The function $u(\cdot)$ is assumed to be finite at x, y, that is $u(x,y,z) = 0$, for, $(x,y) \notin D$, where D is a bounded area on the plane $z = 0$.

Problem 1. In half-space L_D, restore a function of three variables $u(x,y,z)$ if the integrals of it over the surfaces of the family $\{\Upsilon(x,y,z)\}$ are exist:

$$f(x,y,z) = \int\limits_{\Upsilon(x,y,z)} q(z,\zeta) u(\xi,\eta,\zeta) d\omega, \tag{2}$$

where an arbitrary surface of the family is represented by the expression

$$\Upsilon(x,y,z) = \left\{ (\xi,\eta,\zeta):\ z^2 - \zeta^2 = (x-\xi)^2 + (y-\eta)^2,\ 0 \leq \zeta \leq z \leq D \right\} .$$

Let us investigate the uniqueness of solution (2) by reducing it to the Volterra integral equation of the first and then the second type.

Theorem. Let the function $f(x,y,z)$ exist for all x, y, z in the half-space L_D, the weight function $q(z,\zeta) = \dfrac{1}{\sqrt{z^2 - \zeta^2}}$. Then the solution of equation (2) is unique in the class of twice continuously differentiable finite functions supported in half-spaces L_D.

Proof. Let us write equation (2) in the following form:

$$\int_0^z\int_{-\pi}^{\pi}\frac{1}{\rho}u(x+\rho\cos\varphi, y+\rho\sin\varphi,\zeta)d\varphi d\zeta = f(x,y,z), \tag{3}$$

where $\rho=\dfrac{1}{\sqrt{z^2-\zeta^2}}$.

We apply to both sides of equation (4) the Fourier transform in the variable x:

$$\hat{f}(\lambda,y,z)=\int_0^z\int_{-\pi}^{\pi}\left(\int_{-\infty}^{+\infty}e^{i\lambda x}\frac{1}{\rho}u(x+\rho\cos\varphi,y+\rho\sin\varphi,\zeta)dx\right)d\varphi d\zeta=$$

$$=\int_0^z\int_{-\pi}^{\pi}\frac{1}{\rho}e^{-i\lambda\rho\cos\varphi}\hat{u}(\lambda,y+\rho\sin\varphi,\zeta)d\varphi d\zeta. \tag{4}$$

We apply to both sides of equation (4) the Fourier transform in the variable y:

$$\hat{\hat{f}}(\lambda,\mu,z)=\int_0^z\int_{-\pi}^{\pi}\frac{1}{\rho}e^{-i\lambda\rho\cos\varphi}\left(\int_{-\infty}^{+\infty}e^{i\mu y}\hat{u}(\lambda,y+\rho\sin\varphi,\zeta)dy\right)d\varphi d\zeta=$$

$$=\int_0^z\hat{\hat{u}}(\lambda,\mu,\zeta)\int_{-\pi}^{\pi}\frac{1}{\sqrt{z^2-\zeta^2}}e^{-i\sqrt{z^2-\zeta^2}(\lambda\cos\varphi+\mu\sin\varphi)}d\varphi d\zeta.$$

We have obtained the Volterra integral equation of the first type for the function $\hat{\hat{u}}(\lambda,\mu,\zeta)$

$$\int_0^z\hat{\hat{u}}(\lambda,\mu,\zeta)\frac{I(\lambda,\mu,z,\zeta)}{\sqrt{z-\zeta}}d\zeta=\hat{\hat{f}}(\lambda,\mu,z) \tag{5}$$

where

$$I(\lambda,\mu,z,\zeta)=\int_{-\pi}^{\pi}\frac{1}{\sqrt{z+\zeta}}e^{-i\sqrt{z^2-\zeta^2}(\lambda\cos\varphi+\mu\sin\varphi)}d\varphi \tag{6}$$

Let us prove equation (6).

$$\lambda\cos\varphi+\mu\sin\varphi=\gamma\sin(\varphi+k),\quad \gamma=\sqrt{\lambda^2+\mu^2},\quad k\in\left[-\frac{\pi}{2};0\right)\cup\left(0;\frac{\pi}{2}\right].$$

Thus, equation (6) takes the form

$$I(\lambda,\mu,z,\zeta)=\frac{1}{\sqrt{z+\zeta}}\int_{-\pi}^{\pi}e^{-i\gamma\sqrt{z^2-\zeta^2}\sin(\varphi+k)}d\varphi=\frac{1}{\sqrt{z+\zeta}}\left(I_1(z,\zeta,\lambda,\mu)+I_2(z,\zeta,\lambda,\mu)\right).$$

$$I_1(z,\zeta,\lambda,\mu)=\int_{-\gamma\sin k}^{\gamma\sin k}e^{-\left(i\sqrt{z^2-\zeta^2}\right)v}\frac{dv}{\gamma\sqrt{1-\frac{v^2}{\gamma^2}}}. \tag{7}$$

We calculate the integral (7)

$$\left|I_1(z,\zeta,\lambda,\mu)\right|=\left|\int_{-\gamma\sin k}^{\gamma\sin k}e^{-i(\sqrt{z^2-\zeta^2})v}\frac{dv}{\gamma\sqrt{1-\frac{v^2}{\gamma^2}}}\right|\le 2\int_0^{\gamma\sin k}\frac{dv}{\gamma\sqrt{1-\frac{v^2}{\gamma^2}}}=\frac{2}{\gamma}\left(\arcsin(\sin k)-\arcsin 0\right)=\frac{2k}{\gamma}.$$

Thus,

$$\left|I_1(z,\zeta,\lambda,\mu)\right|\le\frac{2k}{\gamma}.$$

Also for $I_1(z,\zeta,\lambda,\mu)$, if we do the same as above, we get the expression

$$\left|I_2(z,\zeta,\lambda,\mu)\right|\le\frac{2k}{\gamma}.$$

Thus,

$$|I(z,\zeta,\lambda,\mu)| \le \frac{1}{\sqrt{z+\zeta}}\left(|I_1(z,\zeta,\lambda,\mu)|+|I_2(z,\zeta,\lambda,\mu)|\right) \le \frac{4k}{\gamma\sqrt{z+\zeta}}$$

or

$$I(z,\zeta,\lambda,\mu) = \frac{4k}{\sqrt{\lambda^2+\mu^2}\sqrt{z+\zeta}}. \qquad (8)$$

Equation (5) has an integrable singularity on the diagonal $z=\zeta$. As can be seen from formula (8), the function $I(\lambda,\mu,z,\zeta)$ is continuous in the region $0<\zeta\le z<D$ and

$$I(\lambda,\mu,z,z) = \frac{4k}{\sqrt{2z}\sqrt{\lambda^2+\mu^2}} \ne 0.$$

Using expression (5) for the function $I(\lambda,\mu,z,\zeta)$, it is easy to show that the first-order partial derivative with respect to the variable z of this function does not have a weak singularity on the diagonal $z=\zeta$:

$$I_z'(\lambda,\mu,z,\zeta) = -\frac{2k\,dz}{\sqrt{\lambda^2+\mu^2}\sqrt{(z+\zeta)^3}}.$$

We can reduce equation (5) to the Volterra equation of the second type, using the Abel method [13]. To do this, we multiply equality (5) by $1/\sqrt{t-z}$ and integrate over z in the range from zero to t. Changing the order of integration in the resulting iterated integral, we find

$$\int_0^t \frac{\hat{\hat{f}}(\lambda,\mu,z)}{\sqrt{t-z}}dz = \int_0^t \frac{dz}{\sqrt{t-z}}\int_0^z \hat{u}(\lambda,\mu,\zeta)I(\lambda,\mu,z,\zeta)\frac{d\zeta}{\sqrt{z-\zeta}} = \int_0^t \hat{u}(\lambda,\mu,\zeta)\left[\int_\zeta^t \frac{I(\lambda,\mu,z,\zeta)}{\sqrt{t-z}\sqrt{z-\zeta}}dz\right]d\zeta\,. \qquad (9)$$

The function under the integral in square brackets here

$$T(t,\zeta,\lambda,\mu) = \int_\zeta^t \frac{I(\lambda,\mu,z,\zeta)}{\sqrt{t-z}\sqrt{z-\zeta}}dz$$

has a finite nonzero value at $t=\zeta$. To verify this, let's change the variable

$$z = t\cos^2\varphi + \zeta\sin^2\varphi.$$

Then the function $T(t,\zeta,\lambda,\mu)$ will take the following form:

$$T(t,\zeta,\lambda,\mu) = \int_\zeta^t \frac{4k}{\sqrt{\lambda^2+\mu^2}\sqrt{z+\zeta}\sqrt{t-z}\sqrt{z-\zeta}}dz = 2\int_0^{\frac{\pi}{2}} \frac{4k\,d\varphi}{\sqrt{\lambda^2+\mu^2}\sqrt{t\cos^2\varphi+(1+\sin^2\varphi)\zeta}}\,.$$

or

$$T(t,\zeta,\lambda,\mu) = 2\int_0^{\frac{\pi}{2}} I(\lambda,\mu,t\cos^2\varphi+\zeta\sin^2\varphi,\zeta)d\varphi, \qquad (10)$$

where

$$I(\lambda,\mu,t\cos^2\varphi+\zeta\sin^2\varphi,\zeta) = \frac{4k}{\sqrt{\lambda^2+\mu^2}\sqrt{t\cos^2\varphi+(1+\sin^2\varphi)\zeta}}\,.$$

From here, assuming $\zeta=t$, we find

$$I(\lambda,\mu,t,t) = 2\int_0^{\frac{\pi}{2}} \frac{4k}{\sqrt{\lambda^2+\mu^2}\sqrt{t\cos^2 t+(1+\sin^2\varphi)t}}d\varphi =$$

$$= 2\frac{4k}{\sqrt{\lambda^2+\mu^2}}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{2t}} = 2\frac{4k}{\sqrt{2t}\sqrt{\lambda^2+\mu^2}}\varphi\Big|_0^{\frac{\pi}{2}} = \frac{4k\pi}{\sqrt{2t}\sqrt{\lambda^2+\mu^2}} \neq 0.$$

Using formula (10), it is also easy to verify that $T(t,\zeta,\lambda,\mu)$ has a continuous derivative with respect t to the variable everywhere except for the diagonal $\zeta = t$. On the diagonal $T_t(t,\zeta,\lambda,\mu)$ has an integrable singularity of the form $1/\sqrt{t-z}$. Differentiating equality (9) and dividing by $T(t,t,\lambda,\mu)$, we obtain for each fixed λ and μ integral Volterra equation of the second kind:

$$\hat{\hat{u}}(\lambda,\mu,t) + \int_0^t \hat{\hat{u}}(\lambda,\mu,t)\frac{T_t(\lambda,t,t)}{T(\lambda,t,t)}d\zeta = \frac{1}{T(\lambda,t,t)}\frac{\partial}{\partial t}\int_0^t \frac{\hat{\hat{f}}(\lambda,\mu,z)}{\sqrt{t-z}}dz$$

with a kernel, an integrable singularity on the diagonal. As follows from the general theory, the solution of such equations is unique [13].

3. Conclusion

The paper considers the problem of recovering a function from families of spheres in space. The uniqueness of the solution of the problem is proved by reducing it to the Volterra integral equation of the first and then the second kind. The proof of the uniqueness theorem is based on the researching of boundary value problems for auxiliary functions. Fourier transform methods are also used. Uniqueness theorems are proved for some new classes of operator equations of Volterra type in three-dimensional space. Problems of such kind arise in geophysics and computerized tomography. The practical significance of the article lies in the fact that the results obtained can be used in the numerical solution of problems of determining the internal structure of objects, arising in the field of medicine and geophysics.

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