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Research Article

Approximating Choquet integral in generalized measure theory: Choquet-midpoint rule

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Abstract

The Choquet integral is an extension of Lebesgue integral and mathematical expectation in generalized measure theory. It's not easy to approximate the Choquet integral in the continuous case on a real line. The Choquet integral should be primarily estimated for non-additive measures. There are few studies on approximating the Choquet integral in the continuous case on real lines. No research has been done on the midpoint rule for the Choquet integral. The main objective of this paper is to propose some applications of the midpoint rule for approximating continuous Choquet integrals. By using the Choquet-midpoint rule, we can numerically solve Choquet integrals, specifically singular and unbounded integrals. Our proposed methodology is illustrated through several numerical examples.

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1. Introduction

The concept of capacity and Choquet integral as important tools in non-additive measure theory [4] were initiated by [2], which later became widely used in many fields such as statistics and financial economics [5,8,9], multicriteria decision making [6,11,13], economic modeling [7], risk measuring [7,8] and fuzzy measure theory [15,16]. Recently, the study of numerical Choquet integration on the real line has been studied using *the statistics software R* by [17,19]. They also proposed several algorithms for some functions with some examples for Hellinger distance between two monotone measures. These studies are often based on Laplace transformation. However, in many problems, it is not possible to provide the Laplace transformation. On the other hands, in many problems, it is not possible to provide the Laplace transformation. Due to the many applications of the Choquet integral for modeling non-deterministic problems, a generalization of the Choquet integral is recently presented in [21]. Study [22] generalizes the generalized Choquet-type integral in terms of a double set-function Choquet integral for a real-valued function based on a set-function and fuzzy measure.

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The midpoint rule is an interesting property in approximation theory[3, 12, 14]. An interesting research topic is the midpoint rule for the Choquet integral, which has not yet been worked out. In this paper, we propose a new method for approximating Choquet integral of continuous functions using Choquet-midpoint rule which helps us to approximate improper integrals with the unbounded intervals $[0,+\infty)$ or $(-\infty,+\infty)$ and singular points. A drawback of the method presented in [1] is that it cannot be used when the integrand is singular in the integration limit. This prompted the authors to develop a new method that allows approximating the solution to the Choquet integral even if the integrand is singular in the integration limit.

The rest of this paper is organized as follows: In section 2, we review some basic concepts of Choquet calculus. In section 3, we introduce our method for numerical Choquet calculus based on Choquet-midpoint rule with several examples. Finally, some conclusions are given in section 4.

2. Preliminaries

In this section, we review some notations needed in the subsequent sections of this paper.

2.1. Choquet integral (expectation)

Let us review the notions of Choquet integral in this paper (see [2, 17, 18]).

Definition 2.1. Let $(\Omega, \mathcal{F})$ be a fixed measurable space. A set function μ defined on $\mathcal{F}$ is said a non-additive measure (or monotone measure) if and only if

- $0 \leq \mu(A) \leq +\infty$ for any $A \in \mathcal{F}$;
- $\mu(\emptyset) = 0$;
- If $E, F \in \mathcal{F}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.

Definition 2.2. A non-additive measure μ is said *finite* if $\mu(\Omega) < \infty$.

Definition 2.3. A non-additive measure μ is called a *capacity* (or *fuzzy measure*), if $\mu(\Omega) = 1$. Moreover, a capacity with σ-additivity assumption is called a *probability measure.*

Distorted probabilities and distorted Lebesgue measures are the well-known examples of monotone measures [15–18].

Definition 2.4. Let $m : [0, 1] \to [0, 1]$ be an increasing and continuous function such that $m(0) = 0, m(1) = 1$. A monotone measure μ_m is called a distorted Lebesgue measure, if

$$\mathbf{P}_m(\cdot) = m(\mathbf{P}(\cdot)),$$

where $\mathbf{P}$ is a probability measure.

Definition 2.5. Let $m : [0, 1] \to [0, +\infty]$ be an increasing and continuous function such that $m(0) = 0$. A non-additive measure μ_m is called a distorted Lebesgue measure, if

$$\mu_m(\cdot) = m(\lambda(\cdot)),$$

where λ is a Lebesgue measure.

Choquet integral [2, 10] is a fundamental concept in potential theory and probability theory which is very important in many problems of information systems.

Definition 2.6. The Choquet integral (expectation) of a nonnegative real-valued measurable function Y with respect to a non-additive measure μ on A is defined by

$$(C)\int_A Y d\mu = \int_0^{+\infty} \mu(A \cap \{\omega | Y(\omega) > r\})\, dr. \tag{2.1}$$

3. Choquet-midpoint rule

The midpoint rule is a basic method in numerical integration on a definite integral. Based on midpoint rule, if f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx \approx (b-a)\, f\left(\frac{a+b}{2}\right).$$

In general, if

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

is a partition of $[a, b]$, and $c_i = \frac{1}{2}(x_{i-1} + x_i) = a + \frac{2i-1}{2n}(b-a)$ is the midpoint of $[x_{i-1}, x_i]$ then

$$\int_a^b f(x)dx \approx \frac{b-a}{n} \sum_{i=1}^{n} f(c_i) = \frac{b-a}{n}[f(c_1) + \cdots + f(c_n)]. \tag{3.1}$$

When no analytical expression can be obtained for Choquet integral, we need the numerical methods. There are a few studies on the numerical Choquet integral in continuous case on real line. In this case, the computing of Choquet integral for a function with respect to a non-additive measure is not very easy. In this section, we propose new methods for approximating Choquet integral of continuous functions using Midpoint Rule. In this section is shown that Choquet integral generalizes the applicability of midpoint rule to the unbounded and singular integrals(improper integrals).

Let $\mathcal{A}^-$ be the class of any non-negative, continuous and decreasing bounded functions on $[a, b]$.

Theorem 3.1. *The approximating Choquet integral of $f \in \mathcal{A}^-$ with respect to a real monotone measure μ on Borel$([a, b])$ by Choquet-midpoint rule can be explained as*

$$(C)\int_a^b f d\mu \cong \mu\left(\left[a, f^{-1}\left(\frac{f(a)+f(b)}{2}\right)\right]\right)(f(a) - f(b)) + f(b)\,\mu[a, b].$$

In general, if $f \in \mathcal{A}^-$,

$$f(a) = f(x_0) > f(x_1) > f(x_2) > \cdots > f(x_n) = f(b)$$

is a partition of $[f(b), f(a)]$, and $r_i = f(b) + \frac{2i-1}{2n}(f(a) - f(b)), i = 1, \ldots n$ is the midpoint then

$$(C)\int_a^b f d\mu \approx \frac{f(a) - f(b)}{n} \sum_{i=1}^{n} \mu\left(\left[a, f^{-1}(r_i)\right]\right) + f(b)\,\mu[a, b]. \tag{3.2}$$

Proof. Since $f \in \mathcal{A}^-$, by definition of Choquet integral, and midpoint rule, it is derived

$$\begin{aligned} (C)\int_a^b f d\mu &= \int_{f(b)}^{f(a)} \mu\left(\left[a, f^{-1}(x)\right]\right) dx + f(b)\,\mu[a, b] \qquad (3.3)\\ &\cong \mu\left(\left[a, f^{-1}\left(\frac{f(a)+f(b)}{2}\right)\right]\right)(f(a) - f(b)) + f(b)\,\mu[a, b]. \end{aligned}$$

If

$$f(a) = f(x_0) > f(x_1) > f(x_2) > \cdots > f(x_n) = f(b)$$

is a partition of $[f(b), f(a)]$, and $r_i = f(b) + \frac{2i-1}{2n}(f(a) - f(b))$ is the midpoint of intreval $[x_{i-1}, x_i], i = 1, \ldots n$, then

$$\begin{aligned} (C)\int_a^b f d\mu &= \int_{f(b)}^{f(a)} \mu\left(\left[a, f^{-1}(x)\right]\right) dx + f(b)\,\mu[a, b] \\ &\approx \frac{f(a) - f(b)}{n} \sum_{i=1}^{n} \mu\left(\left[a, f^{-1}(r_i)\right]\right) + f(b)\,\mu[a, b]. \end{aligned}$$

This completes the proof. ☐

Now we show the applicability of Choquet-midpoint rule (3.2) for unbounded interval.

Example 3.2. The purpose of this example is to show that this method is able to work on semi-infinite intervals, which indicate the superior performance of the method compared to real line problems.
Let $f(x)$ be defined on $x \in [1, +\infty)$ and μ_m be as a Lebesgue measure with the distortion $m(x)$. Note that for $\alpha = 0$, Choquet integral coincide with the classical Lebesgue (Riemann) integral. Midpoint Rule is applied to estimate the approximation solution of $(C)\int_1^{+\infty} f d\mu$ for different values $n = 10$, 100, 1000 and 10000 in **Cases 1-3**.

Case1: For $f(x) = \frac{1}{x^3}$, the results are shown in Table 1.

Table 1. Approximate solutions of $(C)\int_1^{+\infty} f d\mu$ with $f(x) = \frac{1}{x^3}$, $m(x) = x^\alpha, \alpha \geq 0$ for Case 1.

α	Exact	n=10	n=100	n=1000	n=10000
1	0.5	0.445632	0.488258	0.49747	0.499455
2	1	0.441686	0.713741	0.861289	0.934357

Case 2: For $f(x) = e^{-x^2}$, Tables 2 and 3 are given.

Table 2. Approximate solutions of $(C)\int_1^{+\infty} f d\mu$ with $f(x) = e^{-x^2}$, $m(x) = x^\alpha$, $\alpha \geq 0$ for Case 2.

α	Exact	n=10	n=100	n=1000	n=10000
1	0.139403	0.136744	0.139174	0.139383	0.139401
2	0.0890739	0.355283	0.088257	0.0889866	0.0890647
5	0.0918712	0.0454142	0.0799077	0.0895983	0.0915039

Table 3. Approximate solutions of $(C)\int_1^{+\infty} f d\mu$ with $f(x) = e^{-x^2}$, $m(x) = xe^{\alpha x}$, $\alpha \geq 0$ for Case 2.

α	Exact	n=10	n=100	n=1000	n=10000
1	0.284305	0.259736	0.280454	0.283739	0.284225
1.5	0.43288	0.369516	0.420228	0.430597	0.432494
2	0.693904	0.53619	0.654371	0.685226	0.692159

Case 3: For $f(x) = cot^{-1}(x^5)$, Table 4 is devoted for showing the results..

Table 4. Approximate solutions of $(C)\int_1^{+\infty} f d\mu$ with $f(x) = cot^{-1}(x^5)$ and $m(x) = x^\alpha, \alpha \geq 0$ for Case 3.

α	Exact	n=10	n=100	n=1000	n=10000
1	0.231856	0.217701	0.229597	0.231498	0.231799
1.5	0.177035	0.149198	0.170893	0.175741	0.176769
2	0.163551	0.113276	0.14828	0.159296	0.162416

Corollary 3.3. *If μ is a distorted probability $m \circ \mathbf{P}$ then for any $X \in \mathcal{A}^-$,*

$$(C)\int_a^b X dm \circ \mathbf{P} \simeq \frac{X(a) - X(b)}{n} \sum_{i=1}^{n} m \circ \mathbf{P}\left(\left[a, X^{-1}(r_i)\right]\right) + X(b)\, m \circ \mathbf{P}[a, b],$$

where $r_i = X(b) + \frac{2i-1}{2n}(X(a) - X(b)), i = 1, \ldots n$. *In particular, for* $m(x) = x$, *we can approximate the expectation of any* $X \in \mathcal{A}^-$ *in statistics as follows*

$$(C)\int_a^b X d\mathbf{P} \simeq \frac{X(a) - X(b)}{n} \sum_{i=1}^{n} \mathbf{P}\left(\left[a, X^{-1}(r_i)\right]\right) + X(b)\,\mathbf{P}[a,b].$$

Corollary 3.4. *If* μ *is a distorted Lebesgue measure* μ_m *in (3.2), then for any* $f \in \mathcal{A}^-$, *then*

$$(C)\int_a^b f d\mu_m \simeq \frac{f(a) - f(b)}{n} \sum_{i=1}^{n} m\left(f^{-1}(r_i) - a\right) + f(b)\,\mu(b-a).$$

Corollary 3.5. *Let* $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ *be a partition of* $[a,b]$ *and* $c_i = \frac{1}{2}(x_{i-1} + x_i) = a + \frac{2i-1}{2n}(b-a)$ *be the midpoint of* $[x_{i-1}, x_i], i = 1, \ldots n$. *If* $f \in \mathcal{A}^-$ *is a differentiable function, then*

$$(C)\int_a^b f d\mu \simeq \frac{a-b}{n} \sum_{i=1}^{n} f'(c_i)\,\mu([a, c_i]) + f(b)\,\mu[a,b].$$

Proof. Since f is a differentiable, then by (3.3) and using $u = f^{-1}(x)$, we have

$$(C)\int_a^b f d\mu = \int_b^a f'(x)\,\mu([a,x])\,dx + f(b)\,\mu[a,b] \tag{3.4}$$
$$\simeq \frac{a-b}{n} \sum_{i=1}^{n} f'(c_i)\,\mu([a,c_i]) + f(b)\,\mu[a,b].$$

which completes the proof. □

Note 3.6. If $f \in \mathcal{A}^-$ and $\mu[x,b]$ is differentiable with respect to x on $[a,b]$, then Corollary 3.5 implies that

$$(C)\int_a^b f d\mu \simeq \frac{b-a}{n} \sum_{i=1}^{n} \mu'([a,c_i])\,f(c_i) + f(a)\,\mu(\{a\}). \tag{3.5}$$

Proof. By (3.4) and using partial integration, we have

$$(C)\int_a^b f d\mu = \int_b^a f'(x)\,\mu([a,x])\,dx + f(b)\,\mu[a,b]$$
$$= f(x)\,\mu([a,x])\,|_b^a - \int_b^a \mu'([a,x])\,f(x)\,dx + f(b)\,\mu[a,b]$$
$$= f(a)\,\mu(\{a\}) - f(b)\,\mu([a,b]) - \int_b^a \mu'([a,x])\,f(x)\,dx + f(b)\,\mu[a,b]$$
$$= f(a)\,\mu(\{a\}) + \int_a^b \mu'([a,x])\,f(x)\,dx.$$

Then

$$(C)\int_a^b f d\mu \simeq \frac{b-a}{n} \sum_{i=1}^{n} \mu'([a,c_i])\,f(c_i) + f(a)\,\mu(\{a\}).$$

This completes the proof. □

Example 3.7. [17, 20] For distorted Lebesgue measures $\mu_m = m \circ \lambda, \mu_1 = n_1 \circ \lambda$ and $\mu_2 = n_2 \circ \lambda$ with $m(t) = t + \frac{1}{2}t^2$, $n_1(t) = t^2$ and $n_2(t) = e^t - 1$, we have

$$H_{\mu_m}(\mu_1, \mu_2) = \sqrt{\frac{1}{2}(C)\int \left(\sqrt{\frac{d\mu_1}{d\mu_m}} - \sqrt{\frac{d\mu_2}{d\mu_m}}\right)^2 d\mu_m}$$
$$= \sqrt{\frac{1}{2}(C)\int \left(\sqrt{2\left(1 - e^{-t}\right)} - \sqrt{\cosh t}\right)^2 d\mu_m}.$$

Let $f(x) = \left(\sqrt{2\left(1-e^{-x}\right)} - \sqrt{\cosh x}\right)^2$. Then by (3.5), we have

$$H_{\mu_m}(\mu_1,\mu_2) = \sqrt{\frac{1}{2}(C)\int_0^1 f d\mu_m} \simeq \sqrt{\frac{1}{2n}\sum_{i=1}^{n} m'(c_i)\, f(c_i)}$$
$$= \sqrt{\frac{1}{2n}\sum_{i=1}^{n}(c_i+1)\left(\sqrt{2\left(1-e^{-c_i}\right)} - \sqrt{\cosh c_i}\right)^2}$$

where $c_i = \frac{2i-1}{2n}, i = 1, \ldots n$. The exact solution of $H_{\mu_m}(\mu_1,\mu_2)$ is 0.248694. Now by Choquet-midpoint rule for different values $n = 500$, 750, 1000 and 1250 is computed in Table 5.

Table 5. Approximate solutions of $H_{\mu}(\mu_1,\mu_2)$ by Choquet-midpoint rule in Example 3.7.

n=500	n=750	n=1000	n=1250
0.248679	0.248686	0.248689	0.2486901

Let $\mathcal{A}^+$ be the class of any non-negative, continuous and increasing bounded functions on $[a,b]$.

Theorem 3.8. *The approximating Choquet integral of $f \in \mathcal{A}^+$ with respect to a real monotone measure μ on Borel($[a,b]$) by Choquet-midpoint rule can be explained as*

$$(C)\int_a^b f d\mu \cong \mu\left(\left[f^{-1}\left(\frac{f(a)+f(b)}{2}\right), a\right]\right)\left(f(b)-f(a)\right) + f(a)\,\mu[a,b].$$

In general, if $f \in \mathcal{A}^+$,

$$f(a) = f(x_0) < f(x_1) < f(x_2) < \cdots < f(x_n) = f(b)$$

is a partition of $[f(a), f(b)]$, and $r_i = f(a) + \frac{2i-1}{2n}\left(f(b)-f(a)\right)$, $i = 1, \ldots n$ is the midpoint then

$$(C)\int_a^b f d\mu \approx \frac{f(b)-f(a)}{n}\sum_{i=1}^{n}\mu\left(\left[f^{-1}(r_i), b\right]\right) + f(a)\,\mu[a,b]. \tag{3.6}$$

Proof. Since $f \in \mathcal{A}^+$, by definition of Choquet integral, we have

$$(C)\int_a^b f d\mu = \int_{f(a)}^{f(b)} \mu\left(\left[f^{-1}(x), b\right]\right) dx + f(a)\,\mu[a,b] \tag{3.7}$$
$$\simeq \mu\left(\left[f^{-1}\left(\frac{f(a)+f(b)}{2}\right), a\right]\right)\left(f(b)-f(a)\right) + f(a)\,\mu[a,b].$$

and if

$$f(a) = f(x_0) < f(x_1) < f(x_2) < \cdots < f(x_n) = f(b)$$

is a partition of $[f(a), f(b)]$, and $r_i = f(a) + \frac{2i-1}{2n}\left(f(b)-f(a)\right)$ is the midpoint then

$$(C)\int_a^b f d\mu = \int_{f(a)}^{f(b)} \mu\left(\left[f^{-1}(x), b\right]\right) dx + f(a)\,\mu[a,b]$$
$$\approx \frac{f(b)-f(a)}{n}\sum_{i=1}^{n}\mu\left(\left[f^{-1}(r_i), b\right]\right) + f(a)\,\mu[a,b].$$

This completes the proof. □

Corollary 3.9. *If μ is a distorted probability $m \circ \mathbf{P}$ in (3.6), then for any $X \in \mathcal{A}^+$,*

$$(C)\int_a^b X dm \circ \mathbf{P} \simeq \frac{X(b) - X(a)}{n} \sum_{i=1}^{n} m \circ \mathbf{P}\left(\left[X^{-1}(r_i), b\right]\right) + X(a)\, m \circ \mathbf{P}[a,b].$$

In particular, for $m(x) = x$, we can approximate the expectation of any $X \in \mathcal{A}^+$ in statistics as follows

$$(C)\int_a^b X d\mathbf{P} \simeq \frac{X(b) - X(a)}{n} \sum_{i=1}^{n} \mathbf{P}\left(\left[X^{-1}(r_i), b\right]\right) + X(a)\, \mathbf{P}[a,b].$$

Corollary 3.10. *If μ is a distorted Lebesgue measure μ_m in (3.6), then for any $f \in \mathcal{A}^+$, then*

$$(C)\int_a^b f d\mu_m \simeq \frac{f(b) - f(a)}{n} \sum_{i=1}^{n} m\left(b - f^{-1}(r_i)\right) + f(a)\, m(b-a).$$

Corollary 3.11. *Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of $[a,b]$, and $r_i = \frac{1}{2}(x_{i-1} + x_i) = a + \frac{2i-1}{2n}(b-a)$ be the midpoint of $[x_{i-1}, x_i]$, $i = 1, \ldots n$. If $f \in \mathcal{A}^+$ is a differentiable function, then*

$$(C)\int_a^b f d\mu \simeq \frac{b-a}{n} \sum_{i=1}^{n} f'(r_i)\mu([r_i, b]) + f(a)\, \mu[a,b]. \tag{3.8}$$

Proof. Since f is a differentiable, then by (3.7) and using $u = f^{-1}(x)$, we have

$$\begin{aligned}(C)\int_a^b f d\mu &= \int_a^b f'(u)\, \mu([u,b])\, du + f(a)\, \mu[a,b] \\ &\simeq \frac{b-a}{n} \sum_{i=1}^{n} f'(r_i)\mu([r_i, b]) + f(a)\, \mu[a,b]\end{aligned}$$

which completes the proof. □

In Example 3.12, our goal is to investigate the ability and precision of Choquet-midpoint rule when the integrand is singular.

Example 3.12. Assume that $(C)\displaystyle\int_0^2 f d\mu$ which $f(x) = \frac{e^{x^2}-1}{x^2}$. It is noteworthy that the integrand is increasing on $[0,2]$ and singular at point $x = 0$ and $\lim_{x\to 0} f(x) = 1$. Equation (3.8) is implemented for $(C)\int_0^2 f d\mu$ with distortions $m_1(x) = x^\alpha$, $m_2(x) = xe^{\alpha x}, \alpha \geq 0$, respectively. The numerical solutions are disclosed in Tables 6 and 7. Since the values in Tables 6 and 7 are big, then the relative error better shows the high accuracy of the method.

Table 6. Approximate solutions of $(C)\int_0^2 f d\mu$ with $m_1(x) = x^\alpha, \alpha \geq 0$ for Example 3.12.

α	Exact	n=10	n=100	n=1000	n=10000
1	6.1061805	6.174411	6.1069001	6 .1061877	6.1061807
2	6.757357614	6.7656875	6.75742446	6.7573583	6.757357792
5	35.98346425	36.03563966	35.98399746	35.9834696	35.98346568

Example 3.13. [1, 17] To compare our method with other existing methods, we assume $f(x) = \sqrt{\sin\left(\frac{\pi}{2}x\right)}$ and $\mu = m \circ \mathbf{P}$ be a distorted probability with

$$m(x) = \frac{s(x) - s(0)}{s(1) - s(0)}$$

Table 7. Approximate solutions of $(C)\int_0^2 f d\mu$ with $m_2(x) = xe^{\alpha x}, \alpha \geq 0$ for Example 3.12.

α	Exact	n=10	n=100	n=1000	n=10000
1	23.69131815	23.78207567	23.6922508	23.6913274856	23.69131888
2	133.574691636	133.8195834	133.57719763	133.5747167	133.5746966
5	45680.53114279	45746.4473	45681.26524	45680.53850	45680.53311

which $s(x) = \frac{1}{1+e^{-10(x-0.5)}}$ and $\mathbf{P}$ is a truncated normal distribution $N(0.5, 0.1)$ in $[0,1]$ with

$$\mathbf{P}[a,b] = \frac{F(b,0.5,0.1) - F(a,0.5,0.1)}{F(1,0.5,0.1) - F(0,0.5,0.1)}$$

where F is cumulative distribution function of $N(0.5, 0.1)$. Note that the function $f(x)$ is not differentiable at $x = 0$. We use the presented method to find numerical solution of $(C)\int_0^1 \sqrt{\sin(\frac{\pi}{2}x)}\, d\mu$ for various values $n = 10,\ 50, 500, 750, 1000$. The results are shown in Table 8 and show acceptable performance compared to the results in [1, 17]. This method requires fewer node points than the method [1], therefore the proposed method needs less computation. CPU time for Examples 3.7-3.13 are exhibited in seconds in Table 9. These computation times confirm the good efficiency of method.

Table 8. Approximate solutions of $(C)\int_0^1 \sqrt{\sin(\frac{\pi}{2}x)}\, d\mu$ in Example 3.13.

n=10	n=50	n=500	n=750	n=1000
0.830	0.8380	0.838974265	0.838974262868	0.838974262870643

Table 9. CPU time for Examples 3.7-3.13 in terms of second.

	n=10	n=100	n=1000	n=10000
Example 3.7	0.25	1.03	1.84	2.68
Example 3.12	0.86	2.54	3.4	4.12
Example 3.13	0.71	1.49	2.97	4.06

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