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Generalized Riesz Spaces Defined by Using a Sequence of Modulus Functions

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Abstract In this paper, we define a new Riesz sequence space using a sequence of modulus functions. Furthermore, we give that this space is linearly isomorphism with $\ell(p)$ and determine its basis. We also give some inclusion relationships and compute α- and β- duals of this space.

Key words: Paranorm, Riesz sequence space, modulus function, $\alpha-,\beta-$duals, infinite matrices.

1. Introduction

We will denote the set of all sequences with complex terms by ω. With ℓ_∞, c and c_0, we show that the sequence space of all bounded, covergent and null, respectively. Also, we denote by ℓ_1, $\ell(p)$, cs and bs, respectively the spaces of all absolutely, $p-$absolutely convergent, convergent and bounded series. $([6],[14])$

Let $A=(a_{nk})$ be an infinite matrix of real or complex numbers (a_{nk}), $n,k\in\mathbb{N}$. The matrix A define a transformation from X into Y and we denote by $A:X\to Y$, if for every sequence $x=(x_k)\in X$ the sequence $Ax=\{(Ax)_n\}$, the $A-$transform of x, is in Y where

$$(Ax)_n=\sum_k a_{nk}x_k \qquad \text{for each } n\in\mathbb{N}.$$

Let (q_k) be a sequence of positive numbers. We write

$$Q_n=\sum_{k=0}^{n} q_k, \qquad \text{for } n\in\mathbb{N}.$$

Then the matrix $R^q=(r_{nk}^q)$ of the Riesz mean (R,q_n) is defined by

$$r_{nk}^q=\begin{cases}\dfrac{q_k}{Q_n}, & 0\le k\le n,\\ 0, & k>n\end{cases}$$

In [17] and [13] Riesz mean (R,q_n) is regular if and only if $Q_n\to\infty$ is $n\to\infty$.

More recently, in [18] a new concept has been introduced by the following:

$$r^q(u,p)=\left\{x=(x_k)\in\omega:\sum_k\left|\frac{1}{Q_k}\sum_{j=0}^{k}u_jq_jx_j\right|^{p_k}<\infty\right\}$$

in which $0<p_k\le H<\infty$ is involved.

The author defined the following difference sequence spaces $X(\Delta)$, in [7].

$$X(\Delta)=\{x=(x_k)\in\omega:(\Delta x_k)\in X\}$$

where $X=\ell_\infty,c,c_0$ and $\Delta x_k=x_k-x_{k+1}$.

A function $f:[0,\infty)\to[0,\infty)$ is modulus function if

(i) $f(x)=0\Leftrightarrow x=\theta$

(ii) $f(x+y)\le f(x)+f(y),\qquad \forall x,y\ge 0$

(iii) f is increasing

(iv) f is continuous from the right at 0.

Ruckle $[12]$ defined by the following sequence space

$$L(f)=\left\{x=(x_k):\sum_k |f(x_k)|<\infty\right\}.$$

Several authors have studied regard to this subject$([1],[3],[4],[15],[18],[19],[20],[21])$

Finally, Gupkari $[15]$ defined the following sequence space

$$r_f^q(\Delta_s^p)=\left\{x=(x_k)\in\omega:\sum_k\left|f\left(\frac{1}{Q_k^s}\sum_{j=0}^k q_j\Delta x_j\right)\right|^{p_k}<\infty\right\}.$$

2 The Sequence Space $r^q(F,\Delta_s^p)$

In this section, we define the Riesz sequence space $r^q(F,\Delta_s^p)$, as a completely paranormal linear space, which is linearly isomorphic to the space $\ell(p)$. Also we give some topological properties.

The difference sequence space $r^q(F,\Delta_s^p)$ is defined by

$$r^q(F,\Delta_s^p)=\left\{x=(x_k)\in\omega:\sum_k\left|f_k\left(\frac{1}{Q_k^s}\sum_{j=0}^k q_j\Delta x_j\right)\right|^{p_k}<\infty\right\},$$

where $s\ge 0$ and $F=(f_k)$ is a sequence of modulus functions. It can be redefined as

$$r^q(F,\Delta_s^p)=\{\ell(p)\}_{R^q(F,\Delta_s^p)}$$

Theorem 1. The space $r^q(F,\Delta_s^p)$ is a complete linear metric space paranormed by h_Δ defined by

$$h_{\Delta}(x)=\left[\sum_k \left| f_k\left[\frac{1}{Q_k^s}\sum_{j=0}^{k-1}(q_j-q_{j+1})x_j+\frac{q_k}{Q_k^s}x_k\right]\right|^{p_k}\right]^{\frac{1}{M}}$$

where $0<p_k\leq H<\infty$ and $M=\max(1,H)$.

Proof. For the linearity of $r^q\left(F,\Delta_s^p\right)$, we need to show that with respect to coordinate-wise addition and scalar multiplication. Thus we have

$$\left[\sum_k \left| f_k\left[\frac{1}{Q_k^s}\sum_{j=0}^{k-1}(q_j-q_{j+1})(x_j+y_j)+\frac{q_k}{Q_k^s}(x_k+y_k)\right]\right|^{p_k}\right]^{\frac{1}{M}}$$

$$\leq\left[\sum_k \left| f_k\left[\frac{1}{Q_k^s}\sum_{j=0}^{k-1}(q_j-q_{j+1})x_j+\frac{q_k}{Q_k^s}x_k\right]\right|^{p_k}\right]^{\frac{1}{M}}$$

$$+\left[\sum_k \left| f_k\left[\frac{1}{Q_k^s}\sum_{j=0}^{k-1}(q_j-q_{j+1})y_j+\frac{q_k}{Q_k^s}y_k\right]\right|^{p_k}\right]^{\frac{1}{M}}$$

In the case of all $\gamma\in\mathbb{}$ (see [13]),

$$|\gamma|^{p_k}\leq\max\left(1,|\gamma|^M\right)$$

It is clear that $h_{\Delta}(\theta)=0$ and $h_{\Delta}(x)=h_{\Delta}(-x)$, for all $x\in r^q\left(F,\Delta_s^p\right)$. From the above inequalities, yield the subadditivity of h_{Δ} and

$$h_{\Delta}(\gamma x)\leq\max(1,|\gamma|)h_{\Delta}(x).$$

Let $\{x^n\}$ be any sequence of $r^q\left(F,\Delta_s^p\right)$ such that $h_{\Delta}(x^n-x)\to 0$ and (γ_n) is a sequence of scalars such that $\gamma_n\to\gamma$. Then,

$$h_{\Delta}(x^n)\leq h_{\Delta}(x)+h_{\Delta}(x^n-x).$$

$\{h_{\Delta}(x^n)\}$ is bounded. Hence we obtain

$$h_{\Delta}(\gamma_n x^n-\gamma x)=\left[\sum_k \left| f_k\left[\frac{1}{Q_k^s}\sum_{j=0}^{k}(q_j-q_{j+1})(\gamma_n x_j^n-\gamma x_j)\right]\right|^{p_k}\right]^{\frac{1}{M}}$$

$$\leq|\gamma_n-\gamma|^{\frac{1}{M}}h_{\Delta}(x^n)+|\gamma|^{\frac{1}{M}}h_{\Delta}(x^n-x)\qquad(n\to\infty)$$

This implies that the scalar multiplication is continuous. Namely h_{Δ} is paranorm on $r^q\left(F,\Delta_s^p\right)$

Now, let us show completeness of this space. Suppose $\{x^i\}$ is any Cauchy sequence in space $r^q(F,\Delta_s^p)$. Here is $x^i=\{x_k^i\}\in r^q(F,\Delta_s^p)$. Then, there exists a positive integer $n_0(\varepsilon)$,

$$h_\Delta\left(x^i-x^j\right)<\varepsilon \tag{1}$$

for all $i,j\geq n_0(\varepsilon)$. Hence, we have

$$\begin{aligned}&f_k\left[\left(R^q\left(F,\Delta_s^p\right)x^i\right)_k-\left(R^q\left(F,\Delta_s^p\right)x^j\right)_k\right]\\&\leq\left[\sum_k\left|f_k\left[\left(R^q\left(F,\Delta_s^p\right)x^i\right)_k-\left(R^q\left(F,\Delta_s^p\right)x^j\right)_k\right]\right|^{p_k}\right]^{\frac{1}{M}}<\varepsilon\end{aligned}$$

for $i,j\geq n_0(\varepsilon)$. So $\left\{\left(R^q\left(F,\Delta_s\right)x^0\right)_k,\left(R^q\left(F,\Delta_s\right)x^1\right)_k,...\right\}$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, $\left(R^q\left(F,\Delta_s^p\right)x^i\right)_k\to\left(R^q\left(F,\Delta_s^p\right)x\right)_k$ as $i\to\infty$.

From (1) we have

$$\sum_{k=0}^{m}\left|f_k\left[\left(R^q\left(F,\Delta_s\right)x^i\right)_k-\left(R^q\left(F,\Delta_s\right)x^j\right)_k\right]\right|^{p_k}\leq h_\Delta\left(x^i-x^j\right)^M<\varepsilon^M \tag{2}$$

for each $m\in\mathbb{N}$ and $i,j\geq n_0(\varepsilon)$. If we take limit in (2) for $j\to\infty$ and $m\to\infty$, we have

$$h_\Delta\left(x^i-x\right)\leq\varepsilon.$$

Now, if we take $\varepsilon=1$ in (2), we obtain that by Minkowski inequality

$$\left[\sum_{k=0}^{m}\left|f_k\left[\left(R^q\left(F,\Delta_s\right)x\right)_k\right]\right|^{p_k}\right]^{\frac{1}{M}}\leq h_\Delta\left(x^i-x\right)+h_\Delta\left(x^i\right)<1+h_\Delta\left(x^i\right)$$

Thus $r^q(F,\Delta_s^p)$ is complete.

Theorem 2. If (p_k) and (t_k) are bounded sequences of positive real numbers where $0<p_k\leq t_k<\infty$ for any $k\in\mathbb{N}$, then for any sequence of modulus functions $F=(f_k)$, $r^q(F,\Delta_s^p)\subseteq r^q(F,\Delta_s^t)$.

Proof. Let $x\in r^q(F,\Delta_s^p)$. Then

$$\left|f_k\left(\frac{1}{Q_k^n}\sum_{j=0}^{k-1}\left(q_j-q_{j+1}\right)x_j+\frac{q_k}{Q_k^n}a_k\right)\right|^{p_k}<\infty$$

for sufficiently large values of k, say $k\geq k_0$ for some fixed $k_0\in\mathbb{N}$. Hence

$$\left| f_k\left(\frac{1}{Q_k^n}\sum_{j=0}^{k-1}\left(q_j-q_{j+1}\right)x_j+\frac{q_k}{Q_k^n}x_k\right)\right|<\infty$$

Since $F=\left(f_k\right)$ is increasing and $p_k \le t_k$, we obtain

$$\sum_{k\ge k_0}\left| f_k\left(\frac{1}{Q_k^n}\sum_{j=0}^{k-1}q_j-q_{j+1}\right)x_j+\frac{q_k}{Q_k^n}\right|^{t_k}$$

$$\le\sum_{k\ge k_0}\left| f_k\left(\frac{1}{Q_k^n}\sum_{j=0}^{k-1}\left(q_j-q_{j+1}\right)x_j+\frac{q_k}{Q_k^n}x_k\right)\right|^{p_k}<\infty$$

Therefore, $x\in r^q\left(F,\Delta_s^p\right)$.

Theorem 3. Let $F=\left(f_k\right)$, $F^{'}=\left(f_k^{'}\right)$ and $F^{''}=\left(f_k^{''}\right)$ are sequences of modulus functions.

Then we have $r^q\left(F^{'},\Delta_s^p\right)\cap r^q\left(F^{''},\Delta_s^p\right)\subseteq r^q\left(F^{'}+F^{''},\Delta_s^p\right)$.

Proof. Let $x\in r^q\left(F^{'},\Delta_s^p\right)\cap r^q\left(F^{''},\Delta_s^p\right)$. Then, it can be easily seen that

$$\sum_k\left| f_k^{'}\left(\frac{1}{Q_k^s}\sum_{j=0}^{k}q_j\Delta x_j\right)\right|^{p_k}<\infty;$$

and

$$\sum_k\left| f_k^{''}\left(\frac{1}{Q_k^s}\sum_{j=0}^{k}q_j\Delta x_j\right)\right|^{p_k}<\infty$$

From here, we have

$$\sum_k\left|\left(f_k^{'}+f_k^{''}\right)\left(\frac{1}{Q_k^s}\sum_{j=0}^{k}q_j\Delta x_j\right)\right|^{p_k}\le M\sum_k\left| f_k^{'}\left(\frac{1}{Q_k^s}\sum_{j=0}^{k}q_j\Delta x_j\right)\right|^{p_k}+M\sum_k\left| f_k^{''}\left(\frac{1}{Q_k^s}\sum_{j=0}^{k}q_j\Delta x_j\right)\right|^{p_k}$$

which means that $x\in r^q\left(F^{'}+F^{''},\Delta_s^p\right)$.

Theorem 4. Let $F=\left(f_k\right)$, $F^{'}=\left(f_k^{'}\right)$ and $F^{''}=\left(f_k^{''}\right)$ are sequences of modulus functions.

Then we have $r^q\left(F^{'},\Delta_s^p\right)\subseteq r^q\left(F^{'}oF^{''},\Delta_s^p\right)$.

Proof. Let $\varepsilon>0$ and choose δ with $0<\delta<1$ such that $f\left(q\right)<\varepsilon$ for $0\le q\le\delta$. We write

$$y_k=f_k^{'}\frac{1}{Q_k^s}\sum_{j=0}^{k}q_j\Delta x_j$$

and consider

$$\sum_n\left[f_k\left(y_k\right)\right]^{p_k}=\sum_{y_k\le\delta}\left[f_k\left(y_k\right)\right]^{p_k}+\sum_{y_k>\delta}\left[f_k\left(y_k\right)\right]^{p_k}.$$

Since f_k is continuous, we have

$$\sum_{y_k\le\delta}\left[f_k\left(y_k\right)\right]^{p_k}<\varepsilon^H \qquad (3)$$

and for $y_k > \delta \Rightarrow y_k < \frac{y_k}{\delta} \le 1 + \frac{y_k}{\delta}$. We obtain by the definition

$$f_k(y_k) < 2 f_k(1) \frac{y_k}{\delta}$$

and so

$$\sum_{y_k > \delta} \left[f_k(y_k) \right]^{p_k} < \max\left\{1, \left(2 f_k(1) \delta^{-1}\right)^H\right\} \sum_k [y_k]^{p_k} \qquad (4)$$

From inequality (3) and (4) , we have that $r^q\left(F', \Delta_s^p\right) \subseteq r^q\left(F' o F'', \Delta_s^p\right)$.

Theorem 5. If $F = (f_k)$ be a sequence of modulus functions and $\alpha = \lim_{t \to \infty} \frac{f_k(t)}{t} > 0$, then $\left(F, \Delta_s^p\right) \subseteq r^q\left(\Delta_s^p\right)$, where $r^q\left(\Delta_s^p\right) = \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{Q_k^s} \sum_{j=0}^{k} q_j \Delta x_j \right|^{p_k} < \infty \right\}$.

Proof. The definition of α, we obtain $f_k(t) \ge \alpha(t)$, for all $t > 0$ and $\frac{1}{\alpha} f_k(t) \ge t$, for all $t > 0$. Now, for $x \in \left(F, \Delta_s^p\right)$ we have

$$\sum_k \left| \frac{1}{Q_k^s} \sum_{j=0}^{k} q_j \Delta x_j \right|^{p_k} \le \frac{1}{\alpha} \sum_k \left| f_k\left(\frac{1}{Q_k^s} \sum_{j=0}^{k} q_j \Delta x_j \right) \right|^{p_k}$$

which shows that $x \in r^q\left(\Delta_s^p\right)$.

Theorem 6. Let $0 < p_k \le H < \infty$. Then the space $r^q\left(F, \Delta_s^p\right)$ is linearly isomorphic to the space $\ell(p)$.

Proof. To prove this, we need to show that there is a linear bijection between the spaces $r^q\left(F, \Delta_s^p\right)$ and $\ell(p)$ for $0 < p_k \le H < \infty$, using the notation of

$$y_k = f_k \frac{1}{Q_k^s} \sum_{j=0}^{k} q_j \Delta x_j .$$

Let us take $T : r^q\left(F, \Delta_s^p\right) \to \ell(p)$. It is obvious that, T is linear transformation. If we take $x = \theta$ we obtain that $Tx = \theta$ and hence T is injective.

We consider an arbitrary sequence $y \in \ell(p)$ and define the sequence, $x = (x_k)$ by

$$x_k = \sum_{n=0}^{k-1} \left(\frac{1}{q_n} - \frac{1}{q_{n+1}} \right) Q_n y_n + \frac{Q_k}{q_k} y_k \text{ for } k \in \mathbb{N}, \text{ where } Q_n = \sum_{k=0}^{n} q_k .$$

Then we have

$$h_\Delta(x) = \left[\sum_k \left| f_k \left[\frac{1}{Q_k^n} \sum_{j=0}^{k-1} \left(q_j - q_{j+1}\right) x_j + \frac{q_k}{Q_k^n} \right] \right|^{p_k} \right]^{\frac{1}{M}}$$

$$=\left(\sum_k \left| f_k\left(\sum_{j=0}^{k}\delta_{kj}y_j\right)\right|^{p_k}\right)^{\frac{1}{M}}$$

$$=\left(\sum \left| f_k\left(y_k\right)\right|^{p_k}\right)^{\frac{1}{M}}=h_\Delta\left(y\right)<\infty$$

where $$\delta_{kj}=\begin{cases}1, & k=j\\ 0, & k\neq j\end{cases}.$$

Then we have $x\in r^q\left(F,\Delta_s^p\right)$. Hence T is surjective and paranorm preserving. Therefore there is a linear bijection between the spaces $r^q\left(F,\Delta_s^p\right)$ and $\ell\left(p\right)$.

3-The $\alpha-$and $\beta-$ duals of $r^q\left(F,\Delta_s^p\right)$

In the present section, we compute the $\alpha-,\beta-$ duals of the space $r^q\left(F,\Delta_s^p\right)$ and give a basis for this space.

If a sequence space X paranormed by h contains a sequence $\left(y_n\right)$ with the property that for every $x\in X$ there is a unique sequence of scalars $\left(\alpha_n\right)$ such that

$$\lim_{n\to\infty} h\left(x-\sum_{k=0}^{n}\alpha_k y_k\right)=0$$

then $\left(y_n\right)$ is called a Schauder basis (or briefly basis) for X. The series $\sum_{k=0}^{\infty}\alpha_k y_k$ which has the sum x is then called the expansion of x with respect to $\left(y_n\right)$ and is written as

$$x=\sum_{k=0}^{\infty}\alpha_k y_k .$$

For the sequence spaces X and Y, define multipler sequence space $M\left(X:Y\right)$ by

$$M\left(X:Y\right)=\left\{p=\left(p_k\right)\in\omega: px=\left(p_k x_k\right)\in Y, \forall x\in X\right\}$$

Then the $\alpha-,\beta-$duals of X are given by

$$X^\alpha=M\left(X,\ell_1\right), \qquad X^\beta=M\left(X,cs\right)$$

Now we give some lemmas which need to prove our theorems.

Lemma 1.

(i) Let $1<p_k\leq H<\infty$ for every $k\in$. Then $A\in\left(\ell\left(p\right):\ell_1\right)$ if and only if there exists an integer $K>1$ such that

$$\sup_{n\in F}\sum_{k=0}^{\infty}\left|\sum_{n\in K}a_{nk}K^{-1}\right|^{p'_k}<\infty$$

(ii) Let $0<p_k\leq 1$ for every $k\in$. Then $A\in\left(\ell\left(p\right):\ell_1\right)$ if and only if

$$\sup_{K\in F}\sup_{k\in\mathbb{N}}\left|\sum_{n\in K}\alpha_{nk}\right|^{p_k}<\infty$$

Lemma 2.

(i) Let $1<p_k\le H<\infty$ for every $k\in\mathbb{N}$. Then $A\in(\ell(p):\ell_\infty)$ if and only if there exists an integer $K>1$ such that

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{\infty}\left|\alpha_{nk}^{-1}K^{-1}\right|^{p_k'}<\infty$$

(ii) Let $0<p_k\le 1$ for every $k\in\mathbb{N}$. Then $A\in(\ell(p):\ell_\infty)$ if and only if

$$\sup_{n,k\in\mathbb{N}}\left|\alpha_{nk}\right|^{p_k}<\infty$$

Lemma 3. Let $0<p_k\le H<\infty$ for every $k\in\mathbb{N}$. Then $A\in(\ell(p):c)$ if and only if Lemma 2 hold, and

$$\lim_{n\to\infty}\alpha_{nk}=\beta_k$$

Theorem 7. (i) Let $1<p_k\le H<\infty$ for every $k\in\mathbb{N}$. Define the set $R_1(p)$ as follows

$$R_1(p)=\bigcup_{K>1}\left\{x=(x_k)\in\omega:\sup_{N\in F}\sum_k\left|\sum_{n\in\mathbb{N}}f_k\left(\left[\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)x_nQ_k+\frac{x_n}{q_n}Q_n\right]K^{-1}\right)\right|^{p_k'}<\infty\right\}$$

Then

$$\left[r^q\left(F,\Delta_s^p\right)\right]^\alpha=R_1(p)$$

(ii) Let $0<p_k\le 1$ for every $k\in\mathbb{N}$. Define the set $R_2(p)$ by

$$R_2(p)=\left\{x=(x_k)\in\omega:\sup_{N\in F}\sup_{k\in\mathbb{N}}\left|\sum_{n\in\mathbb{N}}f_k\left(\left[\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)x_nQ_k+\frac{x_n}{q_n}Q_n\right]K^{-1}\right)\right|^{p_k}<\infty\right\}$$

Then

$$\left[r^q\left(F,\Delta_s^p\right)\right]^\alpha=R_2(p).$$

Proof. (i) Let $x=(x_k)\in\omega$. We easily derive with the notation $y_k=f_k\dfrac{1}{Q_k^n}\sum_{j=0}^{k}q_j\Delta x_j$ that

$$x_ny_n=\sum_{k=0}^{n-1}\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)x_nQ_k^sy_k+\frac{x_nQ_n^s}{q_n}y_n=\sum_{k=0}^{n}b_{nk}yk=(By)_n \qquad (5)$$

$n\in\mathbb{N}$, where $B=\{b_{nk}\}$ is defined by

$$b_{nk}=\begin{cases}\left(f_k\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)x_nQ_k\right), & (0\le k\le n-1)\\ f_k\left(\frac{x_n}{q_n}Q_n\right), & k=n\\ 0, & k>n\end{cases}$$

for all $k,n\in\mathbb{N}$. Thus we deduce from (5) that $xy=(x_ny_n)\in\ell_1$ whenever $x=(x_k)\in r^q(F,\Delta_s^p)$ if and only if $By\in\ell_1$ whenever $y=(y_k)\in\ell(p)$. This yields, (i) result that

$$\left[r^q\left(F,\Delta_s^p\right)\right]^{\alpha}=R_1(p).$$

(ii) This is easily obtained by proceeding as in the proof of (i) above by using the Lemma 1. So we omit the detail.

Theorem 8. (i) Let $1<p_k\le H<\infty$ for every $k\in\mathbb{N}$. Define the set $R_2(p)$ as follow

$$R_3(p)=\bigcup_{K>1}\left\{x=(x_k)\in\omega:\sum_k\left|f_k\left[\left(\frac{x_k}{q_k}+\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)\sum_{i=k+1}^{n}x_i\right)Q_k^s\right]K^{-1}\right|^{p_k'}<\infty\right\}$$

Then

$$\left[r^q\left(F,\Delta_s^p\right)\right]^{\beta}=R_3(p)\cap cs$$

(ii) Let $0<p_k\le 1$ for every $k\in\mathbb{N}$. Define the set $R_4(p)$ by

$$R_4(p)=\left\{x=(x_k)\in\omega:\sup_{k\in\mathbb{N}}\left|f_k\left[\left(\frac{x_k}{q_k}+\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)\sum_{i=k+1}^{n}x_i\right)Q_k^s\right]K^{-1}\right|^{p_k}<\infty\right\}$$

Then

$$\left[r^q\left(F,\Delta_s^p\right)\right]^{\beta}=R_4(p)\cap cs.$$

Proof. (i) Consider the following equation

$$\sum_{k=0}^{n}x_ky_k=\sum_{k=0}^{n}f_k\left[\left(\frac{x_k}{q_k}\right)+\left(\frac{x_k}{q_k}+\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)\sum_{i=k+1}^{n}x_i\right)Q_k^s\right]y_k \tag{6}$$
$$=(C_ny)$$

where $C=(c_{nk})$ is defined by

$$c_{nk}=\begin{cases}f_k\left(\left(\frac{x_k}{q_k}+\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)\sum_{i=k+1}^{n}x_i\right)Q_k^s\right), & (0\le k\le n)\\ 0, & (k>n)\end{cases}$$

for $k,n\in\mathbb{N}$. By this way, we see from (6) that $xy=(x_n y_n)\in cs$ whenever $x=(x_k)\in r^q(F,\Delta_s^p)$ if and only if $Cy\in c$ whenever $y\in\ell(p)$. Hence we deduce from Lemma 3

$$\sum_k\left|f_k\left[\left(\frac{x_k}{q_k}+\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)\sum_{i=k+1}^{n}x_i\right)Q_k^s\right]K^{-1}\right|^{p_k'}<\infty$$

and $\lim_n c_{nk}$ exists which is show that

$$\left[r^q\left(F,\Delta_s^p\right)\right]^{\beta}=R_3(p)\cap cs\,.$$

(ii) This may be obtained in the similar way as in the proof of (i) above by using the Lemma 2. and Lemma 3. So omit it.

Theorem 9. Let $F=(f_k)$ be a sequence of modulus functions and we define the sequence $y^{(k)}(q)=\left\{y_n^{(k)}(q)\right\}_{n\in\mathbb{N}}$ of the elements of the space $r^q\left(F,\Delta_s^p\right)$ for every fixed $k\in\mathbb{N}_0$ by $(\mathbb{N}_0=\mathbb{N}\cup 0)$

$$y_n^{(k)}(q)=\begin{cases} f_k\left(\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)Q_k^s\right), & (0\le n\le k-1)\\ 0, & (n>k-1)\end{cases}$$

Then the sequence $\left\{y^{(k)}(q)\right\}_{k\in\mathbb{N}}$ is a basis for the space $r^q\left(F,\Delta_s^p\right)$ and any $x\in r^q\left(F,\Delta_s^p\right)$ has a unique representation of the form

$$x=\sum_k\lambda_k(q)y^{(k)}(q) \tag{7}$$

where $\lambda_k(q)=\left(R^q\Delta x\right)_k$ for all $k\in\mathbb{N}$ and $0<p_k\le H<\infty$, $M=\max\{1,H\}$.

Proof It is clear that $y^{(k)}(q)\in r^q\left(F,\Delta_s^p\right)$, since

$$R^q\Delta y^{(k)}(q)=e^{(k)}\in\ell(p),\qquad k\in\mathbb{N}_0 \tag{8}$$

for $0<p_k\le H<\infty$, where $e^{(k)}$ is a sequence whose only non-zero term is 1in k^{th} place for each $k\in\mathbb{N}_0$.

Let $x\in r^q\left(F,\Delta_s^p\right)$ be given. For evey non-negative integer m, we put

$$x^{[m]}=\sum_{k=0}^{m}\lambda_k(q)y^{(k)}(q) \tag{9}$$

We obtain by applying $R^q\Delta$ to (9) with (8) that

$$R^q\Delta x^{[m]}=\sum_{k=0}^{m}\lambda_k(q)R^q\Delta y^{(k)}(q)=\sum_{k=0}^{m}\lambda_k(q)e^{(k)}$$

and

$$\left(R^q\left(x-x^{[m]}\right)\right)_i=\begin{cases}\left(R^q\Delta x\right)_i, & i>m\\ 0, & 0\le i\le m\end{cases}\qquad \left(i,m\in\mathbb{N}\right).$$

given $\varepsilon>0$, then there exists an integer m_0 such that

$$\sum_{i=m}^{\infty}\left[f_k\left(\left|\left(R^q\Delta x\right)_i\right|\right)\right]^{p_k}<\left(\frac{\varepsilon}{2}\right)^M$$

for all $m\ge m_0$. Hence,

$$g_\Delta\left(x-x^{[m]}\right)=\left(\sum_{i=m}^{\infty}\left[f_k\left(\left|\left(R^q\Delta x\right)_i\right|\right)\right]^{p_k}\right)^{\frac{1}{M}}\le\left(\sum_{i=m_0}^{\infty}\left[f_k\left(\left|\left(R^q\Delta x\right)_i\right|\right)\right]^{p_k}\right)^{\frac{1}{M}}<\frac{\varepsilon}{2}<\varepsilon$$

for all $m\ge m_0$, which supplies that $x\in r^q\left(F,\Delta_s^p\right)$ is represented as (7). To show the uniqueness of this representation, we suppose that

$$x=\sum_k \mu_k(q)\,y^{(k)}(q).$$

Since the linear transformation T, from $r^q\left(F,\Delta_s^p\right)$ to $\ell(p)$ used in Theorem 2, is continuous we have

$$\left(R^q\Delta x_n\right)=\sum_k \mu_k(q)\left[f_k\left(R^q\Delta y^{(k)}(q)\right)_n\right]=\sum_k \mu_k(q)e_n^{(k)}=\mu_n(q),\quad n\in\mathbb{N}$$

which contradicts the fact that $\left(R^q\Delta x\right)_n=\lambda_n(q)$ for all $n\in\mathbb{N}$. Hence, the representatıon (7) of $x\in r^q\left(F,\Delta_s^p\right)$ is unique.

REFERENCES

[1] B. Altay and F. Başar, On the paranormed Riesz sequence space of nonabsolute type, Southeast Asian Bull. Math., 26 (2002), pp. 701-715.

[2] B. Altay, and F. Başar, On the space of sequences of *p*-bounded variation and related matrix mappings, Ukranian Math. J., 1(1) (2003), pp. 136-147.

[3] C. Aydın and F. Başar, On the new sequence spaces which include the spaces c_o and c, Hokkaido Math. J., 33 (2002), pp. 383-398.

[4] C. Aydın and F. Başar, Some new paranormed sequence spaces, Inf. Sci., 160 (2004), 27-40.

[5] M. Basarır and M.Öztürk, On the Riesz difference sequence space,¨ Rendiconti del Cirocolo di Palermo, 57 (2008), 377-389.

[6] A. H. Ganie and N. A. Sheikh, On some new sequence spaces of nonabsolute type and matrix transformations, J. Egyp. Math. Soc., (2013)(in Press).

[7] H. Kızmaz, On certain sequence, Canad, Math. Bull., 24(2) (1981), pp.169-176.

[8] I. J. Maddox, Paranormed sequence spaces generated by infinite matrices, Proc. Camb. Phil. Soc., 64 (1968), pp. 335-340.

[9] I. J. Maddox, Elements of Functional Analysis , 2^{nd}ed., The University Press, Cambridge, (1988).

[10] N. Nakano, Concave modulars, J. Math. Soc. Japan, 5(1953), 29-49.

[11] M. Mursaleen, and A. K. Noman, On some new difference sequence spaces of non-absolute type, Math. Comput. Mod., 52 (2010), pp. 603-617.

[12] W. H. Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, Canad. J. Math., 25(1973), 973-978.

[13] S. Toeplitz,¨ Uber allegemeine Lineare mittelbildungen, Prace Math.¨ Fiz., 22 (1991). pp.113-119.

[14] A. Wilansky, Summability through Functional Analysis, North Holland Mathematics Studies, Amsterdam - New York - Oxford, (1984).

[15] Gupkari S.A., Some New Generalized Riesz Spaces, Fasciculi Mathematici, (2023).

[16] K. Raj and S.K Sharma., Difference Sequence Spaces Defined by A Sequence of Modulus Functions, Proyecciones Journal of Mathematics, (2011).

[17] G. M. Petersen, Regular matrix transformations, Mc Graw-Hill, London, (1966).

[18] N. A. Sheikh and A. H. Ganie, A new paranormed sequence space and some matrix transformations, Acta Math. Acad. Paedago. Nygr., 28 (2012), pp. 47-58.

[19] C. S. Wang , On Nörlund sequence spaces, Tamkang J. Math., 9 (1978),¨ pp. 269-274.

[20] P. N. Ng and P. Y. Lee, Cesaro sequences spaces of non-absolute type,´ Comment. Math. Prace Mat. 20(2) (1978), pp. 429-433.

[21] K. G. Erdmann, Matrix transformations between the sequence spaces of Maddox, J. Math. Anal. Appl., 180 (1993), pp. 223- 238.

[22] B. Altay, F. Başar and M. Mursaleen, On the Euler sequence spaces which include the spaces l_p and l_∞-II, Nonlinear Anal., 176 (2006), pp. 1465-1462.

[23] M. Başarır and M.Kayıkçı, On the Generalized B^m-Riesz Difference Sequence Space and β -Property, (2009).