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On the Generalized Francois Numbers

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Abstract. This study introduces the generalized Francois numbers and investigates their some properties. In addition, we provide the basic formulas such as Binet's formula, sums formulas. Also, we obtain some identities among the Fibonacci sequence, the Lucas sequence, and the generalized Francois sequence.

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1. Introduction

Number sequences are an important topic in mathematics. The Fibonacci sequence is one of the most renowned. Leonardo Fibonacci introduced a problem. The Fibonacci sequence emerged the solution of this problem. The sequence has extensive uses in many areas from mathematics to art. In addition, the Fibonacci numbers have in various interesting properties, such as the golden ratio, which arises from the ratio of consecutive the Fibonacci numbers.

For $n \geq 2$, the recurrence relation of the Fibonacci sequence is

$$F_n = F_{n-1} + F_{n-2},$$

with $F_0 = 0, F_1 = 1$. The Fibonacci numbers are associated with the sequence A000045 in OEIS [13]. The Binet's formula of the Fibonacci sequence is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.1}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Also, the negative indices of the Fibonacci numbers are

$$F_{-n} = (-1)^{n+1} F_n. \tag{1.2}$$

The summation formulas for the Fibonacci numbers are

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1, \tag{1.3}$$

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1,$$

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n}.$$

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In addition, there are the following identities of the Fibonacci numbers:

$$F_mF_n - F_{m+k}F_{n-k} = (-1)^{n-k}F_{m+k-n}F_k, \tag{1.4}$$

$$F_nL_m - F_mL_n = 2(-1)^mF_{n-m}, \tag{1.5}$$

$$F_nL_m + F_mL_n = 2F_{n+m}, \tag{1.6}$$

in [5, 14].

One of the other interesting number sequence is Lucas sequence. It is named for honor of the French mathematician Francois Edouard Anatole Lucas. The Lucas numbers share remarkable similarities with the Fibonacci numbers.

For $n \geq 2$, the recurrence relation of the Lucas sequence is given as follows:

$$L_n = L_{n-1} + L_{n-2}$$

with $L_0 = 2, L_1 = 1$. The Lucas numbers correspond to sequence A000032 in OEIS [13]. The Binet's formula of the Lucas sequence is

$$L_n = \alpha^n + \beta^n, \tag{1.7}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Negative subscript of the Lucas numbers is defined as follows:

$$L_{-n} = (-1)^nL_n. \tag{1.8}$$

The summation formulas for the Lucas numbers are given as follows:

$$\sum_{i=1}^{n} L_i = L_{n+2} - 3, \tag{1.9}$$

$$\sum_{i=1}^{n} L_{2i} = L_{2n+1} - 1,$$

$$\sum_{i=1}^{n} L_{2i-1} = L_{2n} - 2.$$

Furthermore, there is the following identity of the Lucas numbers:

$$L_{n+m} + (-1)^mL_{n-m} = L_mL_n, \tag{1.10}$$

in [5, 14].

Some identities between the Fibonacci and the Lucas numbers are as indicated below:

$$F_{n+1} + F_{n-1} = L_n, \tag{1.11}$$

$$L_{n+1} + L_{n-1} = 5F_n, \tag{1.12}$$

$$F_{n+m} + (-1)^mF_{n-m} = L_mF_n, \tag{1.13}$$

$$F_{n+m} - (-1)^mF_{n-m} = F_mL_n, \tag{1.14}$$

$$L_{n+m} - (-1)^mL_{n-m} = 5F_mF_n, \tag{1.15}$$

$$F_{n+2}L_{n+1} - F_{n+1}L_n = F_{2n+2} - 2(-1)^n, \tag{1.16}$$

$$F_nF_{m+1} + F_mF_{n+1} = \frac{2L_{n+m+1} - (-1)^mL_{n-m}}{5} \tag{1.17}$$

in [5, 14].

Catarino and Borges defined the Leonardo sequences, which are related to the Fibonacci sequence in [2]. For $n \geq 2$, the Leonardo sequence is determined by the following recurrence relation:

$$Le_n = Le_{n-1} + Le_{n-2} + 1$$

with $Le_0 = Le_1 = 1$. The Leonardo numbers are associated with the sequence A001595 in OEIS [13]. The Binet's formula of the Leonardo sequence is

$$Le_n = \frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ in [2].

The researchers investigate topics including generalizations of the Leonardo numbers. The generalized Leonardo numbers are defined by the following recurrence relation for $n \geq 2$:

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k,$$

with the initial conditions $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$. Furthermore, the relationship between generalized Leonardo numbers and Fibonacci numbers is presented as

$$\mathcal{L}_{k,n} = (k+1)F_{n+1} - k, \tag{1.18}$$

in [6]. The Binet's formula of the generalized Leonardo sequence is

$$\mathcal{L}_{k,n} = (k+1)\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - k,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ in [7].

Another version of the Leonardo numbers is defined in [3]. The authors introduced a new version of the Leonardo numbers called the Francois numbers, in honor of the French mathematician Francois Edouard Anatole Lucas. For $n \geq 2$, the recurrence relation for Francois sequence is given as follows:

$$\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2} + 1,$$

with the initial conditions $\mathcal{F}_0 = 2, \mathcal{F}_1 = 1$. This sequence corresponds to the sequence A022318 in OEIS that the initial element is eliminated. The Binet's formula of the Francois sequence is

$$\mathcal{F}_n = \alpha^n + \beta^n + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - 1,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The relation among the Francois, the Fibonacci and the Lucas numbers is obtained as

$$\mathcal{F}_n = L_n + F_{n+1} - 1,$$

in [3].

In recent years, the generalizations of the Leonardo numbers have been studied. More information about the Leonardo and the generalized of theLeonardo numbers can be found in [1,3,4,6,8–12,15].

The following table displays the first terms of the Fibonacci, the Lucas, the Leonardo, the Francois, and the generalized Leonardo numbers:

n	F_n	L_n	Le_n	$\mathcal{F}_n$	$\mathcal{L}_{k,n}$
0	0	2	1	2	1
1	1	1	1	1	1
2	1	3	3	4	$2+k$
3	2	4	5	6	$3+2k$
4	3	7	9	11	$5+4k$
5	5	11	15	18	$8+7k$
6	8	18	25	30	$13+12k$
7	13	29	41	49	$21+20k$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Based on the above papers, we introduce the generalized Francois numbers. Furthermore, we derive various identities concerning the generalized Francois numbers. Finally, all the results are reduced to Francois numbers for $k = 1$.

2. Main Results

Firstly, we introduce the generalized Francois numbers.

Definition 2.1. For the positive integer k and $n \geq 2$ the generalized Francois sequence $\mathcal{F}_{k,n}$ is defined as follows:

$$\mathcal{F}_{k,n} = \mathcal{F}_{k,n-1} + \mathcal{F}_{k,n-2} + k, \tag{2.1}$$

with the initial conditions $\mathcal{F}_{k,0} = 2, \mathcal{F}_{k,1} = 1$.

Proposition 2.2. *The Binet-like formula for the generalized Francois sequence is*

$$\mathcal{F}_{k,n} = \alpha^n + \beta^n + k\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - k, \tag{2.2}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$.

Proof. The general solution of the difference equation

$$\mathcal{F}_{k,n} = \mathcal{F}_{k,n-1} + \mathcal{F}_{k,n-2} + k,$$

is given by

$$\mathcal{F}_{k,n} = c_1\alpha^n + c_2\beta^n - k.$$

Considering that $\mathcal{F}_{k,0} = 2$, $\mathcal{F}_{k,1} = 1$, its follows that

$$c_1 = \frac{-(2+k)\beta + 1 + k}{\alpha - \beta} \quad \text{and} \quad c_2 = \frac{(2+k)\alpha - 1 - k}{\alpha - \beta}.$$

From here, the result is obtained. □

Taking $k = 1$ in (2.2), we get the Binet-like formula for the Francois sequence as follows:

$$\mathcal{F}_n = \alpha^n + \beta^n + \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - 1,$$

in [3]. Now, we give the generating function for the generalized Francois numbers in the following proposition.

Proposition 2.3. *The generating function for the generalized Francois numbers is as follows:*

$$\sum_{i=0}^{\infty} \mathcal{F}_{k,n} x^n = \frac{(k+1)x^2 - 3x + 2}{1 - 2x + x^3},$$

where $\mathcal{F}_{k,n}$ *is the nth generalized Francois number.*

Proof. Let us consider the following ordinary generating function:

$$g(x) = \sum_{i=0}^{\infty} \mathcal{F}_{k,n} x^n.$$

From the equation (2.1), we obtain

$$\frac{g(x) - \mathcal{F}_{k,0} - \mathcal{F}_{k,1}x}{x^2} = \frac{g(x) - \mathcal{F}_{k,0}}{x} + g(x) + \frac{k}{1-x}.$$

Hence,

$$g(x) = \frac{(k+1)x^2 - 3x + 2}{1 - 2x + x^3}.$$

□

Taking $k = 1$, we get the generating function for the Francois numbers as follows:

$$\sum_{i=0}^{\infty} \mathcal{F}_n x^n = \frac{2x^2 - 3x + 2}{1 - 2x + x^3}.$$

Proposition 2.4. *For any nonnegative integer n, the following identity holds true:*

$$\mathcal{F}_{k,n} = L_n + kF_{n+1} - k, \tag{2.3}$$

where F_n, L_n, *and* $\mathcal{F}_{k,n}$ *are the nth Fibonacci, Lucas and generalized Francois numbers, respectively.*

Proof. Considering (1.1), (1.7), and (2.2), the result is clear. □

From $k = 1$ in (2.3), we have $\mathcal{F}_n = L_n + F_{n+1} - 1$ in [3].

Proposition 2.5. *The negative subscript of the generalized Francois numbers is given as follows:*

$$\mathcal{F}_{k,-n} = (-1)^n(L_n + kF_{n-1}) - k,$$

where F_n, L_n, and $\mathcal{F}_{k,n}$ are the nth Fibonacci, Lucas and generalized Francois numbers, respectively.

Proof. Using (2.3), we have

$$\mathcal{F}_{k,-n} = L_{-n} + kF_{-n+1} - k.$$

From (1.2) and (1.8), the result is clear. □

Proposition 2.6. *For $n \geq 1$, the following identity holds:*

$$\mathcal{F}_{k,n+1} + \mathcal{F}_{k,n-1} = 5F_n + kL_{n+1} - 2k,$$

where F_n, L_n, and $\mathcal{F}_{k,n}$ are the nth Fibonacci, Lucas and generalized Francois numbers, respectively.

Proof. Using (2.3), we get

$$\mathcal{F}_{k,n+1} + \mathcal{F}_{k,n-1} = (L_{n+1} + L_{n-1}) + k(F_{n+2} + F_n) - 2k.$$

Considering (1.11) and (1.12), the result is obtained. □

Now, we provide the summation formulas for the generalized Francois numbers.

Proposition 2.7. *For $n \geq 0$, the following summation formulas hold true:*

$$\sum_{i=0}^{n} \mathcal{F}_{k,i} = \mathcal{F}_{k,n+2} - 1 - k(n+1),$$

$$\sum_{i=0}^{n} \mathcal{F}_{k,2i} = \mathcal{F}_{k,2n+1} + 1 - kn,$$

$$\sum_{i=0}^{n} \mathcal{F}_{k,2i+1} = \mathcal{F}_{k,2n+2} - 2 - k(n+1),$$

where $\mathcal{F}_{k,n}$ is the nth generalized Francois number.

Proof. From (2.3), the sum of generalized Francois numbers is

$$\sum_{i=0}^{n} \mathcal{F}_{k,i} = \sum_{i=0}^{n} (L_n + kF_{n+1} - k).$$

Using (1.3) and (1.9), we have

$$\sum_{i=0}^{n} \mathcal{F}_{k,i} = L_{n+2} + kF_{n+3} - 1 - 2k - kn.$$

Considering (2.3), the result is obtained. Similarly, the other summation formulas can be proved. □

Taking $k = 1$, the summation formulas for the Francois numbers are obtained as follows:

$$\sum_{i=0}^{n} \mathcal{F}_i = \mathcal{F}_{n+2} - 1 - (n+1),$$

$$\sum_{i=0}^{n} \mathcal{F}_{2i} = \mathcal{F}_{2n+1} + 1 - n,$$

$$\sum_{i=0}^{n} \mathcal{F}_{2i+1} = \mathcal{F}_{2n+2} - 2 - (n+1).$$

Proposition 2.8. *For any nonnegative integer $m \geq 1$ and $n \geq m$, the following identity holds true:*

$$\mathcal{F}_{k,n+m} + (-1)^m \mathcal{F}_{k,n-m} = L_m(\mathcal{F}_{k,n} + k) - k(1 + (-1)^m),$$

$$\mathcal{F}_{k,n+m} - (-1)^m \mathcal{F}_{k,n-m} = F_m(5F_n + kL_{n+1}) - k(1 - (-1)^m),$$

where F_n, L_n, and $\mathcal{F}_{k,n}$ are the nth Fibonacci, Lucas and generalized Francois numbers, respectively.

Proof. From (2.3), we get

$$\mathcal{F}_{k,n+m} + (-1)^m \mathcal{F}_{k,n-m} = (L_{n+m} + (-1)^m L_{n-m}) - k(1 + (-1)^m) + k(F_{n+m+1} + (-1)^m F_{n-m+1}).$$

Using (1.13), (1.10), and (2.3), the first identity is obtained. Similarly, the other identity is derived by using (1.14), (1.15), and (2.3). □

Proposition 2.9. *For nonnegative integers m and r where $m \geq r + 4$, the following identity is valid:*

$$\begin{aligned}\mathcal{F}_{k,m+r}\mathcal{F}_{k,m+r-2} + \mathcal{F}_{k,m-r}\mathcal{F}_{k,m-r-2} &= L_{2m-2}L_{2r} - 5k(F_{m+r-1} + F_{m-r-1}) + 6(-1)^{m+r} \\ &+ k(2F_{2m-1}L_{2r} + 6(-1)^{m+r}) - k^2(L_{m+r} + L_{m-r}) \\ &+ \frac{k^2}{5}(L_{2m}L_{2r} + 6(-1)^{m+r}) + 2k^2,\end{aligned}$$

where F_n, L_n, and $\mathcal{F}_{k,n}$ are the nth Fibonacci, Lucas and generalized Francois numbers, respectively.

Proof. Using (2.2) to LHS, we obtain

$$\begin{aligned}LHS &= \left(\alpha^{m+r} + \beta^{m+r} + k\left(\frac{\alpha^{m+r+1} - \beta^{m+r+1}}{\alpha - \beta}\right) - k\right)\left(\alpha^{m+r-2} + \beta^{m+r-2} + k\left(\frac{\alpha^{m+r-1} - \beta^{m+r-1}}{\alpha - \beta}\right) - k\right) \\ &+ \left(\alpha^{m-r} + \beta^{m-r} + k\left(\frac{\alpha^{m-r+1} - \beta^{m-r+1}}{\alpha - \beta}\right) - k\right)\left(\alpha^{m-r-2} + \beta^{m-r-2} + k\left(\frac{\alpha^{m-r-1} - \beta^{m-r-1}}{\alpha - \beta}\right) - k\right).\end{aligned}$$

Hence, we have

$$\begin{aligned}LHS &= L_{2m+2r-2} + L_{2m-2r-2} + 6(-1)^{m+r} \\ &- k(L_{m+r} + L_{m+r-2}) - k(L_{m-r} + L_{m-r-2}) \\ &+ k(2F_{2m+2r-1} + 2F_{2m-2r-1} + 3(-1)^{m+r} + 3(-1)^{m-r}) \\ &+ 2k^2 + \frac{k^2}{5}(L_{2m+2r} + L_{2m-2r} + 6(-1)^{m+r}) \\ &- k^2(F_{m+r+1} + F_{m-r+1}) - k^2(F_{m+r-1} + F_{m-r-1}).\end{aligned}$$

From (1.10), (1.11), (1.12), and (1.13), the result is clear. □

Now, we provide the identities between the generalized Francois numbers and the generalized Leonardo numbers.

Proposition 2.10. *For any nonnegative integer n, we have*

$$\mathcal{F}_{k,n+1}\mathcal{L}_{k,n+1} - \mathcal{F}_{k,n}\mathcal{L}_{k,n} = (k+1)(F_{2n+2} - 2(-1)^n) - kL_{n-1} + kF_n((k+1)F_{n+3} - 1 - 2k),$$

where F_n, L_n, $\mathcal{L}_{k,n}$, and $\mathcal{F}_{k,n}$ are the nth Fibonacci, Lucas, generalized Leonardo and generalized Francois numbers, respectively.

Proof. Using (2.3) and (1.18) to left hand side (LHS), we get

$$LHS = (L_{n+1} + kF_{n+2} - k)((k+1)F_{n+2} - k) - (L_n + kF_{n+1} - k)((k+1)F_{n+1} - k).$$

Hence, we have

$$LHS = (k+1)(F_{n+2}L_{n+1} - F_{n+1}L_n) - kL_{n-1} + kF_n((k+1)F_{n+3} - 1 - 2k).$$

Considering (1.16), the result is obtained. □

Taking $k = 1$, we derive the following identity between the Leonardo and the Francois numbers:

$$\mathcal{F}_{n+1}Le_{n+1} - \mathcal{F}_n Le_n = (k+1)(F_{2n+2} - 2(-1)^n) - L_{n-1} + F_n(2F_{n+3} - 3).$$

Now, we present the relationships between the generalized Francois and the Fibonacci numbers.

Proposition 2.11. *For $m \geq 1$ and $n \geq m+1$, the following identities hold true:*

$$F_n\mathcal{F}_{k,m} - F_m\mathcal{F}_{k,n} = F_{n-m}(2(-1)^m + k) + k(F_m - F_n),$$
$$F_n\mathcal{F}_{k,m} + F_m\mathcal{F}_{k,n} = 2F_{n+m} - k(F_m + F_n) + k\left(\frac{2L_{n+m+1} - (-1)^m L_{n-m}}{5}\right),$$

where F_n, L_n, and $\mathcal{F}_{k,n}$ are the nth Fibonacci, Lucas and generalized Francois numbers, respectively.

Proof. Using (2.3) to LHS, we get

$$LHS = F_nL_m - F_mL_n + k(F_nF_{m+1} - F_mF_{n+1}) + k(F_m - F_n).$$

From (1.4) and (1.5), the first identity is obtained. Similarly, the second identity can be found by using (1.17) and (1.6). □

Proposition 2.12. *For $m \geq 1$ and $n \geq m+1$, the following identities hold true:*

$$L_n\mathcal{F}_{k,m} - L_m\mathcal{F}_{k,n} = k(-1)^{m+1}F_{n-m} - k(L_n - L_m),$$
$$L_n\mathcal{F}_{k,m} + L_m\mathcal{F}_{k,n} = 2L_nL_m - k(L_n + L_m) + k(2F_{n+m+1} + (-1)^m L_{n-m}),$$

where F_n, L_n, and $\mathcal{F}_{k,n}$ are the nth Fibonacci, Lucas and generalized Francois numbers, respectively.

Proof. Using (2.3) to LHS, we get

$$LHS = k(L_nF_{m+1} - L_mF_{n+1}) - k(L_n - L_m).$$

From (1.5), the first identity is clear. Similarly, the second identity can be obtained. □

3. Conclusion

In this study, the generalized Francois numbers are considered. The basic identities related to these numbers are obtained. In addition, the relations between these numbers, Fibonacci and Lucas numbers are provided. In future studies, other properties of the generalized Francois numbers can be found and the different number systems can be defined with these numbers and their properties can be analyzed.

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

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