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Araştırma Makalesi / Research Article

On the Distribution of a Boundary Functional of the Semi-Markovian Random Walk Process with Two Delaying Barriers

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Abstract

In this study, a process of semi-Markovian random walk with delaying barriers at $0-$ and $\beta-$levels ($\beta > 0$) and first falling moment of the process into the delaying barrier at zero-level, (τ_0), are mathematically constructed, in this case when the random walk happens according to the Laplace's distribution $L(1^-;1^+)$. Then it is given an explicit expression of the Laplace transformation of the distribution of random variable τ_0. Also the simple formulas for expectation and variance of random variable τ_0 are obtained by the means of this Laplace transformation.

***Keywords*:** Semi-Markovian random walk process, Laplace distribution, delaying barrier, expected value, variance, Laplace transformation.

İki Tutan Bariyerli Yarı-Markovian Rastgele Yürüyüş Sürecinin Bir Sinir Fonksiyonalinin Dağılımı Hakkında

Öz

Bu çalışmada, rastgele yürüyüşün $L(1^-;1^+)$ Laplace dağılımına sahip olması durumunda, sıfır ve $\beta(\beta > 0)-$seviyelerinde tutan bariyerlere sahip bir yarı-Markovian rastgele yürüyüş süreci ve bu sürecin sıfır seviyesindeki tutan bariyere ilk kez düşme anı, (τ_0), matematiksel olarak kurulmuştur. Daha sonra τ_0 rastgele değişkeninin Laplace dönüşümünün açık bir ifadesi verilmiştir. Ayrıca bu Laplace dönüşümünü kullanarak, τ_0 rastgele değişkeninin beklenen değer ve varyansı için basit formüller elde edilmiştir.

***Anahtar Kelimeler*:** Yarı-Markovian rastgele yürüyüş süreci, Laplace dağılımı, Tutan bariyer, Beklenen değer, Varyans, Laplace dönüşümü.

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1. Introduction

In recent years, random walks with one or two barriers are used to solve a number of very interesting problems in the fields of inventory, queues and reliability theories, mathematical biology etc. Many good monographs in this field exist in literature (see references Borovkov 1975; Borovkov 1976; Feller 1968 and etc.).

In particular, a number of very interesting problems of stock control, queues and reliability theories can be expressed by means of random walks with two barriers. These barriers can be reflecting, delaying, absorbing, elastic, etc., depending on concrete problems at hand. For instance, it is possible to express random levels of stock in a warehouse with finite volumes or queueing systems with finite waiting time or sojourn time by means of random walks with two delaying barriers. Furthermore, the functioning of stochastics systems with spare equipment can be given by random walks with two barriers, one of them is delaying and the other one is any type barrier. Numerous studies have been done about step processes of semi-Markovian random walk with two barriers of their practical and theoretical importance. But in the most of these studies the distribution of the process has free distribution. Therefore the obtained results in this case are cumbersome and they will not be useful for applications (see references Borovkov 1975; Borovkov 1976; Feller 1968; Khaniev 1984; Khaniev & Ünver 1997; Lotov 1991 and etc.).

For the problem considered in this study, it is considered a semi-Markovian random walk with two delaying barriers, and the process representing the quantity of the stock has been given by using a random walk and a renewal process. Such models were rarely considered in literature (see references Maden & Shamilova 2016; Maden 2016; Maden 2017; Nasirova 1984; Nasirova & Omarova 2007; Nasirova & Sadikova 2009; Nasirova & Shamilova 2014; Nasirova et al.; Omarova & Bakhshiev 2010 and etc.). The practical state of the problem mentioned above is as follows.

Suppose that some quantity of a stock in a certain warehouse is increasing or decreasing in random discrete portions depending to the demands at discrete times. Then, it is possible to characterize the level of stock by a process called the semi-Markovian random walk process. But sometimes some problems occur in stock control theory such that in order to get an adequate solution we have to consider some processes which are more complex than semi-Markovian random walk processes. For example, if the borrowed quantity is demanded to be added to the warehouse immediately when the quantity of demanded stock is more than the total quantity of stock in the warehouse then, it is possible to characterize the level of stock in the warehouse by a stochastic process called as semi-Markovian random walk processes with delaying barrier at zero-level. Also since the volume of warehouse is finite in real cases, the supply coming to the warehouse is stopped until the next demand when the warehouse becomes full. In order to characterize the quantity of stock in the warehouse under these conditions it is necessary to use a stochastic process called as semi-Markovian random walk process with two delaying barriers. Note that semi-Markovian random walk processes with two delaying barriers, have not been considered enough in literature. This type problems may occur, for example, in the control of military stocks, refinery stocks, reserve of oil wells, and etc.

In the following, a semi-Markovian random walk with delaying barriers at $0-$ and $\beta(\beta>0)-$ levels, that has a denumerable state space, is constructed and the main probability characteristics of a boundary functional of this process are considered.

2. Construction of the Process

Suppose that $\{(\xi_i,\eta_i): i=1,2,....\}$ is any sequence of identically and independently distributed random variables, defined on any probability space $(\Omega, F, \mathrm{P}(\cdot))$, such that ξ_i's are positive valued, i.e., $P\{\xi_i>0\}=1$. Also, the random variables ξ_i and η_i are mutually independent as well. Furthermore the numbers $z>0$, $\beta>0$ and $0<z<\beta$ are given. In this case, let us denote the distribution functions of $\xi_1(w)$ and $\eta_1(w)$

$$\Phi(t)=P\{\xi_1(w)<t\},\ F(x)=P\{\eta_1(w)<x\},\ t\in\mathbb{R}^+ \text{ and } x\in\mathbb{R},$$

respectively. By the means of these random variables, we can construct the following process:

$$X_1(t)=\sum_{i=0}^{n-1}\eta_i,\quad \text{if}\quad \sum_{i=0}^{n-1}\xi_i\le t<\sum_{i=0}^{n}\xi_i,\ n=1,2,3,....,$$

where $\xi_0=0$ and $\eta_0=z\in(0,\beta)$. This process forms a step process of the semi-Markovian random walk. Now, let us delay this process by a delaying barrier at zero level as follows:

$$X_2(t)=X_1(t)-\inf_{0<s<t}\{0,\ X_1(s)\}.$$

This process forms a step process of semi-Markovian random walk with delaying barrier on the zero-level. Then, the process $X_2(t)$ is delayed by a delaying barrier on $\beta(\beta>0)-$level:

$$X(t)=X_2(t)-\sup_{0\le s\le t}\{0,\ X_2(s)-\beta\}.$$

The process $X(t)$ forms a step process of semi-Markovian random walk with delaying barriers on the zero-level and on the $\beta(\beta>0)-$level.

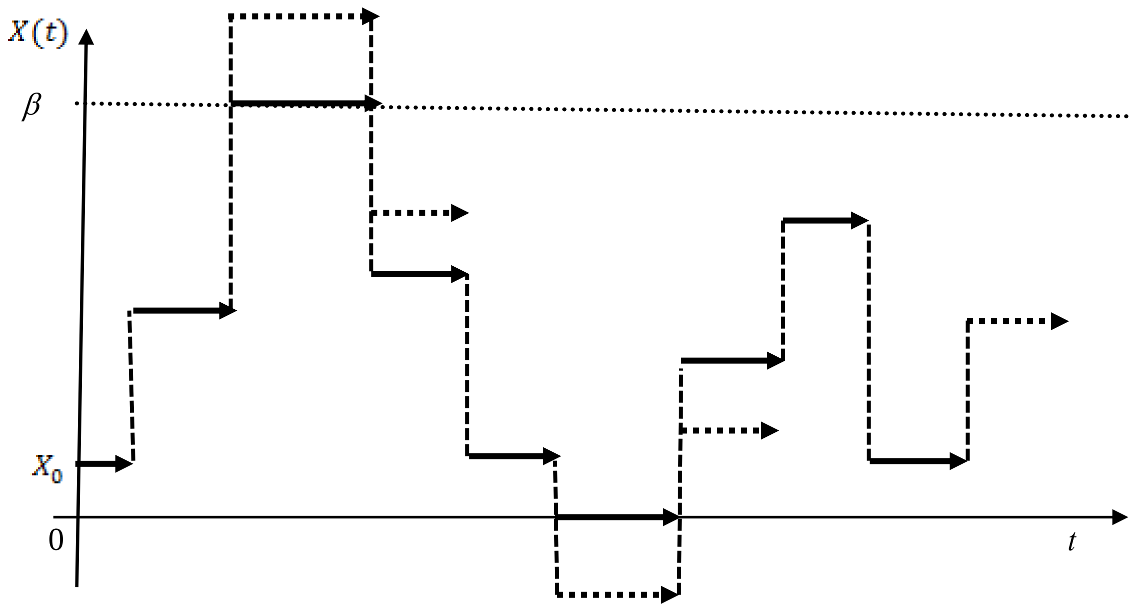

Fig. 1. A View of a Step Process of Semi-Markovian Random Walk with Two Delaying Barriers

Now, we can construct the desired stochastic process in the following way we shall call the step process of the semi-Markovian random walk with delaying barriers at levels $0-$ and $\beta(\beta > 0)-$level.

$$X(t) = X_n, \text{ if } \sum_{i=0}^{n-1} \xi_i \le t < \sum_{i=1}^{n} \xi_i, \text{ n} \ge 1,$$

where

$$X_n = \min\{ \beta, \max\{ 0, X_{n-1} + \eta_n\}\}, n \ge 1, X_0 = z.$$

Let us denote by τ_0 the first falling moment of the process $X(t)$ into the delaying barrier at zero$-$level. In this case, we can write

$$\tau_0 = \inf\{ t : X(t) = 0\}.$$

Also we get $\tau_0 = +\infty$ when $X(t) \ne 0$ for every $t \in \mathbb{R}^+$.

Note that τ_0 is important from a scientific and practical point of view and it is an important boundary functional of the process $X(t)$. This random variable plays an important role in solving of most probability problems arising in control of random levels of stocks in a warehouse which is functioning according to the process $X(t)$. For this reason, the consideration with detailed of random variable τ_0 seems very interesting from scientific and practical point of view.

3. The Laplace Transformation of τ_0

In this section, let us calculate the Laplace transformation of random variable τ_0, because the Laplace transformation is the most important characteristic of a random variable. Let us denote the Laplace transform of the distribution of the random variable τ_0 by

$$L(\theta) = E[e^{-\theta\tau_0}] \tag{3.1}$$

and the Laplace transform of the conditional distribution of the random variable τ_0 by

$$L(\theta|z) = E[e^{-\theta\tau_0}|X(0) = z], \ \theta > 0, \ z \geq 0. \tag{3.2}$$

Let us denote the conditional distribution of random variable of τ_0 and the Laplace transformation of the conditional distribution of it by

$$N(t|z) = P[\tau_0 > t|X(0) = z], \tag{3.3}$$

and

$$\tilde{N}(\theta|z) = \int_{t=0}^{\infty} e^{-\theta t} N(t|z)dt, \tag{3.4}$$

respectively. Then, the Laplace transformation of the absolute distribution has the following form:

$$\tilde{N}(\theta) = \int_{t=0}^{\infty} \tilde{N}(\theta|z)dP(X(0) < z).$$

Finally, let us denote the Laplace transformation of random variable ξ_1 by

$$\varphi(\theta) = E\left[e^{-\theta\xi_1}\right]. \tag{3.5}$$

Thus, we can easily obtain that

$$\tilde{N}(\theta|z) = \frac{1 - L(\theta|z)}{\theta} \quad \text{or} \quad L(\theta|z) = 1 - \theta\tilde{N}(\theta|z).$$

Now, let us give an integral equation for $\tilde{N}(\theta|z)$. For this aim, we can state the following theorem:

Theorem 1: Under above representations, we have

$$\tilde{N}(\theta|z) = \frac{1-\varphi(\theta)}{\theta} + \varphi(\theta) \int_{y=0}^{\infty} \tilde{N}(\theta|y)\, d_y P\{\eta_1 < y - z\}.$$

Proof: According to the total probability formula, it is obvious that

$$N(t|z) = P_z\{\tau_0 > t\} = P\{\tau_0 > t | X(0) = z\}$$

$$= P\{\tau_0 > t; \xi_1 > t | X(0) = z\} + P\{\tau_0 > t; \xi_1 < t | X(0) = z\}$$

$$= P\{\xi_1 > t\} + \int_{s=0}^{t} \int_{y=0}^{\beta} P\{\tau_0 > t - s | X(0) = y\}$$

$$.P\{\xi_1 \in ds; z + \eta_1 > 0; \min[\beta, z + \eta_1] \in dy\}.$$

Thus, we have an integral equation for the distribution of random variable τ_0 as follows:

$$N(t|z) = P\{\xi_1 > t\} + \int_{s=0}^{t} P\{\xi_1 \in ds\} \int_{y=0}^{\beta} N(t - s|y)\, d_y [\varepsilon(\beta - y) P\{\eta_1 < y - z\}] \qquad (3.6)$$

where

$$\varepsilon(\beta - y) = \begin{cases} 1, & \beta \geq y \\ 0, & \beta < y \end{cases}.$$

By applying the Laplace transform to this equation, i.e., multiplying both sides of (3.6) by $e^{-\theta t}$, integrating with respect to t from 0 to ∞, and taking into account the definition of $\tilde{N}(\theta|z)$, we have

$$\tilde{N}(\theta|z) = \int_{t=0}^{\infty} e^{-\theta t} N(\theta|z) dt = \int_{0}^{\infty} e^{-\theta t} P\{\xi_1 > t\} dt$$

$$+ \int_{0}^{\infty} e^{-\theta t} \int_{s=0}^{t} \int_{y=0}^{\beta} P\{\xi_1 \in ds\} N(t - s|y)\, d_y [P\{\eta_1 < y - z\}] dt. \qquad (3.7)$$

On the other hand, it is obviously

$$\int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1 > t\} dt = \frac{1 - \varphi(\theta)}{\theta},$$

where $\varphi(\theta) = E[\exp(-\theta\xi_1)]$. Thus, the equation (3.7) can be rewritten in the following form

$$\tilde{N}(\theta|z) = \int_0^\infty e^{-\theta t} P\{\xi_1 > t\} dt$$

$$= \frac{1-\varphi(\theta)}{\theta} + \varphi(\theta) \int_{y=0}^{\infty} \tilde{N}(\theta|y)\, d_y P\{\eta_1 < y - z\}. \tag{3.8}$$

Therefore, Theorem 1 is proved.

For the arbitrary distributed random variable η_1, the equation (3.8) can be solved with method successive approximations. But such decision is not useful for applications (see references Maden & Shamilova 2016; Maden 2016; Maden 2017; Nasirova 1984; Nasirova & Omarova 2007; Nasirova & Sadikova 2009 and etc.). Therefore we shall solve it, for example, if η_1 has the Laplace distribution as follows:

$$\mathrm{P}\{\eta_1 < x\} = \begin{cases} \dfrac{\lambda}{\lambda+\mu} e^{\mu x}, & x < 0 \\ 1 - \dfrac{\mu}{\lambda+\mu} e^{-\lambda x}, & x > 0 \end{cases}$$

where $\mu > 0$ and $\lambda > 0$. In this case the equation (3.8) can be write as

$$\tilde{N}(\theta|z) = \frac{1-\varphi(\theta)}{\theta} + \frac{\mu\varphi(\theta)}{\lambda+\mu} e^{-\lambda(\beta - z)} \tilde{N}(\theta|\beta)$$

$$+ \frac{\lambda\mu\varphi(\theta)}{\lambda+\mu} e^{-\mu b} \int_{y=0}^{z} \tilde{N}(\theta|y) e^{\mu y} dy + \frac{\lambda\mu\varphi(\theta)}{\lambda+\mu} e^{\lambda z} \int_{y=z}^{\beta} \tilde{N}(\theta|y) e^{-\lambda y} dy. \tag{3.9}$$

Now, we shall following replacement $\tilde{N}(\theta|z) = \dfrac{1 - L(\theta|z)}{\theta}$. Thus, we have

$$L(\theta|z) = \frac{\lambda\varphi(\theta)}{\lambda+\mu} e^{-\mu z} + \frac{\mu\varphi(\theta)}{\lambda+\mu} e^{-\lambda(\beta - z)} L(\theta|\beta)$$

$$+ \frac{\lambda\mu\varphi(\theta)}{\lambda+\mu} e^{-\mu z} \int_{y=0}^{z} L(\theta|y) e^{\mu y} dy + \frac{\lambda\mu\varphi(\theta)}{\lambda+\mu} e^{\lambda z} \int_{y=z}^{\beta} L(\theta|y) e^{-\lambda y} dy \tag{3.10}$$

On the other hand, by considering this integral equation, we can write the following differential equation on z:

$$L''(\theta|z) - (\lambda - \mu) L'(\theta|z) - \lambda\mu\left[1 - \varphi(\theta)\right] L(\theta|z) = 0, \tag{3.11}$$

which has the solution

$$L(\theta|z) = c_1(\theta)\, e^{k_1(\theta) z} + c_2(\theta)\, e^{k_2(\theta) z}, \tag{3.12}$$

where $k_i(\theta),\ \ i = 1,2$ the roots of the characteristic equation of the differential equation (3.11), that is,

$$k^2(\theta) - (\lambda - \mu) k(\theta) - \lambda\mu[1 - \varphi(\theta)] = 0.$$

From the integral equation (3.10) we find the following boundary conditions:

$$\begin{cases} L(\theta|0) = \dfrac{\lambda\varphi(\theta)}{\lambda+\mu} + \dfrac{\mu\varphi(\theta)}{\lambda+\mu} e^{-\lambda\beta)} L(\theta|\beta) + \dfrac{\lambda\mu\varphi(\theta)}{\lambda+\mu} \displaystyle\int_{y=0}^{\beta} L(\theta|y) e^{-\lambda y}\, dy, \\ L'(\theta|0) = -\dfrac{\lambda\mu\varphi(\theta)}{\lambda+\mu} + \dfrac{\lambda\mu\varphi(\theta)}{\lambda+\mu} e^{-\lambda\beta} L(\theta|\beta) + \dfrac{\lambda^2\mu\varphi(\theta)}{\lambda+\mu} \displaystyle\int_{y=b}^{\beta} L(\theta|y)\, e^{-\lambda y}\, dy. \end{cases}$$

Whence we have the following system of the linear algebraic non-homogeneous equations

$$\begin{cases} \left\{ \lambda[\lambda - k_1(\theta)] + [\lambda - k_2(\theta)] k_1(\theta)\, e^{-[\lambda - k_1(\theta)]\beta} \right\} c_1(\theta) \\ \qquad + \left\{ \lambda[\lambda - k_2(\theta)] + [\lambda - k_1(\theta)] k_2(\theta)\, e^{-[\lambda - k_2(\theta)]\beta} \right\} c_2(\theta) = \lambda^2 \varphi(\theta), \\ \left\{ \mu[\lambda - k_1(\theta)] - [\lambda - k_2(\theta)] k_1(\theta)\, e^{-[\lambda - k_1(\theta)]\beta} \right\} c_1(\theta) \\ \qquad + \left\{ \mu[\lambda - k_2(\theta)] - [\lambda - k_1(\theta)] k_2(\theta)\, e^{-[\lambda - k_2(\theta)]\beta} \right\} c_2(\theta) = \lambda\mu\varphi(\theta). \end{cases}$$

After some calculations we get

$$c_1(\theta) = \frac{\lambda\varphi(\theta)[\lambda - k_1(\theta)] k_2(\theta)}{[\lambda - k_1(\theta)]^2 k_2(\theta) - [\lambda - k_2(\theta)]^2 k_1(\theta)\, e^{-[k_2(\theta) - k_1(\theta)]\beta}}$$

and

$$c_2(\theta) = -\frac{\lambda\varphi(\theta)[\lambda - k_2(\theta)]k_1(\theta)}{[\lambda - k_1(\theta)]^2 k_2(\theta) e^{[k_2(\theta)-k_1(\theta)]\beta} - [\lambda - k_2(\theta)]^2 k_1(\theta)}.$$

Substituting values of $c_1(\theta)$ and $c_2(\theta)$ in (3.12) we find the $L(\theta|z)$. Applying the total probability formula, we have

$$L(\theta) = \int_{z=0}^{\beta} L(\theta|z) dP\{X(0) < z\} = L(\theta|\beta)\mathrm{P}\{\eta_1^+ > \beta\} - \int_{z=0}^{\beta} L(\theta|\beta) d\mathrm{P}\{\eta_1^+ > z\}$$

$$= e^{-\lambda\beta} L(\theta|\beta) + \lambda \int_{z=0}^{\beta} e^{-\lambda z} L(\theta|\beta) dz.$$

Now, we shall find $E[\tau_0]$ and $Var[\tau_0]$. By the definitions of expected value and variance and the property of the Laplace transformation, we know that

$$E[\tau_0] = -L'(0)$$

and

$$Var[\tau_0] = L''(0) - [L'(0)]^2.$$

Therefore, we can write the following expression for the expectation of the random variable τ_0

$$E[\tau_0] = \left\{ \frac{\lambda + \mu}{\lambda - \mu}\left(1 - e^{-\lambda\beta}\right) + \frac{\lambda^2 - \lambda\mu - \mu^2}{(\lambda - \mu)^2} e^{-\lambda\beta} + \frac{\mu^3}{\lambda(\lambda - \mu)^2} e^{-(\lambda-\mu)\beta} + \frac{\lambda\mu}{(\lambda - \mu)^2}\left(e^{-\mu\beta} - 1\right) \right\} E[\xi_1].$$

Also we get the following expression for the variance of τ_0:

$$Var[\tau_0] = \left\{ \frac{\lambda + \mu}{\lambda - \mu} - \left(\frac{\lambda + \mu}{\lambda - \mu} + \frac{\lambda\mu}{\lambda - \mu}\beta \right) e^{-\lambda\beta} + \left[\frac{\lambda^2 - \mu(\lambda + \mu)}{(\lambda - \mu)^2} + \frac{\lambda\mu\beta}{\lambda - \mu} \right] e^{-\lambda\beta} - \frac{\mu}{\lambda(\lambda - \mu)^2}\left[\lambda^2 - 2\mu^2 e^{-\lambda\beta}\right] e^{-(\lambda-\mu)\beta} + \frac{\mu(\lambda^2 + \mu^2)}{\lambda(\lambda - \mu)^2} e^{-\lambda\beta} \right\} Var[\xi_1]$$

$$+ \left\{ \left[\frac{\lambda^4 + \mu^4 - \lambda\mu(\lambda - \mu)^2}{(\lambda - \mu)^4} + \frac{\lambda\mu(3\lambda + \mu)}{(\lambda - \mu)^2}\beta + \frac{(\lambda\mu)^2}{(\lambda - \mu)^2}\beta^2 \right] e^{-\lambda\beta} \right.$$

$$-\frac{\lambda^3-\mu^3+\lambda\mu(3\lambda+\mu)}{(\lambda-\mu)^3}\left(e^{-\lambda\beta}-1\right)-\frac{(\lambda\mu)^2}{(\lambda-\mu)^2}\beta^2 e^{-\lambda\beta}$$

$$-\frac{\lambda\mu\left(3\lambda^2-\mu^2\right)}{(\lambda-\mu)^2}\beta e^{-\lambda\beta}-\frac{2\mu^4}{(\lambda-\mu)^3}\beta.e^{-\mu\beta}+\frac{2\mu^6}{\lambda^2(\lambda-\mu)^4}e^{-2(\lambda-\mu)\beta}$$

$$-\left[\frac{\lambda\mu(3\lambda+\mu)}{(\lambda-\mu)^3}+\frac{4(\lambda\mu)^2}{(\lambda-\mu)^3}\beta\right]\left[e^{-(\lambda-\mu)\beta}-e^{-\lambda\beta}\right]-\frac{2\lambda\mu^2}{(\lambda-\mu)^3}\beta e^{-\lambda\beta}$$

$$+\left[\frac{\mu^3\left(7\lambda^2-5\mu^2\right)}{\lambda(\lambda-\mu)^4}+\frac{4\mu^4\beta}{(\lambda-\mu)^3}\right]e^{-(\lambda-\mu)\beta}+\frac{10\mu^4\beta}{(\lambda-\mu)^3}e^{-(2\lambda-\mu)\beta}$$

$$-\left[\frac{\lambda+\mu}{\lambda-\mu}\left(1-e^{-\lambda\beta}\right)-\frac{\lambda^2-\lambda\mu-\mu^2}{(\lambda-\mu)^2}e^{-\lambda\beta}\right.$$

$$\left.\left.-\frac{\mu^3 e^{-(\lambda-\mu)\beta}}{\lambda(\lambda-\mu)^2}-\frac{\lambda\,\mu\, e^{-(\lambda-\mu)\beta}}{(\lambda-\mu)^2}\left(e^{-\mu\beta}-1\right)\right]^2\right\}[E[\xi_1]]^2.$$

In this expressions, by limiting as $\beta\to\infty$ we get

$$E[\tau_0]=\frac{\lambda+\mu}{\lambda-\mu}E[\xi_1]$$

and

$$Var[\tau_0]=\frac{\lambda+\mu}{\lambda-\mu}Var[\xi_1]+\frac{2\lambda\mu(\lambda+\mu)}{(\lambda-\mu)^3}[E[\xi_1]]^2.$$

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