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Operator (h,m)-Convexity and Hermite-Hadamard Type Inequalities

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Abstract

In this paper, firstly we define a new class of operator convex functions in Hilbert space, as operator (h,m)-convex functions. Later we give some properties of this class. Finally, we obtain some new inequalities via Hermite-Hadamard inequality for operator (h,m)-convex function.

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***Keywords*:** Hermite-Hadamard inequality, Operator convex, Operator m-convex, Operator h-convex, Operator (h,m)-convex, Hilbert space.

Operatör (h,m)-Konvekslik ve Hermite-Hadamard Tipi Eşitsizlikler

Öz

Bu çalışmada, ilk olarak Hilbert uzayında yeni bir operatör konveks sınıfı olan "operatör (h,m)-konveks fonksiyonlar sınıfı" tanıtıldı. Daha sonra bu sınıfın bazı özellikleri verildi. Son olarak ise operatör (h,m)-konveks fonksiyonlar için Hermite-Hadamard tipi eşitsizlikler yardımıyla bazı yeni eşitsizlikler elde edildi.

2010 MSC: 26D07, 26D15, 47B65

***Anahtar Kelimeler*:** Hermite-Hadamard tipi eşitsizlik, operatör konveks, operatör m-konveks, operatör h-konveks, operatör (h,m)-konveks, Hilbert uzayı.

1. Introduction and Preliminary

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I and $a, b \in I, a < b$. Then, the following double inequality is well known in literature as Hermite-Hadamard inequality holds for convex functions:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \qquad (1.1)$$

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Matloka (2013) studied classical meaning of (h, m)-convexity, and established several Hermite-Hadamard type inequalities for (h, m)-convex functions. But Özdemir et al.(2011) defined the (h, m)-convex functions. In 2016, Özdemir et al.(2016) published their article. They gave a new class of convex functions as a generalization of convexity which is called (h, m)-convex functions and some properties of this class. Afterwards, they also proved some Hadamard's type inequalities.

Dragomir (2011) obtained some Hermite-Hadamard's type inequalities for operator convex functions of selfadjoint operators in Hilbert spaces and satisfied applications for particular cases of interest.

Salaş et al. (2015) and Erdaş et al. (2015) defined operator p-convex function and operator m-convex function, respectively. Then they obtained some new properties and inequalities of these classes.

Taghavi et al. (2015) introduced the concept of operator h-convex functions for positive linear maps and proved some Hermite-Hadamard's type inequalities for these functions. Later, they gave some applications. Namely, they established several singular value and trace inequalities for operators.

Cortez et al. (2017) introduced the nation of operator h-convex function. Besides, they obtained new Jensen and Hermite-Hadamard inequalities for these operator h-convex functions in Hilbert spaces.

In this paper, firstly we will define operator meaning of (h, m)-convexity. Secondly, we will obtained some properties of operator (h, m)-convex, then finally we will established new inequalities via Hermite-Hadamard inequality for this operator class.

Definition 1.1 (Hudzik et al 1994)

A function $f: \mathbb{R}^+ \to \mathbb{R}$ where $\mathbb{R}^+ = [0, +\infty)$, is said to be s-convex in the second sense if

$$f(\lambda x + (1-\lambda)y) = \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in [0, \infty)$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

Definition 1.2 (Toader 1985)

The function $f: [0, \mathrm{b}] \to \mathbb{R}$, $b > 0$, is said to be m -convex, where $m \in [0,1]$, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0,1]$.

Definition 1.3 (Varośanec 2007)

Let $h: J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function, $h \not\equiv 0$. We say that $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex function or that f belongs to the class SX(h,I), if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

Along this paper, we will use the following symbols;

$<.,.>$ is an inner product,

H is a Hilbert space,

$\mathrm{B(H)} := \{\mathrm{A/A}: H \to H$ bounded, linear operator$\}$,

$B(H)^+$ is all positive operators in B(H),

K is a convex subset of $B(H)^+$

$\rho(A) := \{\lambda \in C: (A - \lambda E)^{-1} \in L(X)\}$

$Sp(A) := C \setminus \rho(A)$,

for $A, B \in K$, $[\mathrm{A,B}] := \{(1-t)A + tB: t \in [0,1]\}$

We review the operator order in B(H) and continuous functional calculus for a bounded selfadjoint operator. For selfadjoint $A, B \in B(H)$ we write, for every $x \epsilon H$,

$$A \leq B \ if \ < Ax, x > \leq < Bx, x >.$$

If A is bounded selfadjoint operator and f is real valued continuous function on Sp(A), , then $f(t) \geq 0$ for any $t \in \mathrm{Sp(A)}$ implies that $f(A) \geq 0$, i.e f(A) is a positive operator or H. Moveover, if both f and g are real valued functions on Sp(A) such that $f(t) \leq g(t)$ for any $t \in \mathrm{Sp(A)}$, then $f(A) \leq g(A)$ in the operator order B(H),

Now we give definitions of operator convex, operator m-convex and operator h-convex functions.

Definition 1.4 (Dragomir 2011)

A real valued continuous function f , $f: I \subseteq \mathbb{R} \to \mathbb{R}$, is said to be operator convex, if

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B)$$

in the operator order in B(H), for all $\mathrm{t} \in [0,1]$ and for every bounded self-adjoint operators A and B in B(H) whose spectras are contained in I.

Definition 1.5 (Erdaş et al 2015)

Let I be an interval in $\mathbb{R}$ and K be convex subset of $B(H)^{+}$. A continuous function $f: I \subset [0, \infty) \to \mathbb{R}$ is said to be operator m-convex function on I for operators in K, if the following inequality is satisfied,

$$f(tA + m(1-t)B) \leq tf(A) + m(1-t)f(B)$$

where m, t $\in [0,1]$ and for every positive operators A and B in K whose spectra are contained in I.

Definition 1.6 (Taghavi et al 2015)

Let I and J be intervals in $\mathbb{R}$ and K be a convex subset of $B(H)^{+}$. A continuous function $f: I \to \mathbb{R}$ is said to be operator h-convex on I for operators in K if

$$f(tA + (1-t)B) \leq h(t)f(A) + h(1-t)f(B)$$

in the operator order in B(H), for all t $\in [0,1]$ and for every positive operators A and B in K whose spectrals are contained in I. $h: J \to \mathbb{R}$ be a non-negative function , if f is an operator h-convex function, we will show as $f \in ES_h O$.

The following definition is given by M. Emin Özdemir et al. (2016) in classical meaning.

Definition 1.7 (Özdemir et al 2016)

Let $h: J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $f: [0, \mathrm{b}] \to \mathbb{R}$ is a (h,m)-convex function, if f is non-negative and for all $x, y \in [0, \mathrm{b}]$, m $\in [0,1]$ and $\alpha \in (0,1)$, we have

$$f(\alpha\, x + m(1 - \alpha)y) \leq h(\alpha)f(x) + mh(1 - \alpha)f(y).$$

If the above inequality is reversed, then f is said to be (h,m)-concave function on [0,b].

2. Main Results

Class of Operator (h,m)-Convexity in Hilbert Space: $\mathrm{ESD}_{(h,m)}\mathrm{O}$

The following Definition 2.1 is firstly defined in here. Let [0,b] and J be intervals in $\mathbb{R}$ and K be a convex subset of $B(H)^{+}$.

Definiton 2.1

Let $h: J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $f: [0, \mathrm{b}] \to \mathbb{R}$ is an operator (h,m)- convex function on [0,b], if f is non-negative and for every positive operators A and B in K whose spectral are contained in I, m ∈ [0,1] and $\alpha \in (0,1)$, then the following inequality holds,

$$f(\alpha A + m(1-\alpha)B) \leq h(\alpha)f(A) + mh(1-\alpha)f(B) \qquad (2.1)$$

If the inequality (2.1) is reversed, then f is said to be operator (h,m)-concave function.

Remark 2.1

If f is an operator *(h,m)*-convex, this class will be shown by $f \in ESD_{(h,m)}\mathrm{O}$.

Theorem 2.1

Let $f: [A, B] \subset [0, \infty) \to (0, \infty)$ be an operator (h,m)-convex function on [A,B] with A <mB. Then the following inequality holds;

$$\frac{1}{mB-A}\int_A^{mB} f(x)dx \leq (f(A) + mf(B))\int_0^1 h(t)dt \qquad (2.2)$$

for m ∈ (0,1] , $h: j \subset \mathbb{R} \to \mathbb{R}$ is a non-negative function.

Proof :

For $x \in H$ with $\| x \| = 1$ and t ∈ [0,1] , we have

$$< \big((1-t)A + tmb\big)x, x >= (1-t) < Ax, x > +tm < Bx, x > \in [A, B], \qquad (2.3)$$

since $< Ax, x > \in Sp(A) \subset [A, B]$ and $< Bx, x > \in Sp(B) \subset [A, B]$. Continuity of f and (2.3) imply that the following integral exists

$$\int_0^1 f(tA + m(1-t)B)dt$$

Since f is an operator (h,m)-convex function and h>0, we can write

$$f(tA + m(1-t)B) \leq h(t)f(A) + mh(1-t)f(B)$$

for all t ∈ [0,1] and m ∈ (0,1] It is easy to observe that

$$\frac{1}{mB-A}\int_A^{mB} f(x)dx = \int_0^1 f(tA + m(1-t)B)dt$$

$$\leq \int_0^1 h(t)f(A) + mh(1-t)f(B)dt.$$

Using the Minkowski Inequality we get

$$\int_0^1 h(t)f(A) + mh(1-t)f(B)dt$$

$$\leq \int_0^1 h(t)f(A)dt$$

$$+ \int_0^1 mh(1-t)f(B)dt = [f(A) + mf(B)] \int_0^1 h(t)dt.$$

Thus

$$\frac{1}{mB-A}\int_A^{mB} f(x)dx \leq [f(A) + mf(B)] \int_0^1 h(t)dt$$

The proof is completed.

Remark 2.2

In (2.2), if we choose operator $m = 1$ and $\mathrm{h(t) = t}$, then we have the following inequality

$$\frac{1}{B-A}\int_A^B f(x)dx \leq \frac{f(A) + f(B)}{2}$$

i.e, obtain the right hand side of Hermite-Hadamard inequality for operator convex function.

Remark 2.3

In (2.2) , if we choose $m = 1$ and $h(t) = t^s, s \in [0,1]$, then we obtain the right hand of operator s-convex function in the second sense in theorem (2.4) in (Ghazanfari 2014).

Theorem 2.2

Let $f: I \subseteq [0,\infty) \to \mathbb{R}$ be a operator (h,m)-convex function with $\mathrm{m} \in (0,1]$. If $0 \leq A < B < \infty$ and $f \in L[A, \frac{B}{m}]$, then the following inequality holds;

$$f(\tfrac{A+B}{2}) \leq \frac{h\left(\frac{1}{2}\right)}{B-A}\int_A^B \left[f(x) + mf\left(\frac{x}{m}\right)\right] dx \leq h(\tfrac{1}{2})[\frac{f(A)+mf\left(\frac{B}{m}\right)+mf\left(\frac{A}{m}\right)+m^2 f\left(\frac{B}{m^2}\right)}{2}] \qquad (2.4)$$

Proof.

For $x \in H$ with $\| x \| = 1$ and $t \in [0,1]$, we have

$$< \big((1-t)A + tmB\big)x, x >= (1-t) < Ax, x > +tm < Bx, x > \in [0,\infty) \qquad (2.5)$$

since $< Ax, x > \in Sp(A) \subseteq [0,\infty)$ and $< Bx, x > \in Sp(B) \subseteq [0,\infty)$. Continuity of f and (2.5) imply that the following integral

$$\int_0^1 f(tA + m(1-t)B)dt,$$

exists.

Since f is an operator (h,m)-convex function and $h > 0$, we can write

$$f(tA + m(1-t)B) \leq h(t)f(A) + mh(1-t)f(B)$$

for all $t \in [0,1]$ and $m \in (0,1]$. For $X, Y \in [0,\infty)$ and $t = \frac{1}{2}$, we can write definition of operator (h,m)-convex function as follows

$$f\left(\frac{X+Y}{2}\right) \leq h\left(\frac{1}{2}\right) f(X) + mh(\tfrac{1}{2})\mathrm{f}(\tfrac{Y}{m})$$

If we choose $X = tA + (1-t)B$ and $Y = tB + (1-t)A$, we get

$$f\left(\frac{A+B}{2}\right) \leq h\left(\frac{1}{2}\right) f(tA + (1-t)B) + mh(\tfrac{1}{2}) f((1-t)\tfrac{A}{m} t\ \tfrac{B}{m}) \qquad (2.6)$$

for all $t \in [0,1]$. By integrating over $[0,1]$ with respect to t, we have the following inequality

$$f\left(\frac{A+B}{2}\right) \leq h\left(\frac{1}{2}\right) \int_0^1 f(\mathrm{tA} + (1-\mathrm{t})\mathrm{B})\, dt + mh(\tfrac{1}{2}) \int_0^1 \mathrm{f}((1-\mathrm{t})\tfrac{A}{m} + t\ \tfrac{B}{m})\mathrm{dt}$$

By the facts that

$$\int_0^1 f(\mathrm{tA} + (1-\mathrm{t})\mathrm{B})\, dt = \frac{1}{B-A}\int_A^B f(x)dx$$

and

$$\int_0^1 \mathrm{f}((1-\mathrm{t})\frac{A}{m} + t\ \frac{B}{m})\, dt = \frac{m}{B-A}\int_{\frac{A}{m}}^{\frac{B}{m}} f(x)dx = \frac{1}{B-A}\int_A^B f(\frac{x}{m})dx$$

Using these inequalities in (2.6), we obtain the first inequality of (2.4). By the operator (h,m)-convex of f, we can write

$$h\left(\frac{1}{2}\right)[f(\mathrm{tA} + (1-\mathrm{t})\mathrm{B} + \mathrm{mf}((1-\mathrm{t})\tfrac{A}{m} + t\ \tfrac{B}{m})] \leq h\left(\frac{1}{2}\right)[tf(A) + m(1-t)f\left(\frac{B}{m}\right) +$$
$$m(1-t)f\left(\frac{A}{m}\right) + m^2tf(\frac{B}{m^2})] \qquad (2.7)$$

integrating the inequality (2.7) on [0,1] with respect to t , we have below inequality

$$\frac{h\left(\frac{1}{2}\right)}{B-A}\int_A^B \left[f(x)+mf\left(\frac{x}{m}\right)\right]dx \le h\left(\frac{1}{2}\right)\left[\frac{f(A)+mf\left(\frac{B}{m}\right)+mf\left(\frac{A}{m}\right)+m^2 ff\left(\frac{B}{m^2}\right)}{2}\right]$$

which completes the proof.

Corollary 1

If we choose $h(t) = 1$- in (2.4), we obtain the following inequality ;

$$f\left(\frac{A+B}{2}\right) \le \frac{1}{B-A}\int_A^B [f(x)+mf\left(\frac{x}{m}\right)]dx \le \left[\frac{f(A)+mf\left(\frac{B}{m}\right)+mf\left(\frac{A}{m}\right)+m^2 f\left(\frac{B}{m^2}\right)}{2}\right].$$

Remark 2.4

If we choose $m = 1$ in (2.4), we obtain the right hand side of the inequality Theorem 2.4 in (Taghavi et al 2015).

Remark 2.5

If we choose $m = 1$and $h(t) = t$ in (2.4), we then obtain right hand side of the Hadamard's inequality.

Theorem 2.3

Let $f: I \subset [0,\infty) \to \mathbb{R}$ be a operator (h,m)-convex function with m ∈ (0,1], t ∈ [0,1]. If $0 \le A < mB < \infty$ and $f \in L[A, mB]$, then the following inequality holds;

$$\frac{1}{m+1}\left[\frac{1}{mB-A}\int_A^{mB} f(x)dx + \frac{1}{B-mA}\int_{mA}^{B} f(x)dx\right] \le$$

$$\frac{f(A)+f(B)}{2}\left[\int_0^1 h(t)dt + \int_0^1 h(1-t)dt\right] . \qquad (2.8)$$

Proof.

Due to definition of operator (h,m)-convex function, we can write

$$\mathrm{f(tA + m(1 - t)B) \leq\ h(t)f(A) + mh(1 - t)f(B)}$$
$$\mathrm{f((1 - t)A + mtB \leq h(1 - t)f(A) + mh(t)f(B)}$$
$$\mathrm{f(tB + m(1 - t)A) \leq\ h(t)f(B) + mh(1 - t)f(A)}$$

and

$$\mathrm{f((1 - t)B + mtA)\ \leq h(1 - t)f(B) + mh(t)f(A)}$$

for all $\mathrm{t} \in [0,1]$. By summing these inequalities and integrating on $[0,1]$ with respect to t, we obtain below inequality

$$\int_0^1 f(\mathrm{tA + m(1-t)B})\,dt + \int_0^1 f(\mathrm{(1-t)A + mtB})\,dt$$
$$+ \int_0^1 f(\mathrm{tB + m(1-t)A})\,dt + \int_0^1 f(\mathrm{(1-t)B + mtA})\,dt \tag{2.9}$$
$$\leq \big(f(A) + f(B)\big)(m+1)\left[\int_0^1 h(t)dt + \int_0^1 h(1-t)dt\right].$$

It is easy to show that

$$\int_0^1 f(\mathrm{tA + m(1-t)B})\,dt = \int_0^1 f(\mathrm{(1-t)A + mtB})\,dt = \frac{1}{mB - A}\int_A^{mB} f(x)dx$$

and

$$\int_0^1 \mathrm{f((1-t)B + mtA)\,dt} = \int_0^1 \mathrm{f(tB + m(1-t)A)\,dt} = \frac{1}{\mathrm{B - mA}}\int_{\mathrm{mA}}^{\mathrm{B}} \mathrm{f(x)dx}.$$

By using these equalities in (2.7), we get the desired result.

Corollary 2

If we choose $h(t) = 1$in (2.8), we obtain the following inequality ;

$$\frac{1}{m+1}\left[\frac{1}{mB - A}\int_A^{mB} f(x)dx + \frac{1}{B - mA}\int_{mA}^{B} f(x)dx\right] \leq f(A) + f(B).$$

Remark 2.6

If we choose $m = 1$ in (2.8), we obtain the right hand side of the inequality Theorem 2.4 in (Taghavi et al 2015).

Remark 2.7

If we choose $m = 1$ and $\mathrm{h(t) = t}$ in (2.8), we obtain the right hand side of the inequality Hermite Hadamard.

Remark 2.8

If we choose $m = 1$ and $\mathrm{h(t)} = t^s$ in (2.8), we obtain the right hand side of the inequality Theorem 2.4 in (Ghazanfari 2014).

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