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RESEARCH ARTICLE

Revisiting Gradient Bach Solitons via Maximum Principles

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

Supposing that the Ricci curvature has an appropriate lower bound and applying suitable maximum principles, we establish triviality results which guarantee that a gradient Bach soliton must be trivial and Bach-flat. Our approach is based on three main cores: convergence to zero at infinity, polynomial volume growth (both related to complete noncompact Riemannian manifolds) and stochastic completeness.

Keywords: *Gradient Bach solitons, triviality, Bach-flat, convergence at infinity, polynomial volume growth, stochastic completeness.*
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1. Introduction

Given an n-dimensional Riemannian manifold (M^n, g), with $n \geq 4$, the *Bach tensor*, which was introduced by Bach in [4] to study conformal geometry in early 1920's, is the symmetric and trace-free tensor defined as

$$B_{ij} = \frac{1}{n-3}\nabla^k\nabla^l W_{ikjl} + \frac{1}{n-2}R^{kl}W_{ikjl}, \tag{1.1}$$

where W_{ikjl} is the Weyl tensor, which is given by

$$W_{ikjl} = R_{ikjl} - \frac{1}{n-2}(g_{ij}R_{kl} - g_{il}R_{kj} - g_{kj}R_{il} + g_{kl}R_{ij}) + \frac{R}{(n-1)(n-2)}(g_{ij}g_{kl} - g_{il}g_{kj}).$$

When $n = 4$, the Bach tensor is conformally invariant of weight 1, but it is not conformally invariant in any other dimension (see, for instance, [12]). Moreover, it is not difficult to see that if (M^n, g) is either locally conformally flat (that is, $W_{ikjl} = 0$) or Einstein, then (M^n, g) is *Bach-flat*, which means that $B_{ij} = 0$. From a physics point of view, the Bach tensor B arises in the theory of conformal gravity, where instead of Einstein's equation one arrives at the Bach equation $B_{\mu\nu} = \kappa T_{\mu\nu}$ by varying the Weyl-squared action with respect to the metric tensor. Conformal gravity has been extensively studied; for more details see, for instance, [24, 25].

In 2012, Das and Kar [9] investigated several aspects of a geometric flow defined using the Bach tensor, the so-called *Bach flow*. The equation of Bach flow is given by

$$\frac{\partial}{\partial t}g_{ij} = -B_{ij}. \tag{1.2}$$

Given a smooth vector field X on an n-dimensional Riemannian manifold (M^n, g), with $n \geq 4$, we say that (M^n, g, X) is a *Bach soliton* if it is a self-similar solution of the Bach flow (1.2), which means that it satisfies the equation

$$B_{ij} + \frac{1}{2}\mathcal{L}_X g_{ij} = \lambda g_{ij}, \tag{1.3}$$

where $\mathcal{L}$ denotes the Lie derivative, λ is a constant and B_{ij} is the Bach tensor defined in (1.1).

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Here, we are interested when the vector field X is gradient, that is, $X = \nabla f$ for some smooth *potential function* $f : M^n \to \mathbb{R}$. In this case, (M^n, g, f) is said a *gradient Bach soliton* and equation (1.3) reads as follows

$$B_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}.$$

If the potential function is constant, then the gradient Bach soliton (M^n, g, f) is called *trivial*.

In [20], Ho studied the solitons to the Bach flow and, in particular, he showed that any compact gradient Bach soliton must be Bach-flat. In the noncompact case, Ho proved the existence of non-trivial gradient solitons to the Bach flow. Furthermore, Ho investigated the Bach flow on a four-dimensional Lie group, in which he considered the convergence of the Bach flow. More recently, Shin [29] proved that, under a finite weighted Dirichlet integral condition, any complete noncompact gradient Bach soliton with harmonic Weyl curvature (which means that the divergence of W vanishes) is Bach-flat. In particular, he also studied complete four-dimensional gradient Bach solitons. Afterwards, the first and second authors joint with Mi [7] improved Shin's results by assuming $\mathrm{Ric}(\nabla f, \nabla f) \geq 0$ instead of harmonic Weyl curvature.

Going a step further, here we establish new characterization results which guarantee that a gradient Bach soliton must be trivial and Bach-flat, under the assumption that such a soliton is either complete noncompact or stochastically complete. For this, we also suppose an appropriate lower bound on the Ricci curvature in the direction of ∇f. Our approach is based on three main cores: convergence to zero at infinity, polynomial volume growth (both related to complete noncompact Riemannian manifolds) and stochastic completeness. These cores are motivated by the maximum principles developed in the works of Alías, Caminha and Nascimento [1, 2] and Pigola, Rigoli and Setti [26, 27].

2. Statements of the main results

This section is devoted to quote the statements of our main results according to the analytical machinery that we use to establish them.

2.1. *Via convergence to zero at infinity*

Let (M^n, g) be a complete noncompact Riemannian manifold and let $d(\,\cdot\,, x_0) : M^n \to [0, +\infty)$ denote the Riemannian distance of M^n, measured from a fixed point $x_0 \in M^n$. According to [1], we say that a continuous function $u \in C^0(M)$ *converges to zero at infinity*, when it satisfies the following condition

$$\lim_{d(x,x_0)\to+\infty} u(x) = 0.$$

Taking into this terminology, we obtain the following result:

Theorem 2.1. *Let (M^n, g, f) be a complete noncompact gradient Bach soliton, whose Ricci curvature satisfies $\mathrm{Ric}(\nabla f, \nabla f) \geq 0$. If $|\nabla f|$ converges to zero at infinity, then (M^n, g, f) must be trivial and Bach-flat.*

Let us observe that if the Weyl curvature tensor is harmonic then $\mathrm{Ric}(\nabla f, \cdot) = 0$ (see [29, Lemma 2.1]). Consequently, from Theorem 2.1 we derive the following consequence, also obtained in [8, Corollary 5.3]:

Corollary 2.1. *Let (M^n, g, f) be a complete noncompact gradient Bach soliton with harmonic Weyl curvature. If $|\nabla f|$ converges to zero at infinity, then (M^n, g, f) must be trivial and Bach-flat.*

2.2. *Via polynomial volume growth*

Let (M^n, g) be a connected, oriented, complete noncompact Riemannian manifold. We denote by $B(p, r)$ the geodesic ball centered at p and with radius r. Given a polynomial function $\sigma : (0, +\infty) \to (0, +\infty)$, we say that (M^n, g) has *polynomial volume growth* like $\sigma(t)$ if there exists $p \in M^n$ such that

$$\mathrm{V}(B(p, r)) = \mathcal{O}(\sigma(r)),$$

as $r \to +\infty$, where V denotes the volume related to the metric g. As it was observed in the beginning of [2, Section 2], if $p, q \in M^n$ are at distance d from each other, we can verify that

$$\frac{\mathrm{V}(B(p, r))}{\sigma(r)} \geq \frac{\mathrm{V}(B(q, r - d))}{\sigma(r - d)}.\frac{\sigma(r - d)}{\sigma(r)}.$$

So, the choice of p in the notion of volume growth is immaterial. For this reason, we will just say that (M^n, g) has polynomial volume growth.

Keeping in mind this previous digression, in the next result we obtain Bach-flat metrics by assuming that the norm of the gradient and of the Hessian of the potential function f are bounded, via a maximum principle related to polynomial volume growth due Alías et al. [2, Theorem 2.1.].

Theorem 2.2. *Let (M^n, g, f) be a complete noncompact gradient Bach soliton, whose Ricci curvature satisfies $\mathrm{Ric}(\nabla f, \nabla f) \geq \alpha|\nabla f|^2$, for some positive constant α. If (M^n, g) has polynomial volume growth and $|\nabla f|, |\nabla^2 f| \in L^\infty(M)$, then (M^n, g, f) must be trivial and Bach-flat.*

2.3. Via stochastic completeness

We recall that a (non necessarily complete) Riemannian manifold (M^n, g) is said to be *stochastically complete* if, for some (and, hence, for any) $(x, t) \in M^n \times (0, +\infty)$, the heat kernel $p(x, y, t)$ of the Laplace operator Δ satisfies the conservation property

$$\int_M p(x, y, t) d\mu(y) = 1. \tag{2.1}$$

From the probabilistic viewpoint, stochastic completeness is the property of a stochastic process to have infinite life time. For the Brownian motion on a manifold, the conservation property (2.1) means that the total probability of the particle to be found in the state space is constantly equal to one (see [11, 16, 17, 30]).

A weaker version of the Theorem 2.2 without asking the hypotheses $|\nabla^2 f| \in L^\infty$ and polynomial volume growth can be obtained by assuming stochastic completeness.

Theorem 2.3. *Let (M^n, g, f) be a stochastically complete gradient Bach soliton, whose Ricci curvature satisfies $\mathrm{Ric}(\nabla f, \nabla f) \geq \alpha|\nabla f|^2$, for some positive constant α. If $|\nabla f| \in L^\infty(M)$, then (M^n, g, f) must be trivial and Bach-flat.*

It is not difficult to verify that from [3, Theorem 2.13] (see also [28, Theorem 2.3]) jointly with Theorem 2.3 we get:

Corollary 2.2. *Let (M^n, g, f) be a complete noncompact gradient Bach soliton, whose Ricci curvature satisfies $\mathrm{Ric}(\nabla f, \nabla f) \geq \alpha|\nabla f|^2$, for some positive constant α, and such that $\mathrm{Ric} \geq -G(r)$, for a function $G \in C^1([0, +\infty))$ obeying*

$$G(0) > 0, \quad G' \geq 0 \quad \text{and} \quad G^{-1/2} \notin L^1([0, +\infty)),$$

where r denotes the Riemannian distance function from a fixed origin in M^n. If $|\nabla f| \in L^\infty(M)$, then (M^n, g, f) must be trivial and Bach-flat.

We recall that a (non necessarily complete) Riemannian manifold (M^n, g) is said to be *parabolic* (with respect to the Laplacian operator) if the constant functions are the only subharmonic functions on M^n which are bounded from above, that is, for a function $u \in \mathcal{C}^2(M)$

$$\Delta u \geq 0 \quad \text{and} \quad u \leq u^* < +\infty \quad \text{implies} \quad u = \text{constant}.$$

It is well known that every parabolic Riemannian manifold is stochastically complete (see [17, Corollary 6.4]). Obviously, every closed Riemannian manifold M^n is parabolic, where by closed we mean compact and without boundary. Moreover, there are several interesting geometric conditions which imply the parabolicity of a Riemannian manifold M^n. As it was observed in [19], when M^n is a complete Riemannian manifold, we can state sufficient conditions for parabolicity and stochastic completeness in terms of the volume function $V(r) = V(B(x_0, r))$, where $B(x_0, r)$ is the geodesic ball of radius r centered at a fixed point $x_0 \in M^n$. Namely, the following implications are true:

$$\int_{r_0}^{\infty} \frac{r dr}{V(r)} = +\infty \quad \Rightarrow \quad M^n \text{ is parabolic}, \tag{2.2}$$

$$\int_{r_0}^{\infty} \frac{r dr}{\log V(r)} = +\infty \quad \Rightarrow \quad M^n \text{ is stochastically complete}. \tag{2.3}$$

For instance, $V(r) \leq Cr^2$ and $V(r) \leq \exp(Cr^2)$ will imply the volume conditions in (2.2) and (2.3), respectively. Cheng and Yau in [6] proved that $V(r) \leq Cr^2$ is a sufficient condition for parabolicity. The sharp sufficient condition (2.2) for parabolicity was proved by several authors in [14], [15], [22] and [33]. Several authors [10], [21], [23], [32] showed that $V(r) \leq \exp(Cr^2)$ is a sufficient condition for stochastic completeness (see also an earlier result [13]), and the sharp result (2.3) was obtained in [15] (see [18] and [31] for its extensions). For a model manifold with pole at x_0, both the parabolicity and stochastic completeness can be characterized solely in terms of the function $V(r)$ and its derivative (see [17] and [5]).

Motivated by this previous discussion, we quote the following consequence of Theorem 2.3:

Corollary 2.3. *Let (M^n, g, f) be a parabolic gradient Bach soliton, whose Ricci curvature satisfies $\mathrm{Ric}(\nabla f, \nabla f) \geq \alpha|\nabla f|^2$, for some positive constant α. If $|\nabla f| \in L^\infty(M)$, then (M^n, g, f) must be trivial and Bach-flat.*

3. Proofs of the Theorems

We start recalling a very nice maximum principle at infinity which corresponds to item (a) of [1, Theorem 2.2].

Lemma 3.1. *Let (M^n, g) be a complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(M)$ be a smooth vector field on M^n. Assume that there exists a nonnegative, non-identically vanishing function $u \in C^\infty(M)$ which converges to zero at infinity and such that $g(\nabla u, X) \geq 0$. If $\mathrm{div}X \geq 0$ on M^n, then $g(\nabla u, X) \equiv 0$ on M^n.*

Now, we are in position to prove our first result.

Proof of Theorem 2.1. Let us assume, by contradiction, that the potential function f is not constant. In this case, we can consider the function $u := |\nabla f|^2$, which is nonnegative, non-identically vanishing and converges to zero at infinity. Also let us consider the smooth vector field $X := \nabla|\nabla f|^2$ on M^n. Then

$$g(\nabla u, X) = |\nabla|\nabla f|^2|^2 \geq 0.$$

Moreover, since the Bach tensor B is trace free and $\Delta f = \lambda n$, from Bochner's formula we have that

$$\frac{1}{2}\mathrm{div}X = \frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + \mathrm{Ric}(\nabla f, \nabla f) \geq 0.$$

Thus, from Lemma 3.1 we obtain

$$|\nabla|\nabla f|^2|^2 = g(\nabla u, X) \equiv 0.$$

Hence, $u = |\nabla f|^2$ is a constant and, since it converges to zero at infinity, we conclude that u vanishes identically, giving us a contradiction. Therefore, (M^n, g, f) must be trivial and, from structural equation $\lambda = 0$, (M^n, g, f) is Bach-flat. □

In order to prove our next results, we quote the following key lemma which corresponds to a particular case of a more general maximum principle due to Alías, Caminha and Nascimento (see [2, Theorem 2.1]).

Lemma 3.2. *Let M^n be a connected, oriented, complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(M)$ be a bounded smooth vector field on M^n. Let $u \in C^\infty(M)$ be a nonnegative smooth function such that $g(\nabla u, X) \geq 0$ and $\mathrm{div}X \geq au$ on M^n, for some positive constant $a \in \mathbb{R}$. If M^n has polynomial volume growth, then u vanishes identically on M^n.*

We use Lemma 3.2 to prove Theorem 2.2 as follows.

Proof of Theorem 2.2. Assuming that (M^n, g, f) is not trivial, we can consider the nonnegative smooth function $u := |\nabla f|^2$ and the smooth vector field $X := \nabla|\nabla f|^2$. The structural equation together Bochner's formula give us

$$\frac{1}{2}\mathrm{div}X = \frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + \mathrm{Ric}(\nabla f, \nabla f) \geq \alpha u,$$

and

$$g(\nabla u, X) = |\nabla|\nabla f|^2|^2 \geq 0.$$

Moreover, from Kato's inequality we get

$$|X| = 2|\nabla f||\nabla|\nabla f|| \leq 2|\nabla f||\nabla^2 f| \ll +\infty,$$

since $|\nabla f|, |\nabla^2 f| \in L^\infty(M)$.

Hence, since (M^n, g) has polynomial volume growth, we can apply Lemma 3.2 to conclude that u vanishes identically on M^n. Therefore, f is constant on M^n, which corresponds to a contradiction with the non-triviality of (M^n, g, f). Therefore, we have that (M^n, g, f) must be trivial with $\lambda = 0$ and, consequently, (M^n, g, f) is Bach-flat. □

Pigola, Rigoli and Setti showed that stochastic completeness turns out to be equivalent to the validity of a weak form of the Omori-Yau maximum principle (see [26, Theorem 1.1] and [27, Theorem 3.1]), as it is expressed below.

Lemma 3.3. *A Riemannian manifold M^n is stochastically complete if, and only if, for every $u \in C^2(M)$ satisfying $\sup_M u < +\infty$ there exists a sequence of points $\{p_k\} \subset M^n$ such that*

$$\lim_k u(p_k) = \sup_M u \qquad \text{and} \qquad \limsup_k \Delta u(p_k) \le 0.$$

In what follows, we will apply Lemma 3.3 to prove our last result.

Proof of Theorem 2.3. From Bochner's formula jointly with Kato's inequality we have that

$$\begin{aligned} |\nabla f|\Delta|\nabla f| + |\nabla|\nabla f||^2 &= \frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + \mathrm{Ric}(\nabla f, \nabla f) \\ &\ge |\nabla|\nabla f||^2 + \alpha|\nabla f|^2, \end{aligned}$$

implying that

$$\Delta|\nabla f| \ge \alpha|\nabla f|.$$

Now, suppose by contradiction that $\sup_M |\nabla f| > 0$. Since $|\nabla f| \in L^\infty(M)$ and M^n is stochastically complete, Lemma 3.3 issues the existence of a sequence of points $\{p_k\} \subset M^n$ such that

$$0 \ge \limsup_k \Delta|\nabla f|(p_k) \ge \alpha \sup_M |\nabla f| > 0,$$

and we reach at a contradiction. This concludes the proof of the theorem. □

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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