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RESEARCH ARTICLE

# New Results on Derivatives of the Shape Operator of Real Hypersurfaces in the Complex Quadric

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

**A real hypersurface $M$ in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ inherits an almost contact metric structure . This structure allows to define, for any nonnull real number $k$, the so called $k$-th generalized Tanaka-Webster connection on $M$ , $\hat{\nabla}^{(k)}$. If $\nabla$ denotes the Levi-Civita connection on $M$ , we introduce the concepts of $(\hat{\nabla}^{(k)}, \nabla)$-Codazzi and $(\hat{\nabla}^{(k)}, \nabla)$-Killing shape operator $S$ of the real hypersurface and classify real hypersurfaces in $Q^m$ satisfying any of these conditions.**

***Keywords:*** *Complex quadric, real hypersurface, shape operator, k-th generalized Tanaka-Webster connection, Cho operators.*
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## 1. Introduction

Suppose that $(\tilde{M}, J, g)$ is a Kähler manifold and $M$ a real hypersurface of $\tilde{M}$, that is, a submanifold of codimension 1 with local normal unit vector field $N$. The Kähler structure $(J, g)$ induces on $M$ an almost constant metric structure $(\phi, \eta, g, \xi)$. Let $\nabla$ be the Levi-Civita connection on $M$ and $S$ the shape operator associated to $N$.

Given such an almost contact metric structure, if $k$ is a nonnull real number we can define the so called $k$-th generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$ on $M$ by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi SX, Y)\xi - \eta(Y)\phi SX - k\eta(X)\phi Y$$

for any $X, Y$ tangent to $M$ (see [3]). Let us denote by $F^{(k)}_X Y = g(\phi SX, Y)\xi - \eta(Y)\phi SX - k\eta(X)\phi Y$, for any $X, Y$ tangent to $M$ and call it the $k$-th Cho operator on $M$ associated to $X$. Notice that if $X \in \mathcal{C}$, the maximal holomorphic distribution on $M$ given by all the vector fields orthogonal to $\xi$, the associated Cho operator does not depend on $k$ and we will denote it simply by $F_X$. If $L$ is a tensor field of type (1,1) on $M$ we sill say that $L$ is $(\hat{\nabla}^{(k)}, \nabla)$-parallel if $\nabla_X L = \hat{\nabla}^{(k)}_X L$ for any vector field $X$ tangent to $M$ . $\hat{\nabla}^{(k)}_X L = \nabla_X L$ for a vector field $X$ tangent to $M$ if and only if $F^{(k)}_X L = LF^{(k)}_X$, that is, the eigenspaces of $L$ are preserved by $F^{(k)}_X$. Let us call $L^{(k)}_F$ to the tensor of type (1,2) on $M$ given by $L^{(k)}_F(X, Y) = [F^{(k)}_X, L](Y) = F^{(k)}_X L(Y) - LF^{(k)}_X(Y)$, for any $X, Y$ tangent to $M$ . Then $L$ is $(\hat{\nabla}^{(k)}, \nabla)$-parallel if and only if the tensor $L^{(k)}_F$ vanishes identically.

In this paper we will consider real hypersurfaces in the complex quadric. The complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ is a compact Hermitian symmetric space of rank 2. $Q^m$ has a Kählerian structure $(J, g)$ and a parallel rank two vector bundle $\mathfrak{A}$ which contains an $S^1$-bundle of real structures, that is, complex conjugations $A$ on the tangent spaces of $Q^m$ satisfying $A^2 = I$ and $AJ = -JA$. This determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle $TM$ for a real hypersurface $M$ in $Q^m$. A nonzero tangent vector $W$ at a point of $Q^m$ is called *singular* if it is tangent to more than one maximal flat in $Q^m$ . There are two types of singular tangent vectors for $Q^m$

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- If there exists a conjugation $A \in \mathfrak{A}$ such that $W$ satisfies $AW = W$, $W$ is singular and is called $\mathfrak{A}$-singular.
- If there exists a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y$ such that $AX = X$, $AY = Y$, with $W/\|W\| = (X + JY)/\sqrt{2}$, $W$ is singular and called $\mathfrak{A}$-isotropic.

The study of real hypersurfaces $M$ in $Q^m$ was initiated by Berndt and Suh in [1]. In this paper the geometric properties of real hypersurfaces $M$ in complex quadric $Q^m$ , which are tubes of radius $r$, $0 < r < \pi/2$, around the totally geodesic $\mathbb{C}P^n$ in $Q^m$ , when $m = 2n$ or tubes of radius $r$, $0 < r < \pi/2\sqrt{2}$, around the totally geodesic $Q^{m-1}$ in $Q^m$ , are presented. The condition of isometric Reeb flow is equivalent to the commuting condition of the shape operator $S$ with the structure tensor $\phi$ of $M$ . The classification of such real hypersurfaces in $Q^m$ is obtained in [2].

A real hypersurface $M$ in $Q^m$ is called Hopf if its Reeb vector field $\xi$ is an eigenvector for $S$.

We will denote by $\mathcal{C}$ the maximal holomorphic distribution on $M$ , $\mathcal{C} = \{X \in TM | g(X, \xi) = 0\}$. The distribution $\mathcal{C}$ is said to be integrable if $[X, Y] \in \mathcal{C}$ for any vector fields $X, Y \in \mathcal{C}$. We say that $M$ is ruled if $\mathcal{C}$ is integrable and its integral manifolds are totally geodesic $Q^{m-1}$ in $Q^m$ . This is equivalent to have $g(SX, Y) = 0$ for any $X, Y \in \mathcal{C}$ (see [4] for examples of ruled real hypersurfaces).

We will say that a tensor field $L$ of type (1,1) on $M$ is $(\hat{\nabla}^{(k)}, \nabla)$-Codazzi if it satisfies

$$(\hat{\nabla}^{(k)}_X L)Y - (\hat{\nabla}^{(k)}_Y L)X = (\nabla_X L)Y - (\nabla_Y L)X$$

for any $X, Y$ tangent to $M$ . Its easy to see that this condition is equivalent to $L^{(k)}_F$ being symmetric, that is, $L^{(k)}_F(X, Y) = L^{(k)}_F(Y, X)$ for any $X, Y$ tangent to $M$ . This condition generalizes the concept of $L$ being $(\hat{\nabla}^{(k)}, \nabla)$-parallel.

In the particular case of $L = S$, in [8] we proved non-existence of real hypersurfaces in $Q^m$ whose shape operator is $(\hat{\nabla}^{(k)}, \nabla)$-parallel, for any nonnull real number $k$. In this paper we will study real hypersurfaces in $Q^m$ for wich $S^{(k)}_F$ is symmetric, that is $S^{(k)}_F(X, Y) = S^{(k)}_F(Y, X)$, either if $X, Y \in \mathcal{C}$ or if $X = \xi$, $Y \in \mathcal{C}$, in the following

**Theorem 1.1.** *Let $M$ be a real hypersurface in $Q^m$ , $m \geq 3$. Then $S^{(k)}_F(X, Y) = S^{(k)}_F(Y, X)$ for some nonnull real number $k$ and any $X, Y \in \mathcal{C}$ if and only if $M$ is locally congruent to either a ruled real hypersurface or to a non Hopf non ruled real hypersurface with four distinct principal curvatures, $0$ with multiplicity $2m-4$, $\frac{\alpha}{4}$ with multiplicity $1$, $\frac{\alpha}{4} + \sqrt{\beta^2 + \frac{9\alpha^2}{16}}$ and $\frac{\alpha}{4} - \sqrt{\beta^2 + \frac{9\alpha^2}{16}}$, each with multiplicity $1$, where $\alpha$ and $\beta$ are nonvanishing functions.*

**Theorem 1.2.** *Let $M$ be a real hypersurface of $Q^m$ , $m \geq 3$. Then $S^{(k)}_F(\xi, Y) = S^{(k)}_F(Y, \xi)$ for some nonnull real number $k$ and any $Y \in \mathcal{C}$ if and only if $M$ is locally congruent to a non Hopf non ruled real hypersurface with, at most, five distinct principal curvatures.*

From both Theorems we can conclude

**Corollary 1.1.** *There does not exist any real hypersurface in $Q^m$ , $m \geq 3$, such that $S^{(k)}_F$ is symmetric, for any nonnull real number $k$.*

We will say that a tensor field $L$ of type (1,1) on $M$ is $(\hat{\nabla}^{(k)}, \nabla)$-Killing if it satisfies

$$(\hat{\nabla}^{(k)}_X L)Y + (\hat{\nabla}^{(k)}_Y L)X = (\nabla_X L)Y + (\nabla_Y L)X$$

for any $X, Y$ tangent to $M$ . This condition also generalizes the condition of $L$ being $(\hat{\nabla}^{(k)}, \nabla)$-parallel and is equivalent to $L^{(k)}_F$ being skewsymmetric, that is, $L^{(k)}_F(X, Y) + L^{(k)}_F(Y, X) = 0$ for any $X, Y$ tangent to $M$ . We will study real hypersurfaces in $Q^m$ such that $S^{(k)}_F$ is skewsymmetric either if $X, Y \in \mathcal{C}$ or if $X = \xi$, $Y \in \mathcal{C}$.

We will prove

**Theorem 1.3.** *Let $M$ be a real hypersurface in $Q^m$ , $m \geq 3$. Then $S^{(k)}_F(X, Y) = -S^{(k)}_F(Y, X)$ for some nonnull real number $k$ and any $X, Y \in \mathcal{C}$ if and only if either*

1. *$M$ is Hopf with $S\xi = 0$ and $N$ is $\mathfrak{A}$-isotropic, or*
2. *$M$ is locally congruent to a tube around $\mathbb{C}P^l$, $m = 2l$, or*

3. *$M$ is locally congruent to a ruled real hypersurface.*

**Theorem 1.4.** *Let $M$ be a real hypersurface in $Q^m$ , $m \geq 3$. Then $S_F^{(k)}(\xi, Y) = -S_F^{(k)}(Y, \xi)$ for some nonnull real number $k$ and any $Y \in \mathcal{C}$ if and only if $M$ is locally congruent to a non Hopf non ruled real hypersurface with, at most, five distinct principal curvatures.*

**Corollary 1.2.** *There does not exist any real hypersurface $M$ in $Q^m$ , $m \geq 3$, such that $S_F^{(k)}$ is skewsymmetric, for any nonnull real number $k$.*

## 2. The space $Q^m$.

For more details in this section we refer to [5], [6], [9], [11], [12], and [13]. The complex quadric $Q^m$ is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $z_1, \ldots, z_{m+2}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip $Q^m$ with the Riemannian metric which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure $(J, g)$ on the complex quadric.

The complex projective space $\mathbb{C}P^{m+1}$ is defined by using the Hopf fibration

$$\pi : S^{2m+3} \to \mathbb{C}P^{m+1}, \quad z \to [z],$$

which is said to be a Riemannian submersion. Then we can consider the following diagram for the complex quadric $Q^m$ :

$$\begin{array}{ccc} \tilde{Q} = \pi^{-1}(Q) & \xrightarrow{\ \tilde{i}\ } & S^{2m+3} \subset \mathbb{C}^{m+2} \\ \downarrow \pi & & \downarrow \pi \\ Q = Q^m & \xrightarrow{\ \ i\ \ } & \mathbb{C}P^{m+1} \end{array}$$

The submanifold $\tilde{Q}$ of codimension 2 in $S^{2m+3}$ is called the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{m+2}$, which is given by

$$\tilde{Q} = \{x + iy \in \mathbb{C}^{m+2} | g(x, x) = g(y, y) = \frac{1}{2} \text{ and } g(x, y) = 0\},$$

where $g(x, y) = \sum_{i=1}^{m+2} x_i y_i$ for any $x = (x_1, \ldots, x_{m+2}), y = (y_1, \ldots, y_{m+2}) \in \mathbb{R}^{m+2}$. Then the tangent space is decomposed as $T_z S^{2m+3} = H_z \oplus F_z$ and $T_z \tilde{Q} = H_z(Q) \oplus F_z(Q)$ at $z = x + iy \in \tilde{Q}$ respectively, where the horizontal subspaces $H_z$ and $H_z(Q)$ are given by $H_z = (\mathbb{C}z)^\perp$ and $H_z(Q) = (\mathbb{C}z \oplus \mathbb{C}\bar{z})^\perp$, and $F_z$ and $F_z(Q)$ are fibers which are isomorphic to each other. Here $H_z(Q)$ is a subspace of $H_z$ of real codimension 2 and orthogonal to the two unit normals $-\bar{z}$ and $-J\bar{z}$. Explicitly, at the point $z = x + iy \in \tilde{Q}$ it can be described as

$$H_z = \{u + iv \in \mathbb{C}^{m+2} | \quad g(x, u) + g(y, v) = 0, \quad g(x, v) = g(y, u)\}$$

and

$$H_z(Q) = \{u + iv \in H_z | \quad g(u, x) = g(u, y) = g(v, x) = g(v, y) = 0\},$$

where $\mathbb{C}^{m+2} = \mathbb{R}^{m+2} \oplus i\mathbb{R}^{m+2}$, and $g(u, x) = \sum_{i=1}^{m+2} u_i x_i$ for any $u = (u_1, \ldots, u_{m+2})$, $x = (x_1, \ldots, x_{m+2}) \in \mathbb{R}^{m+2}$.

These spaces can be naturally projected by the differential map $\pi_*$ as $\pi_* H_z = T_{\pi(z)}\mathbb{C}P^{m+1}$ and $\pi_* H_z(Q) = T_{\pi(z)}Q$ respectively. Thus at the point $\pi(z) = [z]$ the tangent subspace $T_{[z]}Q^m$ becomes a complex subspace of $T_{[z]}\mathbb{C}P^{m+1}$ with complex codimension 1. The unit normal fields $-\pi_*\bar{z}$ and $-\pi_* J\bar{z}$ span the normal space of $Q^m$ in $\mathbb{C}P^m$ at every point (see Reckziegel [9]).

Then let us denote by $A_{\bar{z}}$ the shape operator of $Q^m$ in $\mathbb{C}P^{m+1}$ with respect to the unit normal $\pi_*\bar{z}$. It satisfies $A_{\bar{z}}\pi_* w = \tilde{\nabla}_{\pi_* w}\bar{z} = \pi_*\bar{w}$ for every $w \in H_z(Q)$, where $\tilde{\nabla}$ denotes the covariant derivative of $\mathbb{C}P^{m+1}$ induced by its

Fubini-Study metric. That is, the shape operator $A_{\bar{z}}$ is just a complex conjugation restricted to $T_{[z]}Q^m$. Moreover, it satisfies the following for any $w \in T_{[z]}Q^m$ and any $\lambda \in S^1 \subset \mathbb{C}$

$$\begin{aligned} A^2_{\lambda\bar{z}}w =& A_{\lambda\bar{z}}A_{\lambda\bar{z}}w = A_{\lambda\bar{z}}\lambda\bar{w} \\ =& \lambda A_{\bar{z}}\lambda\bar{w} = \lambda\bar{\nabla}_{\lambda\bar{w}}\bar{z} = \lambda\bar{\lambda}\bar{\bar{w}} \\ =& |\lambda|^2 w = w. \end{aligned}$$

Accordingly, $A^2_{\lambda\bar{z}} = I$ for any $\lambda \in S^1$. So the shape operator $A_{\bar{z}}$ becomes an anti-commuting involution such that $A^2_{\bar{z}} = I$ and $AJ = -JA$ on the complex vector space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \pi_*(\mathbb{R}^{m+2} \cap H_zQ)$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = \pi_*(i\mathbb{R}^{m+2} \cap H_z(Q))$ is the $(-1)$-eigenspace of $A_{\bar{z}}$. That is, $A_{\bar{z}}X = X$ and $A_{\bar{z}}JX = -JX$, respectively, for any $X \in V(A_{\bar{z}})$.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of $Q^m$ can be described in terms of the complex structure $J$ and any complex conjugation $A \in \mathfrak{A}$:

$$\begin{aligned} \bar{R}(X,Y)Z \quad = \quad & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY. \end{aligned}$$

Note that $J$ and each complex conjugation $A$ anti-commute, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

## 3. Real hypersurfaces in $Q^m$ .

Consider a real hypersurface $M$ in $Q^m$ with unit local normal vector field $N$. For any vector field $X$ tangent to $M$ we write

$$JX = \phi X + \eta(X)N \tag{3.1}$$

where $\phi X$ denotes the tangential component of $JX$. $\phi$ defines on $M$ a skew-symmetric tensor field of type (1,1) called the structure tensor. The vector field $\xi = -JN$ is called the Reeb vector field of $M$ . Consider on $M$ the 1-form given by $\eta(X) = g(X,\xi)$ for any vector field $X$ tangent to $M$ . We have that $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $M$ . Therefore we have the following relations

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{3.2}$$

for any $X, Y$ tangent to $M$. From (3.2) we also have

$$\phi\xi = 0, \quad \eta(X) = g(X,\xi).$$

The tangent bundle $TM$ of $M$ splits orthogonally into

$$TM = \mathcal{C} \oplus \mathcal{F},$$

where $\mathcal{C} = ker(\eta) = \{X \in TM | g(X,\xi) = 0\}$ is the maximal complex (holomorphic) subbundle of $TM$ and $\mathcal{F} = \mathbb{R}\xi$. Notice that the structure tensor field $\phi$ restricted to $\mathcal{C}$ coincides with the complex structure $J$.

At each point $z \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of $T_zM$ as

$$\mathcal{Q}_z = \{X \in T_zM | AX \in T_zM, \forall A \in \mathfrak{A}_z\}.$$

Then we have, [2],

**Lemma 3.1.** *Let $M$ be a real hypersurface in $Q^m$ . Then the following are equivalent*

1. *The normal vector $N_z$ of $M$ at $z$ is $\mathfrak{A}$-principal.*
2. $\mathcal{Q}_z = \mathcal{C}_z$.

If the normal vector $N_z$ of $M$ at $z$ is not $\mathfrak{A}$-principal there exists a real structure $A \in \mathfrak{A}_{[z]}$ such that

$$\begin{aligned} N_{[z]} &= \cos(t)Z_1 + \sin(t)JZ_2, \\ AN_{[z]} &= \cos(t)Z_1 - \sin(t)JZ_2, \end{aligned} \tag{3.3}$$

where $Z_1$, $Z_2$ are orthonormal eigenvectors of $\mathfrak{A}$ with eigenvalue 1 and $0 < t \le \frac{\pi}{4}$. As $\xi = -JN$ (3.3) implies

$$\begin{aligned} \xi_{[z]} &= -\cos(t)JZ_1 + \sin(t)Z_2, \\ A\xi_{[z]} &= \cos(t)JZ_1 + \sin(t)Z_2. \end{aligned} \tag{3.4}$$

So we have $g(AN_{[z]}, \xi_{[z]}) = 0$, $g(N_z, AN_z) = cos(2t) = -g(\xi_z, A\xi_z)$.

The shape operator of a real hypersurface $M$ in $Q^m$ is denoted by $S$. The real hypersurface is called *Hopf hypersurface* if the Reeb vector field is an eigenvector of the shape operator, i.e.

$$S\xi = \alpha\xi, \tag{3.5}$$

where $\alpha = g(S\xi, \xi)$ is the Reeb function. The Codazzi equation of $M$ is given by

$$\begin{aligned} &g((\nabla_X S)Y - (\nabla_Y S)X, Z) = \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ &+g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z) \end{aligned} \tag{3.6}$$

for any $X, Y, Z$ tangent to $M$ . To be used later we have, see [2], the following

**Proposition 3.1.** *The following statements hold for a tube M of radius r, $0 < r < \pi/2$ around the totally geodesic $\mathbb{C}P^l$ in $Q^m$ , $m = 2l$:*

1. *M is a Hopf hypersurface.*
2. *The normal bundle of M consists of $\mathfrak{A}$-isotropic singular tangent vectors of $Q^m$ .*
3. *M has four distinct principal curvatures, unless $m = 2$ in which case M has two distinct principal curvatures, which are given in the following matrix*

| *Principal curvature* | *Eigenspace* | *Multiplicity* |
|---|---|---|
| $-2\cot(2r)$ | $\mathcal{F}$ | 1 |
| $-\cot(r)$ | $\nu_z\mathbb{C}P^n \ominus [\xi]$ | $2l-2$ |
| $\tan(r)$ | $T_z\mathbb{C}P^n \ominus [A\xi]$ | $2l-2$ |
| 0 | $[A\xi]$ | 2 |

4. *The shape operator commutes with the structure tensor field $\phi$, i.e. $S\phi = \phi S$.*
5. *M is a homogeneous hypersurface.*

**Proposition 3.2.** *Let $M$ be a Hopf real hypersurface in $Q^m$ , $m \ge 3$. Then the tensor field $2S\phi S - \alpha(\phi S + S\phi)$ leaves $\mathcal{Q}$ and $\mathcal{C} \ominus \mathcal{Q}$ invariant and we have $2S\phi S - \alpha(\phi S + S\phi) = 2\phi$ on $\mathcal{Q}$ and $2S\phi S - \alpha(\phi S + S\phi) = 2\mu^2\phi$ on $\mathcal{C} \ominus \mathcal{Q}$, where $\mu = g(A\xi, \xi) = -cos(2t)$.*

Recently, Lee and Suh, [7], have proved the following

**Proposition 3.3.** *Let $M$ be a Hopf real hypersurface in $Q^m$ , $m \ge 3$. Then $M$ has an $\mathfrak{A}$-principal normal vector field in $Q^m$ if and only if $M$ is locally congruent to a tube of radius $r$, $0 < r < \frac{\pi}{2\sqrt{2}}$ around the $m$-dimensional sphere $S^m$, embedded in $Q^m$ as a real form of $Q^m$ .*

The Reeb function of such a tube is $\alpha = -\sqrt{2}cot(\sqrt{2}r)$.

## 4. Proof of Theorem 1.1

Let $M$ be a real hypersurface in $Q^m$ such that $S_F^{(k)}(X,Y) = S_F^{(k)}(Y,X)$ for any $X,Y \in \mathcal{C}$. This yields

$$g(\phi SX, SY)\xi - \eta(SY)\phi SX - g(\phi SX, Y)S\xi = g(\phi SY, SX)\xi - \eta(SX)\phi SY - g(\phi SY, X)S\xi \tag{4.1}$$

for any $X,Y \in \mathcal{C}$. Suppose first that $M$ is Hopf and write $S\xi = \alpha\xi$. From (4.1) we get $g(\phi SX, SY)\xi - \alpha g(\phi SX, Y)\xi = g(\phi SY, SX)\xi - \alpha g(\phi SY, X)\xi$, for any $X,Y \in \mathcal{C}$. Therefore , for any $X \in \mathcal{C}$ we have

$$2S\phi SX = \alpha(\phi S + S\phi)X. \tag{4.2}$$

As $M$ is Hopf we know from Proposition 3.2 that $2S\phi SX - \alpha(\phi S + S\phi)X = 2\phi X$ for any $X \in \mathcal{Q}$ and $2S\phi SX - \alpha(\phi S + S\phi)X = 4cos^2(2t)\phi X$ for any $X \in \mathcal{C} \ominus \mathcal{Q}$. From (4.2) we have $\mathcal{Q} = 0$ and $m = 2$, which is impossible.

Thus $M$ must be non Hopf. We write $S\xi = \alpha\xi + \beta U$, for a unit $U \in \mathcal{C}$, being $\beta$ a nonvanishing function at least on an open neighborhood of a point of $M$ . The calculations are made on such a neighborhood.

The scalar product of (4.1) and $\phi U$ gives $-\eta(SY)g(SX,U) = -\eta(SX)g(SY,U)$ for any $X,Y \in \mathcal{C}$. Take $Y \in \mathcal{C}$ and orthogonal to $U$. Then we have $\eta(SX)g(SY,U) = 0$ for any $X \in \mathcal{C}$ and such a $Y$. If now $X = U$ we obtain $\beta g(SU,Y) = 0$ for any $Y \in \mathcal{C}$ orthogonal to $U$. This yields

$$SU = \beta\xi + \gamma U \tag{4.3}$$

for a certain function $\gamma$.

The scalar product of (4.1) and $U$ implies

$$\eta(SY)g(SX,\phi U) - \beta g(\phi SX, Y) = \eta(SX)g(SY,\phi U) - \beta g(\phi SY, X) \tag{4.4}$$

for any $X,Y \in \mathcal{C}$.

If we take $X = U$ in (4.4) we get $-\beta g(\phi SU, Y) = \beta(S\phi U, Y) + \beta g(S\phi U, Y)$ for any $Y \in \mathcal{C}$ and, as $\beta \neq 0$ this yields $2S\phi U = -\phi SU = -\gamma\phi U$. Thus

$$S\phi U = -\frac{\gamma}{2}\phi U. \tag{4.5}$$

The scalar product of (4.1) and $\xi$ gives

$$g(\phi SX, SY) - \alpha g(\phi SX, Y) = g(\phi SY, SX) - \alpha g(\phi SY, X) \tag{4.6}$$

for any $X,Y \in \mathcal{C}$. Take $X = U$, $Y = \phi U$ in (4.6). We obtain $2g(\phi SU, S\phi U) - \alpha g(SU,U) = \alpha g(S\phi U, \phi U)$, that is, $-\gamma^2 - \alpha\gamma = -\frac{\alpha\gamma}{2}$, or $\gamma(\gamma + \frac{\alpha}{2}) = 0$. Therefore, either $\gamma = 0$ or $\gamma = -\frac{\alpha}{2}$.

Now we take $X,Y \in \mathcal{C}_U = \{Z \in \mathcal{C} | g(Z,U) = g(Z,\phi U) = 0\}$ in (4.4) and, as $\beta \neq 0$, we get $g(\phi SX, Y) = g(\phi SY, X)$ for any $X,Y \in \mathcal{C}_U$. From (4.3) and (4.5) this yields

$$S\phi X + \phi SX = 0 \tag{4.7}$$

for any $X \in \mathcal{C}_U$. Suppose $X \in \mathcal{C}_U$ is unit and satisfies $SX = \lambda X$, then from (4.7), $S\phi X = -\lambda\phi X$. Moreover, from (4.6) $S\phi SX - \alpha\phi SX = -S\phi SX + \alpha S\phi X$. That is, $2S\phi SX = \alpha(\phi S + S\phi)X = 0$. Therefore, $-2\lambda^2 = 0$. Thus the unique principal curvature on $\mathcal{C}_U$ is $0$. Then if $\gamma = 0$, $M$ is a ruled real hypersurface.

If $\gamma = -\frac{\alpha}{2}$ we have $S\xi = \alpha\xi + \beta U$, $SU = \beta\xi - \frac{\alpha}{2}U$, $S\phi U = \frac{\alpha}{4}\phi U$ and $SX = 0$ for any $X \in \mathcal{C}_U$. In this case, if $\alpha = 0$, $M$ is ruled and minimal. If $\alpha \neq 0$, $M$ is a non Hopf, no ruled real hypersurface with four distinct principal curvatures, $0$ with multiplicity $2m-4$, $\frac{\alpha}{4}$, with multiplicity $1$, $\frac{\alpha}{4} + \sqrt{\beta^2 + \frac{9\alpha^2}{16}}$ and $\frac{\alpha}{4} - \sqrt{\beta^2 + \frac{9\alpha^2}{16}}$, each with multiplicity $1$, finishing the proof.

## 5. Proof of Theorem 1.2

Suppose now that $S_F^{(k)}(\xi,Y) = S_F^{(k)}(Y,\xi)$ for any $Y \in \mathcal{C}$. therefore

$$\begin{array}{c} g(\phi S\xi, SY)\xi - \eta(SY)\phi S\xi - k\phi SY - g(\phi S\xi, Y)S\xi \\ +kS\phi Y = g(\phi SY, S\xi)\xi - \eta(S\xi)\phi SY + S\phi SY \end{array} \tag{5.1}$$

for any $Y \in \mathcal{C}$. Suppose first that $M$ is Hopf with $S\xi = \alpha\xi$. From (5.1) we have $-k\phi SY + kS\phi Y = -\alpha\phi SY + S\phi SY$ for any $Y \in \mathcal{C}$. Thus

$$S\phi SY = \alpha\phi SY + k(S\phi - \phi S)Y \tag{5.2}$$

for any $Y \in \mathcal{C}$. If we take the scalar product of (5.2) and $X \in \mathcal{C}$ and change $X$ and $Y$ we obtain

$$-S\phi SY = -\alpha S\phi Y + k(S\phi - \phi S)Y \tag{5.3}$$

for any $Y \in \mathcal{C}$.

If we substract (5.3) from (5.2) we get $2S\phi SY = \alpha(\phi S + S\phi)Y$ for any $Y \in \mathcal{C}$ and, as in previous Theorem, we arrive to a contradiction. Therefore, $M$ must be non Hopf.

We write as above $S\xi = \alpha\xi + \beta U$. Then (5.1) becomes

$$\begin{array}{c} \beta g S(\phi U, Y)\xi - \beta^2 g(U, Y)\phi U - k\phi SY - \beta g(\phi U, Y)S\xi \\ +kS\phi Y = -\beta g(S\phi U, Y)\xi - \alpha\phi SY + S\phi SY \end{array} \tag{5.4}$$

for any $Y \in \mathcal{C}$. The scalar product of (5.4) with $\xi$, bearing in mind that $\beta \neq 0$, gives $g(S\phi U, Y) - (\alpha + k)g(\phi U, Y) = -2g(S\phi U, Y)$ for any $Y \in \mathcal{C}$. Thus $3S\phi U = (\alpha + k)\phi U$, that is

$$S\phi U = \frac{\alpha + k}{3}\phi U. \tag{5.5}$$

Taking $Y = \phi U$ in (5.4) we obtain $\beta g(S\phi U, \phi U)\xi - k\phi S\phi U - \beta S\xi - kSU = \beta g(\phi S\phi U, U)\xi - \alpha\phi S\phi U + S\phi SU$. From (5.5) this gives $\beta(\frac{\alpha+k}{3})\xi + k(\frac{\alpha+k}{3})U - \alpha\beta\xi - \beta^2 U - kSU = -\beta(\frac{\alpha+k}{3})\xi + \alpha(\frac{\alpha+k}{3})U - (\frac{\alpha+k}{3})SU$, that is, $(2k - \alpha)SU = \beta(2k - \alpha)\xi + (k^2 - \alpha^2 - 3\beta^2)U$. If $\alpha = 2k$, we get $-3k^2 - 3\beta^2 = 0$, which is impossible. Thus $\alpha \neq 2k$ and

$$SU = \beta\xi + \frac{k^2 - \alpha^2 - 3\beta^2}{2k - \alpha}U. \tag{5.6}$$

Taking $Y = U$ in (5.4) we get $\beta^2\phi U - k\phi SU + kS\phi U = -\alpha\phi SU + S\phi SU$. From (5.5) and (5.6) we obtain

$$3\beta^2 - k^2 + 2\alpha^2 + \alpha k = 0. \tag{5.7}$$

From (5.7), $k^2 - \alpha^2 - 3\beta^2 = \alpha^2 + \alpha k$ and we can write (5.6) as

$$SU = \beta\xi + \frac{\alpha^2 + \alpha k}{2k - \alpha}U. \tag{5.8}$$

We also have obtained that $\mathcal{C}_U$ is $S$-invariant. Take $Y \in \mathcal{C}_U$ in (5.4). Then we get

$$-k\phi SY + kS\phi Y = -\alpha\phi SY + S\phi SY \tag{5.9}$$

for any $Y \in \mathcal{C}_U$. Suppose $SY = \lambda Y$. From (5.9) we have $-k\lambda\phi Y + kS\phi Y = -\alpha\lambda\phi Y + \lambda S\phi Y$. That is, $(\lambda - k)S\phi Y = \lambda(\alpha - k)\phi Y$. If $\lambda = k$ we get $k(\alpha - k) = 0$, and as $k \neq 0$, $\alpha = k$. From (5.7) $3\beta^2 + 2k^2 = 0$, which is impossible. Thus $\lambda \neq k$ and

$$S\phi Y = \lambda(\frac{\alpha - k}{\lambda - k})\phi Y \tag{5.10}$$

for any $Y \in \mathcal{C}_U$ such that $SY = \lambda Y$.

If we take $\phi Y$ instead of $Y$ in (5.9) we obtain $-k\phi S\phi Y - kSY = -\alpha\phi S\phi Y + S\phi SY$. As $\lambda \neq k$ this yields $k\lambda(\alpha - \lambda) = (\alpha - k)\lambda(\alpha - \lambda)$. Therefore we can have

- $\alpha = \lambda$. In this case $S\phi Y = \alpha\phi Y$.
- $\alpha \neq \lambda$. As $2k - \alpha \neq 0$, then $\lambda = 0$ and $S\phi Y = 0$.
- $\alpha \neq \lambda$, $\lambda \neq 0$. Then $\alpha = 2k$, which is impossible.

Moreover, if $\alpha = -k$, from (5.7), $3\beta^2 - k^2 + 2\alpha^2 + \alpha k = 3\beta^2 = 0$, which is impossible. This yields that our real hypersurface is not ruled, finishing the proof.

## Remark

The real hypersurfaces appearing in Theorem 1.2 are not the ones appearing in Theorem 1.1, because if $\frac{\alpha}{4} = \frac{\alpha + k}{3}$, we have $\alpha = -4k$. Then from (5.7) we obtain $3\beta^2 + 27k^2 = 0$, which is impossible, proving the first Corollary.

## 6. Proof of Theorem 1.3

Let now $M$ be a real hypersurface satisfying $S_F^{(k)}(X,Y) = -S_F^{(k)}(Y,X)$ for any $X, Y \in \mathcal{C}$. Then we have

$$-\eta(SY)\phi SX - g(\phi SX, Y)S\xi - \eta(SX)\phi SY - g(\phi SY, X)S\xi = 0 \tag{6.1}$$

for any $X, Y \in \mathcal{C}$. Suppose first that $M$ is Hopf and write $S\xi = \alpha\xi$. The scalar product of (6.1) and $\xi$ gives $\alpha g(\phi SX, Y)\xi + \alpha g(\phi SY, X)\xi = 0$. That is, $\alpha(\phi S - S\phi)X = 0$ for any $X \in \mathcal{C}$. Therefore either $\alpha \neq 0$ and then $\phi S = S\phi$, thus $M$ is locally congruent to a tube around $\mathbb{C}P^l$, $m = 2l$, or $\alpha = 0$. In this case take $X \in \mathcal{C}$ such that $SX = \lambda X$. Codazzi equation yields $(\nabla_X S)\xi - (\nabla_\xi S)X = -S\phi SX - \nabla_\xi \lambda X + S\nabla_\xi X$. If we take its scalar product with $\xi$ we obtain $-g(\nabla_\xi \lambda X, \xi) = g(\lambda X, \phi S\xi) = 0 = g(X, AN)g(A\xi, \xi) - g(\xi, AN)g(AX, \xi) + g(X, A\xi)g(JA\xi, \xi) - g(\xi, AN)g(JAX, \xi) = 2g(X, AN)g(A\xi, \xi)$. Thus either $g(A\xi, \xi) = 0$ and $N$ is $\mathfrak{A}$-isotropic or $g(AN, X) = 0$ for any $X \in \mathcal{C}$ and $N$ is $\mathfrak{A}$-principal. In this case $M$ is locally congruent to a tube of radius $r < \frac{\pi}{2\sqrt{2}}$ around $S^m$. But as $\alpha = 0$, $cot(\sqrt{2}r) = 0$ and this yields $r = \frac{\pi}{2\sqrt{2}}$, which is impossible. Therefore $N$ is $\mathfrak{A}$-isotropic.

Suppose now that $M$ is non Hopf and write again $S\xi = \alpha\xi + \beta U$. The scalar product of (6.1) and $\phi U$ implies

$$-\eta(SY)g(SX, U) - \eta(SX)g(SY, U) = 0 \tag{6.2}$$

for any $X, Y \in \mathcal{C}$. Let $Y \in \mathcal{C}$ be orthogonal to $U$ and $X = U$. Then from (6.2) we have $-\beta g(SU, Y) = 0$ for any $Y \in \mathcal{C}$ orthogonal to $U$. Thus

$$SU = \beta\xi + \gamma U \tag{6.3}$$

for a certain function $\gamma$.

Taking $Y = U$ in (6.1) it follows $-\beta\phi SX - g(\phi SX, U)S\xi - \eta(SX)\phi SU - g(\phi SU, X)S\xi = 0$ for any $X \in \mathcal{C}$. Its scalar product with $U$ yields $2\beta g(S\phi U, X) - \beta g(\phi SU, X) = 0$ for any $X \in \mathcal{C}$. As $\beta \neq 0$, this gives $2S\phi U = \phi SU = \gamma\phi U$. Therefore

$$S\phi U = \frac{\gamma}{2}\phi U. \tag{6.4}$$

The scalar product of (6.1) and $\xi$ implies $-\alpha g(\phi SX, Y) - \alpha g(\phi SY, X) = 0$, for any $X, Y \in \mathcal{C}$. If $\alpha \neq 0$ we have $g(\phi SX, Y) + g(\phi SY, X) = 0$ for any $X, Y \in \mathcal{C}$. Taking $X = U, Y = \phi U$, we get $g(\phi SU, \phi U) + g(\phi S\phi U, U) = 0$. That is, $g(SU, U) - g(S\phi U, \phi U) = \gamma - \frac{\gamma}{2} = \frac{\gamma}{2} = 0$. Therefore $\gamma = 0$, and $SU = \beta\xi$, $S\phi U = 0$. Take now $X \in \mathcal{C}_U$, $Y = U$ in (6.1) and obtain $-\beta\phi SX = 0$. As $\beta \neq 0$, $SX = 0$ for any $X \in \mathcal{C}_U$ and $M$ is ruled.

The other possibility is to have $\alpha = 0$. Then $S\xi = \beta U$. Taking $Y = U$, $X \in \mathcal{C}_U$ in (6.1) we also obtain $-\beta\phi SX = 0$. That is, $SX = 0$ for any $X \in \mathcal{C}_U$. Therefore if $\gamma = 0$, we have a minimal ruled real hypersurface. If $\gamma \neq 0$, take $X = Y \in \mathcal{C}$ in (6.2). Then $\eta(SX)g(SX, U) = 0$ for any $X \in \mathcal{C}$. Taking $X = U$ we get $\beta g(SU, U) = 0$. Thus $\gamma = 0$ and we arrive at a contradiction, finishing the proof.

## 7. Proof of Theorem 1.4

If $M$ is a real hypersurface such that $S_F^{(k)}(\xi, Y) = -S_F^{(k)}(Y, \xi)$ for any $Y \in \mathcal{C}$ we get

$$-\eta(SY)\phi S\xi - k\phi SY - g(\phi S\xi, Y)S\xi + kS\phi Y - \eta(S\xi)\phi SY + S\phi SY = 0 \tag{7.1}$$

for any $Y \in \mathcal{C}$. If we suppose that $M$ is Hopf and write $S\xi = \alpha\xi$, from (7.1) we get $-k\phi SY + kS\phi Y - \alpha\phi SY + S\phi SY = 0$ for any $Y \in \mathcal{C}$. That is

$$-(k+\alpha)\phi SY + kS\phi Y + S\phi SY = 0 \tag{7.2}$$

for any $Y \in \mathcal{C}$. If we take the scalar product of (7.2) and $X \in \mathcal{C}$ and change $X$ by $Y$ and $Y$ by $X$ we have

$$(k+\alpha)S\phi Y - k\phi SY - S\phi SY = 0 \tag{7.3}$$

for any $Y \in \mathcal{C}$. Substracting (7.3) from (7.2) we obtain

$$2S\phi SY - \alpha(\phi S + S\phi)Y = 0 \tag{7.4}$$

for any $Y \in \mathcal{C}$. As we have seen above, this yields $m = 2$ and it is impossible.

Therefore we suppose $M$ is non Hopf and write $S\xi = \alpha\xi + \beta U$. Then (7.1) yields

$$-\beta\eta(SY)\phi U - k\phi SY - \beta g(\phi U, Y)S\xi + kS\phi Y - \alpha\phi SY + S\phi SY = 0 \tag{7.5}$$

for any $Y \in \mathcal{C}$. The scalar product of (7.5) and $\xi$ gives $-\alpha\beta g(\phi U, Y) + k\beta g(\phi Y, U) + \beta g(\phi SY, U) = 0$ for any $Y \in \mathcal{C}$ and as $\beta \neq 0$ we get $-(\alpha+k)g(\phi U, Y) - g(S\phi U, Y) = 0$ for any $Y \in \mathcal{C}$. Then

$$S\phi U = -(\alpha+k)\phi U. \tag{7.6}$$

The scalar product of (7.1) and $\phi U$ gives $-\beta\eta(SY) - (k+\alpha)g(SY, U) + kg(\phi Y, S\phi U) + g(\phi SY, S\phi U) = 0$ for any $Y \in \mathcal{C}$, and bearing in mind (7.6) we get $-\beta^2 g(Y, U) - 2(\alpha+k)g(SU, Y) - k(\alpha+k)g(Y, U) = 0$ for any $Y \in \mathcal{C}$. If $\alpha + k = 0$, we should have $\beta = 0$, which is impossible. Therefore

$$\alpha + k \neq 0. \tag{7.7}$$

Moreover, if $Y \in \mathcal{C}_U$ we have $(\alpha+k)g(SU, Y) = 0$ and from (7.7), $g(SU, Y) = 0$ for any $Y \in \mathcal{C}_U$. If $Y = U$, it follows $2(\alpha+k)g(SU, U) = -k(\alpha+k) - \beta^2$. Thus

$$SU = \beta\xi - \left(\frac{k}{2} + \frac{\beta^2}{2(\alpha+k)}\right)U. \tag{7.8}$$

If we take $Y = \phi U$ in (7.5) we get $-k\phi S\phi U - \beta S\xi - kSU - \alpha\phi S\phi U + S\phi S\phi U = 0$, that is, $-(k+\alpha)\phi S\phi U - \beta S\xi - kSU + (\alpha+k)SU = 0$. Then, $-(k+\alpha)^2 U - \beta S\xi + \alpha SU = 0$ and its scalar product with $U$ gives $-(k+\alpha)^2 - \beta^2 + \alpha g(SU, U) = 0$. If $\alpha = 0$ we obtain $-k^2 - \beta^2 = 0$, which is impossible. Therefore

$$\alpha \neq 0 \tag{7.9}$$

and

$$SU = \beta\xi + \frac{(k+\alpha)^2 + \beta^2}{\alpha}U. \tag{7.10}$$

From (7.8) and (7.10), $-\frac{k}{2} - \frac{\beta^2}{2(\alpha+k)} = \frac{(k+\alpha)^2 + \beta^2}{\alpha}$ and this yields $-k(\alpha+k)\alpha - \alpha\beta^2 = 2(k+\alpha)^3 + 2(\alpha+k)\beta^2$. Therefore

$$(\alpha+k)(-\alpha k - 2(\alpha+k)^2) = (3\alpha+2k)\beta^2. \tag{7.11}$$

If $3\alpha + 2k = 0$, $k = -\frac{3}{2}\alpha$ and from (7.11), $(\alpha - \frac{3\alpha}{2})(\frac{3\alpha^2}{2} - 2(-\frac{3\alpha}{2} + \alpha)^2) = 0$. Then $-\frac{\alpha}{2}(\frac{3\alpha^2}{2} - \frac{2\alpha^2}{4}) = -\frac{\alpha^3}{2} = 0$. Then $\alpha = 0$, a contradiction with (7.9). Therefore,

$$3\alpha + 2k \neq 0. \tag{7.12}$$

We also know that $\mathcal{C}_U$ is $S$-invariant. Let $Y \in \mathcal{C}_U$. From (7.5) we obtain

$$-k\phi SY + k\phi Y - \alpha\phi SY + S\phi SY = 0. \tag{7.13}$$

As $\mathcal{C}_U$ is $S$-invariant, if we take the scalar product of (7.13) and $X \in \mathcal{C}_U$ and change $X$ by $Y$ we obtain

$$kS\phi Y - k\phi SY + \alpha S\phi Y - S\phi SY = 0. \tag{7.14}$$

Substracting (7.14) from (7.13) we get

$$-\alpha(\phi S + S\phi)Y + 2S\phi SY = 0 \tag{7.15}$$

for any $Y \in \mathcal{C}_U$ and adding (7.13) and (7.14) we have

$$(2k + \alpha)(S\phi Y - \phi SY) = 0 \tag{7.16}$$

for any $Y \in \mathcal{C}_U$. If we suppose $2k + \alpha = 0$, from (7.11) we have $-k(2k^2 - 2(-k)^2) = 0 = -4\beta^2$, which is impossible. Then

$$2k + \alpha \neq 0 \tag{7.17}$$

and

$$S\phi Y = \phi SY \tag{7.18}$$

for any $Y \in \mathcal{C}_U$. Let us suppose that $Y \in \mathcal{C}_U$ satisfies $SY = \lambda Y$. Then (7.18) yields $S\phi Y = \lambda\phi Y$ and from (7.15) $-2\alpha\lambda\phi Y + 2\lambda^2\phi Y = 0$. Thus $\lambda(\lambda - \alpha) = 0$ and either $\lambda = 0$ or $\lambda = \alpha$. Therefore on $\mathcal{C}_U$ we have at most two distinct principal curvatures, $\alpha$ and $0$. From (7.7) our real hypersurface is not ruled and the proof is finished.

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**Author's contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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