Fundamental Journal of Mathematics and Applications, 7(4) (2024), 218-225
Research Paper / Open Access

Fundamental Journal of Mathematics and Applications

ISSN Online: 2645-8845
www.dergipark.org.tr/en/pub/fujma
https://doi.org/10.33401/fujma.1466353

Some Fixed Point Theorems for the New Generalizations of P-Contractive Maps

Aynur Şahin [1,†,*] **and Buse Demir** [1,‡]

[1] *Sakarya University, Faculty of Science, Department of Mathematics, Sakarya, Türkiye*
[†] *ayuce@sakarya.edu.tr*, [‡] *buse.demir5@ogr.sakarya.edu.tr*
[*] *Corresponding Author*

Article Information

Keywords: P-Contractive map; Fixed point; Enriched contractive maps; Compact metric space

AMS 2020 Classification: 47H10; 54E45

Abstract

In this paper, we introduce the enriched P-contractive and the enriched Suzuki-type P-contractive maps, and for such maps, we establish the existence and uniqueness of fixed points (fps) in the setting of normed spaces. Also, we introduce the generalized Suzuki-type P-contractive map and prove some fp theorems for this map in compact metric spaces. These results unify, generalize, and complement various comparable results in the literature.

1. Introduction

Suppose that $\mathcal{H}$ is any nonempty set and $\varphi : \mathcal{H} \to \mathcal{H}$ is any map. If there is a point $x \in \mathcal{H}$ such that $\varphi(x) = x$, then x is known as a fixed point (fp) of φ. The set of all fps of φ is denoted by $F(\varphi)$, that is $F(\varphi) = \{x \in \mathcal{H} : \varphi(x) = x\}$. In mathematics, the fp theory, which examines the fps of maps, has a wide range of application areas: economics, physics, engineering, etc., as well as geometry, analysis, and topology, among other fields of mathematics (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]). That is why this topic is essential to work and contribute.

In 1922, Banach [11] was the first who observed that if one takes a complete metric space, namely, $(\mathcal{H}, d)$, then any contraction map on $\mathcal{H}$, that is, a map $\varphi : \mathcal{H} \to \mathcal{H}$, which has the property that for all $x, y \in \mathcal{H}$, a constant $\theta \in [0, 1)$ exists with

$$d(\varphi(x), \varphi(y)) \leq \theta d(x, y)$$

admits a unique fp.

In 1962, Edelstein [12] generalized Banach's fp theorem as: any contractive map on a compact metric space $(\mathcal{H}, d)$, that is, a map $\varphi : \mathcal{H} \to \mathcal{H}$, which has the property that for all $x, y \in \mathcal{H}$ with $x \neq y$, one has

$$d(\varphi(x), \varphi(y)) < d(x, y)$$

admits a unique fp.

By weakening the concept of contractivity, in 2009, Suzuki [13] generalized Edelstein's fp theorem as: any Suzuki-type contractive map (SC) on a compact metric space $(\mathcal{H}, d)$, that is, a map $\varphi : \mathcal{H} \to \mathcal{H}$ such that for all $x, y \in \mathcal{H}$ with $x \neq y$, one has

$$\frac{1}{2} d(x, \varphi(x)) < d(x, y) \Longrightarrow d(\varphi(x), \varphi(y)) < d(x, y)$$

admits a unique fp.

In this direction, Popescu [14] introduced the concept of P-contraction maps as a generalization of contraction maps. A self-map φ on a metric space $(\mathcal{H}, d)$ is called P-contraction if there exists $\theta \in [0, 1)$ such that

$$d(\varphi(x), \varphi(y)) \leq \theta [d(x, y) + |d(x, \varphi(x)) - d(y, \varphi(y))|]$$

Received: 07-04-2024 **Revised:** 29-08-2024 **Accepted:** 05-11-2024 **Online:** 31-12-2024 **Published:** 31-12-2024

for all $x, y \in \mathcal{H}$.

On the other hand, in 2020, Berinde and Păcurar [15] introduced the enriched version of contraction maps. A self-map φ on a nonempty subset $\mathcal{K}$ of a normed space $(\mathcal{H}, \|\cdot\|)$ is called enriched contraction if there exist $k \in [0, +\infty)$ and $\theta \in [0, k+1)$ such that

$$\|k(x-y) + \varphi(x) - \varphi(y)\| \leq \theta \|x-y\|$$

for all $x, y \in \mathcal{K}$. Also, we can conclude that if we take $k = 0$ in this inequality, we get the contraction map.

By following this, in 2022, Abbas, Anjum, and Rakočević [16] introduced the enriched version of SC maps and proved the following theorem.

Theorem 1.1. *[16, Theorem 6] Let $(\mathcal{H}, \|\cdot\|)$ be a compact normed space and $\varphi : \mathcal{H} \to \mathcal{H}$ be a map. If there exists $k \in [0, \infty)$ with $\lambda = \frac{1}{k+1}$ such that for all $x, y \in \mathcal{H}$,*

$$\frac{\lambda}{2}\|x - \varphi(x)\| < \|x-y\| \Longrightarrow \|k(x-y) + \varphi(x) - \varphi(y)\| < \|x-y\|.$$

Then, φ has a fp.

In 2024, Altun, Hançer, and Ateş [17] introduced the concept of enriched P-contraction maps as a generalization of P-contraction maps. A self-map φ on a nonempty subset $\mathcal{K}$ of a normed space $(\mathcal{H}, \|\cdot\|)$ is called enriched P-contraction if there exist $k \in [0, +\infty)$ and $\theta \in [0, k+1)$ such that

$$\|k(x-y) + \varphi(x) - \varphi(y)\| \leq \theta \|x-y\| + \frac{\theta}{k+1} \left| \|x - \varphi(x)\| - \|y - \varphi(y)\| \right|$$

for all $x, y \in \mathcal{K}$. It is clear that every P-contraction is an enriched P-contraction with $k = 0$.

Motivated by the above results, we first introduce the enriched P-contractive and the enriched Suzuki-type P-contractive maps and prove the generalizations of the results of [18, 19] in the setting of a normed space. Also, we introduce the generalized Suzuki-type P-contractive maps and prove the generalizations of the results of [19] in a compact metric space.

2. Preliminaries

In 2018, Altun, Durmaz and Olgun [18] introduced the following generalization of contractive maps.

Definition 2.1. *[18, Definition 2.2] Let $(\mathcal{H}, d)$ be a metric space and $\varphi : \mathcal{H} \to \mathcal{H}$ be a map. Then, the map φ is said to be P-contractive (PC) if*

$$d(\varphi(x), \varphi(y)) < d(x,y) + |d(x, \varphi(x)) - d(y, \varphi(y))|$$

for all $x, y \in \mathcal{H}$ with $x \neq y$.

Now, we give two examples of PC maps.

Example 2.2. *(see [20]) Let $\mathcal{H} = [0,1]$ with the usual metric and $\varphi : \mathcal{H} \to \mathcal{H}$ be defined by*

$$\varphi(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 0, \\ \frac{x}{2}, & \text{if } x \neq 0. \end{cases}$$

Then, the map φ is PC but not contractive.

Example 2.3. *(see [18])*

(i) Let $\mathcal{H} = [0,2]$ with the usual metric and $\varphi : \mathcal{H} \to \mathcal{H}$ be defined by

$$\varphi(x) = \begin{cases} 1, & \text{if } x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Then, the map φ is PC but not SC.

(ii) *Let* $\mathcal{H} = \{(0,0),(4,0),(0,4),(4,5),(5,4)\} \subset \mathbb{R}^2$ *with the metric*

$$d(x,y) = d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

for $x = (x_1,x_2), y = (y_1,y_2) \in \mathcal{H}$. *Define a map* $\varphi : \mathcal{H} \to \mathcal{H}$ *by*

$$\varphi = \begin{pmatrix} (0,0) & (4,0) & (0,4) & (4,5) & (5,4) \\ (0,0) & (0,0) & (0,0) & (4,0) & (0,4) \end{pmatrix}. \tag{2.1}$$

Then, the map φ *is SC but not PC.*

Moreover, the following fp theorem has been presented by Altun, Durmaz, and Olgun [18].

Theorem 2.4. *[18, Theorem 2.14] Let* $(\mathcal{H},d)$ *be a compact metric space and* $\varphi : \mathcal{H} \to \mathcal{H}$ *be a continuous PC map. Then,* φ *has a unique fp.*

Recently, in 2023, Altun [19] introduced a new class of map called Suzuki-type P-contractive as a generalization of SC and PC maps inspired by the previous maps we mentioned above.

Definition 2.5. *[19, Definition 2.1] Let* $(\mathcal{H},d)$ *be a metric space and* $\varphi : \mathcal{H} \to \mathcal{H}$ *be a map. Then* φ *is said to be Suzuki-type P-contractive (SPC) if*

$$\tfrac{1}{2}d(x,\varphi(x)) < d(x,y) \Longrightarrow d(\varphi(x),\varphi(y)) < d(x,y) + |d(x,\varphi(x)) - d(y,\varphi(y))|$$

for all $x,y \in \mathcal{H}$.

Remark 2.6. *The map* φ *in Example 2.3-(i) is SPC, but not SC. Also, the map* φ *in Example 2.3-(ii) is SPC, but not PC. Hence, we can see that the class of SPC maps generalizes the SC and PC maps.*

After that, he proved an fp theorem for the class of the new map above.

Theorem 2.7. *[19, Theorem 2.1] Let* $(\mathcal{H},d)$ *be a compact metric space and* $\varphi : \mathcal{H} \to \mathcal{H}$ *be a continuous SPC map. Then,* φ *has a unique fp.*

Lemma 2.8. *[21] Let* $\mathcal{H}$ *be a compact topological space and* $f : \mathcal{H} \to \mathbb{R}$ *be a lower semi-continuous function. Then, there exists an element* $x_0 \in \mathcal{H}$ *such that* $f(x_0) = \inf\{f(x) : x \in \mathcal{H}\}$.

Also, by Lemma 2.8, Altun [19] obtained the following result by assuming the lower semi-continuity of the function $f(x) = d(x,\varphi(x))$ instead of the continuity of φ.

Theorem 2.9. *[19, Theorem 2.2] Let* $(\mathcal{H},d)$ *be a compact metric space and* $\varphi : \mathcal{H} \to \mathcal{H}$ *be an SPC map. Then* φ *has a unique fp provided the function* f *defined by* $f(x) = d(x,\varphi(x))$ *is lower semi-continuous.*

3. Enriched versions of PC and SPC maps

First, we introduce the concept of an enriched P-contractive map as a generalization of PC maps.

Definition 3.1. *Let* $\mathcal{K}$ *be a nonempty subset of a normed space* $(\mathcal{H},\|\cdot\|)$, $\varphi : \mathcal{K} \to \mathcal{K}$ *be a map and* $k \in [0,\infty)$. *Then, the map* φ *is said to be enriched P-contractive (EPC) if*

$$\|k(x-y) + \varphi(x) - \varphi(y)\| < \|x-y\| + |\|x-\varphi(x)\| - \|y-\varphi(y)\||$$

for all $x,y \in \mathcal{K}$ *with* $x \neq y$.

Remark 3.2. *Obviously, we get the PC map if* $k = 0$ *in the above definition.*

Example 3.3. *Let* $\mathcal{K} = \{(0,0),(4,0),(0,4),(4,5),(5,4)\}$ *be a subset of the normed space* $\mathbb{R}^2$ *endowed with the norm* $\|x\| = |x_1| + |x_2|$ *for* $x = (x_1,x_2) \in \mathbb{R}^2$ *and* $\varphi : \mathcal{K} \to \mathcal{K}$ *be a map defined by (2.1). Since*

$$\|\varphi(x) - \varphi(y)\| = 8 > 2 = \|x-y\| + |\|x-\varphi(x)\| - \|y-\varphi(y)\||$$

for $x = (4,5)$ *and* $y = (5,4)$, *we find that* φ *is not a PC map. On the other hand, we can see that, for* $k = 4$,

$$\|k(x-y) + \varphi(x) - \varphi(y)\| = 0 < 2 = \|x-y\| + |\|x-\varphi(x)\| - \|y-\varphi(y)\||$$

for $x = (4,5)$ *and* $y = (5,4)$. *Then,* φ *is an EPC map with* $k = 4$.

We prove the following theorem, which is more general than Theorem 2.4, in the setting of normed spaces.

Theorem 3.4. *Let $\mathcal{K}$ be a nonempty, compact and convex subset of a normed space $(\mathcal{H}, \|\cdot\|)$ and $\varphi : \mathcal{K} \to \mathcal{K}$ be a continuous EPC map with $k \in [0,\infty)$. Then, φ has a unique fp.*

Proof. We consider $\varphi_\lambda(x) = (1-\lambda)x + \lambda\varphi(x)$, which is called the average map. Now, we may take $\lambda = \frac{1}{k+1}$. Then $k = \frac{1}{\lambda} - 1$. It is easy to see that $\lambda \in (0,1]$. Since φ is a EPC map, then we have

$$\|(\frac{1}{\lambda} - 1)(x-y) + \varphi(x) - \varphi(y)\| < \|x-y\| + |\|x-\varphi(x)\| - \|y-\varphi(y)\||.$$

By multiplying both sides of this inequality with λ, we obtain

$$\|(1-\lambda)(x-y) + \lambda(\varphi(x) - \varphi(y))\| < \lambda\|x-y\| + |\|\lambda x - \lambda\varphi(x)\| - \|\lambda y - \lambda\varphi(y)\||$$

or

$$\|[(1-\lambda)x + \lambda\varphi(x)] - [(1-\lambda)y + \lambda\varphi(y)]\| < \lambda\|x-y\| + |\|x - [(1-\lambda)x + \lambda\varphi(x)]\| - \|y - [(1-\lambda)y + \lambda\varphi(y)]\||.$$

From the definition of the average map, we get

$$\begin{aligned}\|\varphi_\lambda(x) - \varphi_\lambda(y)\| &< \lambda\|x-y\| + |\|x-\varphi_\lambda(x)\| - \|y-\varphi_\lambda(y)\|| \\ &\le \|x-y\| + |\|x-\varphi_\lambda(x)\| - \|y-\varphi_\lambda(y)\||.\end{aligned}$$

Hence, we have

$$\|\varphi_\lambda(x) - \varphi_\lambda(y)\| < \|x-y\| + |\|x-\varphi_\lambda(x)\| - \|y-\varphi_\lambda(y)\||,$$

that is, φ_λ is a PC map. On the other hand, by the continuity of φ, it is easy to show that φ_λ is continuous. Since $\mathcal{K}$ is compact and φ_λ is continuous, then, by Lemma 2.8, there exists $u \in \mathcal{K}$ such that

$$\|u - \varphi_\lambda(u)\| = \inf\{\|x - \varphi_\lambda(x)\| : x \in \mathcal{K}\}. \tag{3.1}$$

We claim that $\|u-\varphi_\lambda(u)\| = 0$. On the contrary, we assume that $\|u-\varphi_\lambda(u)\| > 0$. By the equality (3.1) and the P-contractivity of φ_λ, we have

$$\begin{aligned}\|\varphi_\lambda(u) - \varphi_\lambda^2(u)\| &< \|u-\varphi_\lambda(u)\| + |\|u-\varphi_\lambda(u)\| - \|\varphi_\lambda(u) - \varphi_\lambda^2(u)\|| \\ &= \|u-\varphi_\lambda(u)\| + \|\varphi_\lambda(u) - \varphi_\lambda^2(u)\| - \|u-\varphi_\lambda(u)\| \\ &= \|\varphi_\lambda(u) - \varphi_\lambda^2(u)\|\end{aligned}$$

which is a contradiction. Therefore, we get $\|u-\varphi_\lambda(u)\| = 0$, that is, u is a fp of φ_λ. Now, assume that there is another fp of φ_λ, called v. Since both of u and v are fps of φ_λ, then we have

$$\begin{aligned}\|u-v\| &= \|\varphi_\lambda(u) - \varphi_\lambda(v)\| \\ &< \|u-v\| + |\|u-\varphi_\lambda(u)\| - \|v-\varphi_\lambda(v)\|| \\ &= \|u-v\|.\end{aligned}$$

This is a contradiction. Consequently, we get $F(\varphi_\lambda) = \{u\}$. Then

$$u = \varphi_\lambda(u) \Longleftrightarrow u = (1-\lambda)u + \lambda\varphi(u) \Longleftrightarrow u = \varphi(u).$$

Therefore, $F(\varphi) = \{u\}$. This completes the proof. □

If we take $k = 0$ in Theorem 3.4, we obtain $\lambda = 1$ and so $\varphi_1 = \varphi$. Then, we can derive Theorem 2.4 in the setting of normed spaces as follows.

Corollary 3.5. *Let $\mathcal{K}$ be a nonempty, compact subset of a normed space $(\mathcal{H}, \|\cdot\|)$ and $\varphi : \mathcal{K} \to \mathcal{K}$ be a continuous PC map. Then, φ has a unique fp.*

We introduce the concept of an enriched Suzuki-type P-contractive map as a generalization of SPC maps.

Definition 3.6. *Let $\mathcal{K}$ be a nonempty subset of a normed space $(\mathcal{H}, \|\cdot\|)$, $\varphi : \mathcal{K} \to \mathcal{K}$ be a map and $k \in [0,\infty)$. Then φ is said to be enriched Suzuki-type P-contractive (ESPC) if*

$$\frac{1}{2}\|x-\varphi(x)\| < (k+1)\|x-y\| \Longrightarrow \|k(x-y) + \varphi(x) - \varphi(y)\| < \|x-y\| + |\|x-\varphi(x)\| - \|y-\varphi(y)\||$$

for all $x, y \in \mathcal{K}$ with $x \neq y$.

Remark 3.7. *We can see that we get the SPC map if $k=0$ in the above definition.*

Remark 3.8. *Also, to observe the connections between the generalizations, we can give the following diagram:*

$$\begin{array}{ccccc} & & \Delta & & \\ & PC & \Longrightarrow & EPC & \\ \Omega & \Downarrow & & \Downarrow & \delta. \\ & SPC & \Longrightarrow & ESPC & \\ & & \nabla & & \end{array}$$

Here, Altun [19] showed the implication given by Ω. Also, the implications presented by Δ and ∇ can be seen from Remark 3.2 and Remark 3.7, respectively. Finally, it remains an open problem whether the converse of the implication given by δ is true.

We prove the following theorem, which is more general than Theorem 2.7, in the setting of normed spaces.

Theorem 3.9. *Let $\mathcal{K}$ be a nonempty, compact and convex subset of a normed space $(\mathcal{H},\|\cdot\|)$ and $\varphi:\mathcal{K}\to\mathcal{K}$ be a continuous ESPC map. Then, φ has a unique fp.*

Proof. We consider the average map $\varphi_\lambda(x)=(1-\lambda)x+\lambda\varphi(x)$. Now, we may take $\lambda=\dfrac{1}{k+1}$. Then $k=\dfrac{1}{\lambda}-1$. It is easy to see that $\lambda\in(0,1]$. Since φ is an ESPC map, then we have

$$\frac{1}{2}\|x-\varphi(x)\|<(k+1)\|x-y\|\Longrightarrow\|k(x-y)+\varphi(x)-\varphi(y)\|<\|x-y\|+|\|x-\varphi(x)\|-\|y-\varphi(y)\||.$$

From the left side of this implication, we have

$$\begin{aligned}\frac{1}{2}\|x-\varphi(x)\|<\frac{1}{\lambda}\|x-y\|&\Longrightarrow\frac{1}{2}\|\lambda x-\lambda\varphi(x)\|<\|x-y\|\\&\Longrightarrow\frac{1}{2}\|x-\varphi_\lambda(x)\|<\|x-y\|.\end{aligned}$$

The right side of this implication becomes

$$\|(\frac{1}{\lambda}-1)(x-y)+\varphi(x)-\varphi(y)\|<\|x-y\|+|\|x-\varphi(x)\|-\|y-\varphi(y)\||.$$

By multiplying both sides of the above inequality with λ, we obtain

$$\|(1-\lambda)(x-y)+\lambda(\varphi(x)-\varphi(y))\|<\|\lambda x-\lambda y\|+|\|\lambda x-\lambda\varphi(x)\|-\|\lambda y-\lambda\varphi(y)\||.$$

From the definition of the average map, we get

$$\begin{aligned}\|\varphi_\lambda(x)-\varphi_\lambda(y)\|&<\lambda\|x-y\|+|\|x-\varphi_\lambda(x)\|-\|y-\varphi_\lambda(y)\||\\&\leq\|x-y\|+|\|x-\varphi_\lambda(x)\|-\|y-\varphi_\lambda(y)\||.\end{aligned}$$

Then, we have

$$\frac{1}{2}\|x-\varphi_\lambda(x)\|<\|x-y\|\Longrightarrow\|\varphi_\lambda(x)-\varphi_\lambda(y)\|<\|x-y\|+|\|x-\varphi_\lambda(x)\|-\|y-\varphi_\lambda(y)\||,$$

that is, φ_λ is an SPC map. Also, by the continuity of φ, it is easy to show that φ_λ is continuous. Similarly to the proof of Theorem 3.4, since $\mathcal{K}$ is compact and φ_λ is continuous, then, by Lemma 2.8, there exists $u\in\mathcal{K}$ such that satisfies the equality (3.1). We claim that $\|u-\varphi_\lambda(u)\|=0$. On the contrary, we assume that $\|u-\varphi_\lambda(u)\|>0$. In this case, since $0<\dfrac{1}{2}\|u-\varphi_\lambda(u)\|<\|u-\varphi_\lambda(u)\|$, then, by the equality (3.1), we have

$$\begin{aligned}\|\varphi_\lambda(u)-\varphi_\lambda^2(u)\|&<\|u-\varphi_\lambda(u)\|+|\|u-\varphi_\lambda(u)\|-\|\varphi_\lambda(u)-\varphi_\lambda^2(u)\||\\&=\|u-\varphi_\lambda(u)\|+\|\varphi_\lambda(u)-\varphi_\lambda^2(u)\|-\|u-\varphi_\lambda(u)\|\\&=\|\varphi_\lambda(u)-\varphi_\lambda^2(u)\|,\end{aligned}$$

which is a contradiction. Therefore, we get $\|u-\varphi_\lambda(u)\|=0$, that is, u is a fp of φ_λ. Now, assume that there is another fp of φ_λ, called v. Since $u\neq v$, then we have $0=\dfrac{1}{2}\|u-\varphi_\lambda(u)\|<\|u-v\|$. Hence, we get

$$\begin{aligned}\|u-v\|&=\|\varphi_\lambda(u)-\varphi_\lambda(v)\|\\&<\|u-v\|+|\|u-\varphi_\lambda(u)\|-\|v-\varphi_\lambda(v)\||\\&=\|u-v\|.\end{aligned}$$

This is a contradiction. Consequently, we obtain $F(\varphi_\lambda)=\{u\}$. Hence, $F(\varphi)=\{u\}$, that is, φ has a unique fp u. □

If we take $k = 0$ in Theorem 3.9, we obtain $\lambda = 1$ and so $\varphi_1 = \varphi$. Then, we can derive Theorem 2.7 in the setting of normed spaces as described below.

Corollary 3.10. *Let $\mathcal{K}$ be a nonempty, compact subset of a normed space $(\mathcal{H}, \|\cdot\|)$ and $\varphi : \mathcal{K} \to \mathcal{K}$ be a continuous SPC map. Then, φ has a unique fp.*

4. A direct generalization of SPC maps

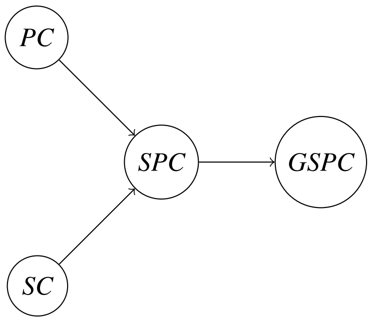
We begin by defining the generalized Suzuki-type P-contractive map, which is a direct generalization of SPC maps.

Definition 4.1. *Let $(\mathcal{H}, d)$ be a metric space and $\varphi : \mathcal{H} \to \mathcal{H}$ be a map. Then, the map φ is said to be generalized Suzuki-type P- contractive (GSPC) if*

$$\lambda d(x, \varphi(x)) < d(x,y) \Longrightarrow d(\varphi(x), \varphi(y)) < d(x,y) + |d(x, \varphi(x)) - d(y, \varphi(y))|$$

for all $x, y \in \mathcal{H}$ and $\lambda \in (0,1)$.

Remark 4.2. *We get the SPC map if we take $\lambda = \dfrac{1}{2}$ in the above definition. Hence, we generalize the SPC maps, as well as the SC and PC maps. We can express this in the below diagram:*



Here, Example 2.3 shows that the classes of SC and PC maps are distinct. Also, Altun [19] showed the implication between PC and SPC and the implication between SC and SPC. Finally, the last implication can be clearly seen when we take $\lambda = \frac{1}{2}$ in the definition of GSPC.

By Lemma 2.8, we prove the following theorem, which is a generalization of Theorem 2.9.

Theorem 4.3. *Let $(\mathcal{H}, d)$ be a compact metric space and $\varphi : \mathcal{H} \to \mathcal{H}$ be a GSPC map. Then φ has a unique fp in $\mathcal{H}$ provided the function $f(x) = d(x, \varphi(x))$ is lower semi-continuous.*

Proof. Since $\mathcal{H}$ is compact and $f : \mathcal{H} \to \mathbb{R}$ is lower semi-continuous, and by Lemma 2.8, then there exists $u \in \mathcal{H}$ such that $f(u) = \inf\{f(x) : x \in \mathcal{H}\}$, that is, we have

$$d(u, \varphi(u)) = \inf\{d(x, \varphi(x)) : x \in \mathcal{H}\}. \tag{4.1}$$

We now show that $d(u, \varphi(u)) = 0$. On the contrary, we suppose that $d(u, \varphi(u)) > 0$. In this case, since $\lambda \in (0,1)$, we get $0 < \lambda d(u, \varphi(u)) < d(u, \varphi(u))$. Then, by the equality (4.1), we have

$$\begin{aligned} d(\varphi(u), \varphi^2(u)) &< d(u, \varphi(u)) + |d(u, \varphi(u)) - d(\varphi(u), \varphi^2(u))| \\ &= d(u, \varphi(u)) + d(\varphi(u), \varphi^2(u)) - d(u, \varphi(u)) \\ &= d(\varphi(u), \varphi^2(u)), \end{aligned}$$

which is a contradiction. Therefore, we get $d(u, \varphi(u)) = 0$; that is, u is a fp of φ. Now, we need to show that the fp is unique. Assume that there is another fp of φ, called v. In this case, since $0 = \lambda d(u, \varphi(u)) < d(u, v)$, then we have

$$\begin{aligned} d(u, v) &= d(\varphi(u), \varphi(v)) \\ &< d(u, v) + |d(u, \varphi(u)) - d(v, \varphi(v))| \\ &= d(u, v), \end{aligned}$$

which leads to a contradiction. In conclusion, the fp of φ is unique. □

As a generalization of Theorem 2.7, we can give the following theorem using Theorem 4.3.

Theorem 4.4. *Let $(\mathcal{H}, d)$ be a compact metric space and $\varphi : \mathcal{H} \to \mathcal{H}$ be a continuous GSPC map. Then, φ has a unique fp.*

Proof. Since φ is continuous, then $f(x) = d(x, \varphi(x))$ is also continuous by the following inequality

$$|d(x,\varphi(x)) - d(y,\varphi(y))| \leq d(x,y) + d(\varphi(x),\varphi(y)).$$

So, it will be directly lower semi-continuous. Therefore, by Theorem 4.3, φ has a unique fp. □

Remark 4.5. *We can see that the constant, which is $\frac{1}{2}$ in the SPC maps, is to be $\lambda \in (0,1)$ in the GSPC maps. Consequently, when $\lambda = \frac{1}{2}$, Theorems 4.3 and 4.4 are reduced to Theorems 2.9 and 2.7, respectively.*

5. Conclusions

We define three new generalizations of PC maps and establish the existence theorems of a unique fp for these maps. We note that the EPC and ESPC maps include the PC and SPC maps, respectively, and the GSPC maps include the SPC maps. Our results improve and extend the corresponding results of [18, 19].

In forthcoming research, the existence of a unique fp for the multi-valued versions of the EPC and ESPC maps can be proved. Furthermore, the enriched version of GSPC maps may be introduced, and some fixed point theorems for this map might be offered.

Declarations

Acknowledgements: The authors are thankful to the reviewers and the editor for their constructive comments. The second author is grateful to The Scientific and Technological Research Council of Türkiye (TÜBİTAK) for the support within the scope of 2209-A-Research Project Support Programme for Undergraduate Students. Also the authors are grateful to the anonymous referee for helpful suggestions to improve the paper.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: The authors declare no conflict of interest.

Supporting/Supporting Organizations: This research received no external funding.

Ethical Approval and Participant Consent: This article does not contain any studies with human or animal subjects. It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

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Availability of Data and Materials: Data sharing not applicable.

Use of AI tools: The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

ORCID
Aynur Şahin https://orcid.org/0000-0001-6114-9966
Buse Demir https://orcid.org/0009-0008-7337-5934

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Fundamental Journal of Mathematics and Applications (FUJMA), (Fundam. J. Math. Appl.)
https://dergipark.org.tr/en/pub/fujma

How to cite this article: A. Şahin and B. Demir, *Some fixed point theorems for the new generalizations of P-contractive maps*, Fundam. J. Math. Appl., **7**(4) (2024), 218-225. DOI 10.33401/fujma.1466353