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Lattice of Subinjective Portfolios of Modules

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Abstract — Given a ring R, we study its right subinjective profile $\mathfrak{si}\mathfrak{P}(R)$ to be the collection of subinjectivity domains of its right R-modules. We deal with the lattice structure of the class $\mathfrak{si}\mathfrak{P}(R)$. We show that the poset $(\mathfrak{si}\mathfrak{P}(R), \subseteq)$ forms a complete lattice, and an indigent R-module exists if $\mathfrak{si}\mathfrak{P}(R)$ is a set. In particular, if R is a generalized uniserial ring with $J^2(R) = 0$, then the lattice $(\mathfrak{si}\mathfrak{P}(R), \subseteq, \wedge, \vee)$ is Boolean.

Keywords *Subinjectivity domain, subinjective profile, complete lattice of subinjectivity domains*

Mathematics Subject Classification (2020) 16D50, 06B10

1. Introduction

Throughout this paper, every ring R is associative with unity, and all modules are unitary. $Mod-R$ stands for the category of right R-modules. Flat modules, injective modules, and projective modules are among the most studied structures of module and ring theory, and they occur naturally in many algebra fields, such as homological algebra, category theory, representation theory, and algebraic geometry. Researchers conduct numerous studies on projective, injective, and flat modules. Many of these studies explore ideas based on relative projectivity, injectivity, and flatness. Recently, instead of simply categorizing modules as having a specific homological property, each module is allocated a relative domain that gauges the degree to which it possesses that particular property. In particular, several research papers have been devoted to the study of the injectivity, flatness, and projectivity level of modules [1–11].

Subinjectivity domain of a module (Definition 2.2) was originally introduced in [12] in order to study in a way the degree of injectivity of modules. In this article, we shift our focus from the subjective domain of modules to examining the collection of these domains using a fresh approach. This collection is called the (right) subinjective profile (si-profile, for short) of R, and is denoted by $\mathfrak{si}\mathfrak{P}(R)$. $\mathfrak{si}\mathfrak{P}(R) = \{Mod-R\}$ if and only if R is a semisimple Artinian ring if and only if there exists an injective indigent right (or left) R-module. Semisimple Artinian rings stand out as the most straightforward type of rings regarding their subinjective characteristics. Another straightforward case arises from rings that are not semisimple Artinian; these rings exhibit only two possible domains of subinjectivity: injective modules and all modules. Such rings have no subinjective middle class [12, 13].

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We organize the paper as subsequent. In Section 2, we provide brief definitions and properties. In Section 3, we study the class $\mathfrak{siP}(R)$ under the condition that it is a set. We show that if $\mathfrak{siP}(R)$ is a set, then the class $\mathcal{IN}$ of injective modules is an si-portfolio and $\mathfrak{siP}(R)$ is closed under intersections. R has no subinjective middle class if and only $|\mathfrak{siP}(R)| = 2$. We show that the poset $(\mathfrak{siP}(R), \subseteq)$ forms a complete lattice if $\mathfrak{siP}(R)$ is a set (Theorem 3.6). In particular, if R is a generalized uniserial ring with $J^2(R) = 0$, then the lattice $(\mathfrak{siP}(R), \subseteq, \wedge, \vee)$ is Boolean (Theorem 3.9).

2. Preliminaries

This section provides some basic notions to be required for the following section. The paper uses the books [14–16] for the basic definitions. The references [7, 8, 12, 17] cover various aspects of this topic.

Definition 2.1. A module E is injective if for any morphism $f : A \to E$ and any monomorphism $q : A \to B$, f factors through q by some morphism $B \to E$.

By fixing the module A, the notion of A-subinjective module is introduced in [12].

Definition 2.2. A module X is called B-subinjective if for every monomorphism $h : B \to K$ and every homomorphism $f : B \to X$, there exists a homomorphism $g : K \to X$ such that $gh = f$.

Definition 2.3. For an R-module X, the subinjectivity domain of X, denoted as $\underline{\mathfrak{In}}^{-1}(X)$, encompasses all modules with respect to which X exhibits subinjective properties, i.e.,

$$\underline{\mathfrak{In}}^{-1}(X) = \{N \in Mod - R : X \text{ is } N\text{-subinjective}\}$$

Every subinjectivity domain contains the class $\mathcal{IN}$ of injective modules. Therefore, the class $\mathcal{IN}$ serves as a minimum benchmark for the subinjectivity domains of R-modules. The following result follows from Lemma 2.2 in [12].

Proposition 2.4. An R-module X is injective if and only if $\underline{\mathfrak{In}}^{-1}(X) = Mod - R$ if and only if X is X-subinjective.

Proposition 2.4 naturally leads to considering the degree to which a specific module exhibits injectiveness, as injective modules epitomize the highest level of injectiveness. The notion of indigent modules was introduced by Aydoğdu and López-Permouth [12].

Definition 2.5. A module M is called indigent if $\underline{\mathfrak{In}}^{-1}(M) = \mathcal{IN}$.

The existence of indigent modules within any arbitrary ring remains uncertain, although an affirmative answer is established for certain rings, such as Noetherian rings (for more details, see Proposition 3.4 in [2]). When considering the degree of injectivity in modules, we encounter two extremes: at one end, we find injective modules, and at the other, we have what are known as indigent modules.

Example 2.6. This example exhibits an indigent module. Let R be a commutative hereditary Noetherian ring. Let U be the direct sum of a representative set of all (nonprojective) simple modules. U is indigent module by [18, Proposition 2.12].

In this article, we focus on the study of the class of subinjectivity domains.

Definition 2.7. [19] A class $\mathcal{A}$ of R-modules is called si-portfolio if there exists an R-module M such that $\mathcal{A} = \underline{\mathfrak{In}}^{-1}(M)$.

Definition 2.8. [19] The class $\{\mathcal{A} \subseteq Mod-R : \mathcal{A} \text{ is an sp-portfolio}\}$ is called the (right) subinjective profile (si-profile, for short) of R and is denoted by $\mathfrak{siP}(R)$.

The class $Mod - R$ is an obvious example of an si-portfolio. Note that it is still unknown whether $\mathcal{IN}$ is an si-portfolio on non-Noetherian rings.

For a module T, we denote its injective hull, singular submodule, and radical by $E(T)$, $Z(T)$, and $\mathrm{Rad}(T)$, respectively. The Jacobson radical of a ring R is denoted by $J(R)$. We use the notations $\leq$ and $\subseteq$ in order to indicate submodules and set inclusion, respectively.

3. Lattice Structure

The poset of si-portfolios is denoted by $(\mathfrak{si}\mathfrak{P}(R), \subseteq)$ where the partial order is given by containment $\subseteq$. Note that $\mathfrak{si}\mathfrak{P}(R)$ need not actually form a set but we still use the term poset by abuse of language when $\mathfrak{si}\mathfrak{P}(R)$ is a class. The poset $(\mathfrak{si}\mathfrak{P}(R), \subseteq)$ always contains a unique maximal element, the class of all the modules $Mod-R$. It is unknown whether $\mathcal{IN}$ is an si-portfolio. Moreover, it is unknown whether $\mathfrak{si}\mathfrak{P}(R)$ is closed under intersections.

Theorem 3.1. If $\mathfrak{si}\mathfrak{P}(R)$ is a set, then $\mathcal{IN}$ is an si-portfolio and $\mathfrak{si}\mathfrak{P}(R)$ is closed under intersections.

PROOF. Assume that $\mathfrak{si}\mathfrak{P}(R)$ is a set. Consider the function $\underline{\mathfrak{In}}^{-1} : Mod{-}R \to \mathfrak{si}\mathfrak{P}(R)$, $A \to \underline{\mathfrak{In}}^{-1}(A)$. The function $\underline{\mathfrak{In}}^{-1}$ is onto. Since $\mathfrak{si}\mathfrak{P}(R)$ is a set, there is a set $\mathcal{I}$ of R-modules such that $\underline{\mathfrak{In}}^{-1}_{|\mathcal{I}}$ is one-to-one and onto. We show that the R-module

$$\mathrm{O} := \underset{M \in \mathcal{I}}{\oplus} M$$

is indigent. Let C be an R-module from $\underline{\mathfrak{In}}^{-1}(\mathrm{O})$. Then, by Proposition 2.4 [12],

$$C \in \bigcap_{M \in \mathcal{I}} \underline{\mathfrak{In}}^{-1}(M)$$

Moreover, since

$$\underline{\mathfrak{In}}^{-1}(C) \in \mathfrak{si}\mathfrak{P}(R)$$

and

$$\underline{\mathfrak{In}}^{-1}(C) = \underline{\mathfrak{In}}^{-1}(X)$$

for a module $X \in \mathcal{I}$. Then, $C \in \underline{\mathfrak{In}}^{-1}(C)$, and thus C is injective R-module. This implies that O is indigent.

Let $\mathcal{M}$ be a family of si-portfolios. Since $\mathfrak{si}\mathfrak{P}(R)$ is a set, $\mathcal{M}$ is a set. Let I be a complete set of non-isomorphic modules whose subprojectivity domains are in $\mathcal{M}$. Set

$$\mathcal{O} := \underset{M \in I}{\oplus} M$$

Then,

$$\bigcap_{\mathcal{A} \in \mathcal{M}} \mathcal{A} = \underline{\mathfrak{In}}^{-1}(\mathcal{O})$$

by Proposition 2.4 in [12]. $\Box$

In [13], the authors investigate rings for which the si-profile consists of $\mathcal{IN}$ and $Mod-R$. They called these rings as having no subinjective middle class. By Theorem 3.1, we have the following result.

Corollary 3.2. R has no subinjective middle class if and only $|\mathfrak{si}\mathfrak{P}(R)| = 2$.

The subprojectivity domains of two non-isomorphic modules may be the same. For example,

$$\underline{\mathfrak{In}}^{-1}(0) = \underline{\mathfrak{In}}^{-1}(R) = Mod - R$$

For the remaining discussions, let δ be a complete set of representatives of non-isomorphic non-injective simple modules.

Proposition 3.3. Let $\mathcal{I}$ and $\mathcal{J}$ be subsets of δ. Then,

$$\underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right) = \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right) \quad \text{if and only if} \quad \mathcal{I}=\mathcal{J}$$

Proof. To show the necessity, assume that

$$\underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right) = \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right) \quad \text{and} \quad \mathcal{I}\neq\mathcal{J}$$

Without loss of generality, we may assume that a simple R-module A exists in $\mathcal{I}\setminus\mathcal{J}$. Then, since $A\notin\mathcal{J}$, $\operatorname{Hom}(A,S)=0$, for all $S\in\mathcal{J}$. Thus,

$$A\in\bigcap_{S\in\mathcal{J}}\underline{\mathfrak{In}}^{-1}(S) = \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right)$$

Then, by assumption,

$$A\in\underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right) = \bigcap_{S_i\in\mathcal{I}}\underline{\mathfrak{In}}^{-1}(S)$$

Since $A\in\mathcal{I}$, $A\in\underline{\mathfrak{In}}^{-1}(A)$, and thus A is injective, a contradiction. The sufficiency is clear. □

A ring R is called a semilocal ring if $R/J(R)$ is semisimple Artinian. Note that any semilocal ring has only finitely many simple R-modules up to isomorphism [14]. Define

$$\mathfrak{si}\mathfrak{P}(\delta):=\left\{\underline{\mathfrak{In}}^{-1}\left(\underset{S_i\in\mathcal{I}}{\oplus} S_i\right):\mathcal{I}\subseteq\delta\right\}$$

Corollary 3.4. If R is a semilocal ring, then $|\mathfrak{si}\mathfrak{P}(\delta)|=2^{|\delta|}$.

Proof. Since R is a semilocal ring, R has only finitely many simple R-modules up to isomorphism. Put $\delta:=\{S_1,S_2,\dots,S_n\}$ and let

$$\mathfrak{si}\mathfrak{P}(\delta):=\left\{\underline{\mathfrak{In}}^{-1}\left(\underset{S_i\in\mathcal{I}}{\oplus} S_i\right):\mathcal{I}\subseteq\delta\right\}$$

Since $|\delta|=n$, $|\mathfrak{si}\mathfrak{P}(\delta)|=2^n$ by Proposition 3.3. □

Note that R is a generalized uniserial ring with $J^2(R)=0$ if and only if every right (or left) R-module is a direct sum of a semisimple module and an injective module [20].

Lemma 3.5. Let R be a generalized uniserial ring with $J^2(R)=0$. Then,

$$\mathfrak{si}\mathfrak{P}(R)=\left\{\underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right):\mathcal{I}\subseteq\delta\right\} \quad \text{and} \quad |\mathfrak{si}\mathfrak{P}(R)|=2^{|\delta|}$$

Proof. Since R is a semilocal ring, R has only finitely many simple R-modules up to isomorphism. Put $\delta:=\{S_1,S_2,\dots,S_n\}$. By Corollary 3.4,

$$\mathfrak{si}\mathfrak{P}(\delta)=\left\{\underline{\mathfrak{In}}^{-1}\left(\underset{S_i\in\mathcal{I}}{\oplus} S_i\right):\mathcal{I}\subseteq\delta\right\}$$

and $|\mathfrak{si}\mathfrak{P}(\delta)|=2^n$. We claim that $\mathfrak{si}\mathfrak{P}(\delta)=\mathfrak{si}\mathfrak{P}(R)$. Clearly, $\mathfrak{si}\mathfrak{P}(\delta)\subseteq\mathfrak{si}\mathfrak{P}(R)$. Note that, for $\mathcal{I}=\emptyset$,

$$\underline{\mathfrak{In}}^{-1}\left(\underset{S_i\in\mathcal{I}}{\oplus} S_i\right)=\underline{\mathfrak{In}}^{-1}(\{0\})=Mod-R\in\mathfrak{si}\mathfrak{P}(\delta)$$

Let M be any R-module. If M is injective, then $\underline{\mathfrak{In}}^{-1}(M)=Mod-R$, and thus $\underline{\mathfrak{In}}^{-1}(M)$ is in $\mathfrak{si}\mathfrak{P}(\delta)$. If M is not injective, then, by [20], $M=A\oplus E$, where A is semisimple and E is injective. Without loss of generality, we may assume that A has no injective direct summands. Further, we have that

$$\underline{\mathfrak{In}}^{-1}(M)=\underline{\mathfrak{In}}^{-1}(A)\cap\underline{\mathfrak{In}}^{-1}(E)=\underline{\mathfrak{In}}^{-1}(A)\cap Mod-R=\underline{\mathfrak{In}}^{-1}(A)$$

Let $\mathcal{C}$ be a complete set of non-isomorphic simple submodules of A. Since

$$\underline{\mathfrak{In}}^{-1}(A) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{C_i \in \mathcal{C}} C_i\right)$$

and each $C_i \in \mathcal{C}$ is isomorphic to one of S_γ in δ, $\underline{\mathfrak{In}}^{-1}(M) = \underline{\mathfrak{In}}^{-1}(A)$ must be in $\mathfrak{siP}(\delta)$. Thus, $\mathfrak{siP}(\delta) = \mathfrak{siP}(R)$, as claimed. □

A partially ordered set P is called a lattice if every pair of elements a and b in P has both a supremum $a \vee b$ (called join) and an infimum $a \wedge b$ (called meet). A partially ordered set P is called a complete lattice if its subsets have a join and a meet. A lattice P is said to be bounded if it has the greatest element and the least element [21].

Theorem 3.6. Assume that $\mathfrak{siP}(R)$ is a set. The poset $(\mathfrak{siP}(R), \subseteq)$ forms a complete lattice under the following meet and join operations:

i. The meet $\wedge$ is defined by $P_1 \wedge P_2 = P_1 \cap P_2$.

ii. The join $\vee$ is defined by $P_1 \vee P_2 = \bigcap\{P \in \mathfrak{siP}(R) : P_1 \subseteq P \text{ and } P_2 \subseteq P\}$.

Proof. $(\mathfrak{siP}(R), \subseteq, \wedge)$ is a meet-semilattice by Proposition 2.4 in [12]. Using the same technique in Theorem 3.1, it can be easily seen that $(\mathfrak{siP}(R), \subseteq, \vee)$ is a join-semilattice. By Theorem 3.1, every subset of $\mathfrak{siP}(R)$ has a meet. Again, by the same idea used in Theorem 3.1, it can be seen that every subset of $\mathfrak{siP}(R)$ has a join. Hence, $(\mathfrak{siP}(R), \subseteq, \vee)$ is a complete lattice. The class $\mathcal{IN}$ is the minimal element of $\mathfrak{siP}(R)$ by Theorem 3.1. On the other hand, for any injective module E, $\underline{\mathfrak{In}}^{-1}(E) = Mod - R$, and thus $Mod - R$ is the maximal element of $\mathfrak{siP}(R)$. □

Let P be a lattice with 0, 1, and $t \in P$. An element $t' \in P$ is called a complement of t if $t \wedge t' = 0$ and $t \vee t' = 1$. P is called complemented if each element in P has at least one complement. A complemented lattice is called Boolean if it is distributive. We claim that $(\mathfrak{siP}(R), \subseteq)$ is Boolean if R is a generalized uniserial ring with $J^2(R) = 0$.

Lemma 3.7. Let R be a generalized uniserial ring with $J^2(R) = 0$. Let $\mathcal{I}$ and $\mathcal{J}$ be any two subsets of δ. Then,

$$\underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I}} S\right) \vee \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{J}} S\right) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I} \cap \mathcal{J}} S\right)$$

Proof. If

$$\underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I}} S\right) \subseteq \underline{\mathfrak{In}}^{-1}(T)$$

for a non-injective simple T, then $T \cong S$, for some $S \in \mathcal{I}$, since otherwise, $\mathrm{Hom}(T, S) = 0$, for every $S \in \mathcal{I}$, and hence

$$T \in \bigcap_{S \in \mathcal{I}} \underline{\mathfrak{In}}^{-1}(S) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I}} S\right) \subseteq \underline{\mathfrak{In}}^{-1}(T)$$

which implies T is injective, a contradiction. Therefore,

$$\underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I}} S\right) \vee \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{J}} S_\gamma\right) \subseteq \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I} \cap \mathcal{J}} S\right)$$

Suppose that

$$\underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{I}} S\right) \vee \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{J}} S_\gamma\right) = \underline{\mathfrak{In}}^{-1}(X)$$

for a module X. As noted in the proof of Lemma 3.5,

$$\underline{\mathfrak{In}}^{-1}(X) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S \in \mathcal{T}} S\right)$$

where $\mathcal{T} \subseteq \delta$. The containment

$$\underline{\mathfrak{In}}^{-1}(X) \subset \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}\cap\mathcal{J}}{\oplus} S\right)$$

follows by the definition of $\vee$. Repeating the first paragraph, it can be shown that $\mathcal{I} \cap \mathcal{J} \subseteq \mathcal{T} \subseteq \mathcal{I}$ and $\mathcal{I} \cap \mathcal{J} \subseteq \mathcal{T} \subseteq \mathcal{J}$. Then, $\mathcal{I} \cap \mathcal{J} = \mathcal{T}$, which proves the assertion. □

For the sake of completeness, we provide the following result.

Corollary 3.8. [12, Theorem 4.2] If R is right-left hereditary Artinan serial ring, then

$$\underline{\mathfrak{In}}^{-1}\left(\underset{S\in\delta}{\oplus} S\right) = \mathcal{IN}$$

Theorem 3.9. If R is a generalized uniserial ring with $J^2(R) = 0$, then the lattice $(\mathfrak{si}\mathfrak{P}(R), \subseteq, \wedge, \vee)$ is Boolean.

Proof. Recall that $\mathfrak{si}\mathfrak{P}(R)$ is a set by Lemma 3.5. We first show that $(\mathfrak{si}\mathfrak{P}(R), \subseteq, \wedge, \vee)$ is complemented. Let $\mathcal{P} \in \mathfrak{si}\mathfrak{P}(R)$. If either $\mathcal{P} = \mathcal{IN}$ or $\mathcal{P} = Mod-R$, then the proof is completed. Assume that neither $\mathcal{P} = \mathcal{IN}$ nor $\mathcal{P} = Mod-R$. Then, a non-injective module H exists, such as $\underline{\mathfrak{In}}^{-1}(H) = \mathcal{P}$. Since R is a generalized uniserial ring with $J^2(R) = 0$, we get $H = A \oplus B$ where A is a direct sum of non-injective simple modules, and B is an injective module by [20]. Then,

$$\underline{\mathfrak{In}}^{-1}(H) = \underline{\mathfrak{In}}^{-1}(A) \cap \underline{\mathfrak{In}}^{-1}(P) = \underline{\mathfrak{In}}^{-1}(A) \cap Mod-R = \underline{\mathfrak{In}}^{-1}(A)$$

Let $\mathcal{C}$ be an exact set of non-isomorphic simple direct summands of A. Define a subset

$$\mathcal{I} := \{S \in \delta : S \cong C, C \in \mathcal{C}\} \subseteq \delta$$

Then,

$$\underline{\mathfrak{In}}^{-1}(A) = \underline{\mathfrak{In}}^{-1}\left(\underset{C\in\mathcal{C}}{\oplus} C\right) = \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right)$$

We note that $\mathcal{J} := \delta - \mathcal{I} \neq \emptyset$, since otherwise, we would have

$$\underline{\mathfrak{In}}^{-1}(H) = \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right) = \mathcal{IN}$$

by Corollary 3.8, a contradiction. By Proposition 2.4 in [12] and Corollary 3.8,

$$\underline{\mathfrak{In}}^{-1}(H) \wedge \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right) = \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\delta}{\oplus} S\right) = \mathcal{IN}$$

To show that

$$\underline{\mathfrak{In}}^{-1}(H) \vee \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right) = Mod-R$$

we assume that

$$\underline{\mathfrak{In}}^{-1}(H) \vee \underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right) = \underline{\mathfrak{In}}^{-1}(X)$$

for some module X. If X is injective, then we are done. Assume that X is not injective. Since R is a generalized uniserial ring with $J^2(R) = 0$, by [20], X has a non-injective simple direct summand, say T. Since

$$\underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{I}}{\oplus} S\right) = \underline{\mathfrak{In}}^{-1}(H) \subseteq \underline{\mathfrak{In}}^{-1}(T)$$

and

$$\underline{\mathfrak{In}}^{-1}\left(\underset{S\in\mathcal{J}}{\oplus} S\right) \subseteq \underline{\mathfrak{In}}^{-1}(T)$$

$T \in \mathcal{J} \cap \mathcal{I}$. But $\mathcal{J} \cap \mathcal{I} = \emptyset$, and this means that X has no non-injective simple direct summand, and so it is injective by [20].

For the distributive property, we only need to show that

$$\left(\underline{\mathfrak{In}}^{-1}(A) \vee \underline{\mathfrak{In}}^{-1}(V)\right) \wedge \underline{\mathfrak{In}}^{-1}(Z) = \left(\underline{\mathfrak{In}}^{-1}(A) \wedge \underline{\mathfrak{In}}^{-1}(Z)\right) \vee \left(\underline{\mathfrak{In}}^{-1}(V) \wedge \underline{\mathfrak{In}}^{-1}(Z)\right)$$

for any modules A, V, and Z. Without lost of generality we may assume that A, V, and Z have no projective simple direct summands. Since R is a generalized uniserial ring with $J^2(R) = 0$,

$$\underline{\mathfrak{In}}^{-1}(A) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S\in\mathcal{I}_A} S\right)$$

$$\underline{\mathfrak{In}}^{-1}(V) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S\in\mathcal{I}_V} S\right)$$

and

$$\underline{\mathfrak{In}}^{-1}(Z) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S\in\mathcal{I}_Z} S\right)$$

by Lemma 3.5 where $\mathcal{I}_A$, $\mathcal{I}_V$, and $\mathcal{I}_Z$ are subsets of δ. By Proposition 2.4 in [12] and Lemma 3.7,

$$\left(\underline{\mathfrak{In}}^{-1}(A) \vee \underline{\mathfrak{In}}^{-1}(V)\right) \wedge \underline{\mathfrak{In}}^{-1}(Z) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S\in\mathcal{J}_1} S\right)$$

where

$$\mathcal{J}_1 = (\mathcal{I}_A \cap \mathcal{I}_V) \cup \mathcal{I}_Z$$

Similarly,

$$\left(\underline{\mathfrak{In}}^{-1}(A) \wedge \underline{\mathfrak{In}}^{-1}(Z)\right) \vee \left(\underline{\mathfrak{In}}^{-1}(V) \wedge \underline{\mathfrak{In}}^{-1}(Z)\right) = \underline{\mathfrak{In}}^{-1}\left(\bigoplus_{S\in\mathcal{J}_2} S\right)$$

where

$$\mathcal{J}_2 = (\mathcal{I}_A \cup \mathcal{I}_Z) \cap (\mathcal{I}_V \cup \mathcal{I}_Z)$$

Obviously, $\mathcal{J}_1 = \mathcal{J}_2$, which proves the assertion.□

Example 3.10. Let K be any field. Let $R = T_3(K)$ denote the ring of all upper triangular 3×3 matrices with entries in K and let S denote the left socle of R. R/S is a generalized uniserial ring with $J^2(R/S) = 0$ by Example 13.6 in [20]. Then, $(\mathfrak{si}\mathfrak{P}(R/S), \subseteq)$ is a Boolean lattice by Theorem 3.9.

4. Conclusion

The objective of this paper is to commence exploration into an alternative viewpoint regarding the subinjective profile of rings. Differing from recent examinations focusing on the subinjective profile of rings, our approach delves into the lattice theoretical perspective of this concept. In future studies, researchers can consider how profile properties may determine the rings' structure. Specifically, they can investigate the necessary and sufficient conditions for rings to exhibit distributive and modular properties within this profile.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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