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Relationship Between a Homoderivation and a Semi-Derivation

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Abstract — Let $\wp$ be a ring. It is shown that if an additive mapping ϑ is a zero-power valued on $\wp$, then $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is a bijective mapping of $\wp$. The main aim of this study is to prove that ϑ is a homoderivation of $\wp$ if and only if $\vartheta : \wp \to \wp$ such that $\vartheta = \alpha - 1$ is a semi-derivation associated with α, where $\alpha : \wp \to \wp$ is a homomorphism of $\wp$. Moreover, if ϑ is a zero-power valued homoderivation on $\wp$, then ϑ is a semi-derivation associated with α, where $\alpha : \wp \to \wp$ is an automorphism of $\wp$ such that $\alpha = \vartheta + 1$.

Keywords *Ring, semi-derivation, homoderivation*

Mathematics Subject Classification (2020) 16N60, 16W25

1. Introduction

The definition of homoderivation is given by El Sofy in [1]. Many problems have been solved using homoderivation since the definition of homoderivation is given, but in solving the problems addressed, the necessity of the function being zero-power valued has emerged most of the time. For this reason, the most general results in the literature are generally found when homoderivation is zero-power valued. This situation brings to mind whether there is a relationship between being a zero-power valued mapping and being a bijective mapping. In this study, it is first shown that if ϑ is both an additive and a zero-power valued mapping on $\wp$, then $\alpha = \vartheta + 1$ is one-to-one and onto mapping on $\wp$. However, it is shown that a homoderivation ϑ is a semi-derivation of the ring $\wp$ associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ where 1 is the identity mapping of $\wp$. In [2], it is proved by Chang that if there exists a nonzero semi-derivation f of a prime ring $\wp$ associated with a not necessarily surjective function g, then g is a homomorphism of $\wp$. It is shown in this paper that every homoderivation ϑ is a semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ and α is a homomorphism without the condition of primeness of $\wp$. In addition, if ϑ is a zero-power valued homoderivation on $\wp$, then ϑ is a semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is an automorphism without the condition of primeness of $\wp$. This means that the results have been provided for semi-derivation in the literature [2–6], but associated functions have to be surjective can be used for a zero-power valued homoderivation [7–11].

Almost every result for semi-derivation in the literature becomes applicable to homoderivation, but it is necessary to ensure that each condition is actually needed. To see this better, the results in [2],

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and in [5] for semi-derivation can be compared with El Sofy's results in [1] for homoderivation. For example, Herstein is proved in [12] that if d is a nonzero derivation of a prime ring $\wp$ with characteristic, not 2 and $a \in \wp$ is such that $[a, d(x)] = 0$, for all $x \in \wp$, then a must be in the center of $\wp$. This result of Herstein is generalized by Theorem 4 in [2] by using a nonzero semi-derivation f associated with a surjective function g of $\wp$ and by Theorem 3.3.1 in [1] by using a nonzero homoderivation ϑ of $\wp$. Every homoderivation ϑ of $\wp$ is a semi-derivation associated with $\alpha = \vartheta + 1$, but α doesn't have to be a surjective mapping of $\wp$. If ϑ is a nonzero homoderivation and $a \in \wp$ is such that $[a, \vartheta(x)] = 0$, for all $x \in \wp$, then it holds that, for all $x \in \wp$,

$$[a, \alpha(x)] = [a, \vartheta(x) + x] = [a, \vartheta(x)] + [a, x] = [a, x]$$

The last equation is in equipoise that α must be a surjective function in the proof of Theorem 4 in [2]. Hence, it is obtained from Theorem 4 in [2] that if ϑ is a nonzero homoderivation of a prime ring $\wp$ with characteristic not 2 and $a \in \wp$ is such that $[a, \vartheta(x)] = 0$, for all $x \in \wp$, then a must be in the center of $\wp$. This means that the result of Chang is more general than the result of El Sofy. A similar situation is observed between Theorem 6 [2] and Theorem 3.3.3 [1]. Moreover, the results of in [3] for semi-derivation on an ideal of the ring can be compared with El Sofy's results in [1] for homoderivation on an ideal of the ring. Other examples can also be found.

In [4] and [5], it is proved that if f is a semi-derivation associated with function g of a prime ring $\wp$, then f is an ordinary derivation of $\wp$ or f satisfies that $f(x) = \lambda (1 - g)(x)$, for all $x \in \wp$, where λ is an element in the extended centroid of $\wp$ and g is an endomorphism of $\wp$. Since every homoderivation ϑ is a nonzero semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ and α is a homomorphism of $\wp$ without the condition of primeness of $\wp$, it is clear that homoderivation ϑ satisfies the result of Bresar and Chuang in case of $\lambda = -1$ without the condition of primeness of $\wp$.

It is shown in [6] that every semi-derivation $f : R \to R$ associated with a function g is both a $(1, g)$-derivation and a $(g, 1)$-derivation. Since every homoderivation ϑ is a semi-derivation associated with $\alpha = \vartheta + 1$, then ϑ is both a $(1, \alpha)$-derivation and $(\alpha, 1)$-derivation. To illustrate, let $\wp$ be a noncommutative ring, β be a $\wp$-module homomorphism, and α be a non-additive mapping of $\wp$. Assume that $f, g : \wp \times \wp \to \wp \times \wp$ are defined as $f((x, y)) = (0, \beta(y))$ and $g((x, y)) = (\alpha(x), 0)$, respectively. Then, f is both a $(1, g)$-derivation and $(g, 1)$-derivation where 1 is the identity mapping of $\wp$ but f is neither a semi-derivation associated with g of $\wp$ nor a homoderivation of $\wp$.

In the main section, it is proved that every homoderivation ϑ is a semi-derivation associated with α where $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is a homomorphism of $\wp$ but the converse is generally not true (see Examples 3.7-3.9). Thus, it is shown that a homoderivation ϑ is equivalent to the function given, for example, of semi-derivation by Bergen in [13]. It is clear that homoderivation ϑ satisfies the result of Bresar and Chuang in the case of $\lambda = -1$ without the condition of primeness of $\wp$.

2. Preliminaries

Let $\wp$ be a ring. Then, $\wp$ is called prime if a, b in $\wp$ such that $a\wp b = (0)$ implies that either a or b is zero. An additive mapping $\alpha : \wp \to \wp$ is called a homomorphism if $\alpha(uv) = \alpha(u)\alpha(v)$, for all $u, v \in \wp$. A mapping ϑ such that $\vartheta(A) \subseteq A$ is called a zero-power valued on A if there is a positive integer $n(a) > 1$ such that $\vartheta^{n(a)}(a) = 0$, for all $a \in A$.

In [14], derivation is defined on $\wp$ as follows: An additive mapping $d : \wp \to \wp$ is called a derivation if $d(uv) = d(u)v + ud(v)$, for all $u, v \in \wp$. Semi-derivation [13] is defined on $\wp$ as follows: An additive mapping ϑ is called a semi-derivation if there is a function $\alpha : \wp \to \wp$ such that

i. $\vartheta(uv) = \vartheta(u)\alpha(v) + u\vartheta(v) = \vartheta(u)v + \alpha(u)\vartheta(v)$, for all $u, v \in \wp$

ii. $\vartheta(\alpha(u)) = \alpha(\vartheta(u))$, for all $u \in \wp$

It is clear that any derivation is a semi-derivation associated with 1 which is the identitiy mapping of $\wp$. Conversely, in the the same article, it is shown by Bergen that $\alpha : \wp \to \wp$ such that $\alpha \neq 1$ is a homomorphism, $\vartheta = \alpha - 1$ is a semi-derivation which is not a derivation of $\wp$. $(1, g)$-derivation and $(g, 1)$-derivation are defined on $\wp$ in [6] as follows, respectively: An additive mapping ϑ is called a $(1, g)$-derivation if there is a function $g : \wp \to \wp$ such that $\vartheta(uv) = \vartheta(u)v + g(u)\vartheta(v)$, for all $u, v \in \wp$, and an additive mapping ϑ is called a $(g, 1)$-derivation if there is a function $g : \wp \to \wp$ such that $\vartheta(uv) = \vartheta(u)g(v) + u\vartheta(v)$, for all $u, v \in \wp$. The definition of homoderivation is introduced in [1] as follows: An additive mapping $\vartheta : \wp \to \wp$ is a homoderivation if $\vartheta(uv) = \vartheta(u)\vartheta(v) + \vartheta(u)v + u\vartheta(v)$, for all $u, v \in \wp$.

Lemma 2.1. [2] Let $\wp$ be a prime ring. If f is a nonzero semi-derivation of a prime ring $\wp$ associated with a not necessarily surjective function g, then g is a homomorphism of $\wp$.

Lemma 2.2. [1] Let ϑ be a homoderivation of $\wp$. Then, $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is an endomorphism of $\wp$.

3. Main Results

Unless otherwise stated throughout this paper, $\wp$ is a noncommutative ring, and 1 is the identity mapping of $\wp$.

Lemma 3.1. Let ϑ be an additive mapping of $\wp$. Then, $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is an additive mapping and $\alpha\vartheta = \vartheta\alpha$.

Proof. Assume that $\alpha : \wp \to \wp$ is defined as $\alpha = \vartheta + 1$ where ϑ is an additive mapping of $\wp$. Therefore, it holds that, for all $u, v \in \wp$

$$\begin{aligned} \alpha(u+v) &= (\vartheta + 1)(u+v) \\ &= \vartheta(u+v) + u + v \\ &= \vartheta(u) + \vartheta(v) + u + v \\ &= \alpha(u) + \alpha(v) \end{aligned}$$

which implies that α is an additive mapping of $\wp$. Moreover,

$$\begin{aligned} (\alpha\vartheta)(u) &= \alpha(\vartheta(u)) \\ &= \vartheta(\vartheta(u)) + \vartheta(u) \\ &= \vartheta^2(u) + \vartheta(u) \end{aligned}$$

and using that ϑ is an additive mapping

$$(\vartheta\alpha)(u) = \vartheta(\alpha(u)) = \vartheta(\vartheta(u) + u) = \vartheta(\vartheta(u)) + \vartheta(u) = \vartheta^2(u) + \vartheta(u)$$

for all $u \in \wp$. Thus, it implies that

$$(\alpha\vartheta)(u) = (\vartheta\alpha)(u)$$

for all $u \in \wp$, which means that

$$\alpha\vartheta = \vartheta\alpha$$

□

Theorem 3.2. Let ϑ be an additive mapping of $\wp$ and $\alpha : \wp \to \wp$ be defined as $\alpha = \vartheta + 1$ where 1 is the identity mapping of $\wp$. Assume that $(A, +)$ is a nonzero subgroup of $\wp$.

i. If ϑ is a zero-power valued on A, then α is a surjective mapping on A

ii. If ϑ is a zero-power valued on A and $\ker\alpha \subset A$, then α is an injective mapping of $\wp$

PROOF. *i.* Assume that ϑ is an additive and a zero-power valued mapping on A. Then, $\vartheta(A) \subset A$ and there exists a positive integer $n(u) > 1$ such that $\vartheta^{n(u)}(u) = 0$, for all $u \in A$. Since ϑ satisfies that $\vartheta(A) \subset A$, it holds that, for all $u \in A$,

$$\alpha(u) = \vartheta(u) + u \subset A$$

that is,

$$\alpha(A) \subset A$$

For all $u \in A$,

$$\begin{aligned}\alpha(u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u)) =&\vartheta(u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u)) \\ &+ u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u) \\ =&\vartheta(u) - \vartheta^2(u) + \vartheta^3(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)}(u) \\ &+ u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u) \\ =&u + (-1)^{n(u)-1}\,\vartheta^{n(u)}(u)\end{aligned}$$

Thus,

$$\alpha(u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u)) = u + (-1)^{n(u)-1}\,\vartheta^{n(u)}(u)$$

for all $u \in A$. Using that ϑ is a zero-power valued mapping of A, it implies that, for all $u \in A$,

$$\alpha(u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u)) = u$$

Since there exists at least one $a = u - \vartheta(u) + \vartheta^2(u) + \cdots + (-1)^{n(u)-1}\,\vartheta^{n(u)-1}(u) \in A$ such that $\alpha(a) = u$, for all $u \in A$. Then, it is obtained that α is a surjective mapping on A.

ii. Let ϑ is a zero-power valued on A and $\ker\alpha \subset A$. Assume that $a \in \ker\alpha \subset A$. Then, it holds that

$$\begin{aligned}\alpha(a) = 0 &\Rightarrow \vartheta(a) + a = 0 \\ &\Rightarrow \vartheta(a) = -a\end{aligned}$$

Thus,

$$\vartheta(a) = -a$$

which means that

$$\vartheta(\ker\alpha) \subset \ker\alpha \tag{3.1}$$

Moreover, using that ϑ is an additive mapping of $\wp$

$$\vartheta^2(a) = \vartheta\,(\vartheta(a)) = \vartheta(-a) = -\vartheta(a) = -(-a) = a$$
$$\vartheta^3(a) = \vartheta\left(\vartheta^2(a)\right) = \vartheta(a) = -a$$
$$\vartheta^4(a) = \vartheta\left(\vartheta^3(a)\right) = \vartheta(-a) = -(\vartheta(a)) = -(-a) = a$$
$$\vdots$$
$$\vartheta^{n(a)}(a) = (-1)^{n(a)}a$$

for all $a \in \ker\alpha$. Thus, it follows that, for all $a \in \ker\alpha$,

$$\vartheta^{n(a)}(a) = (-1)^{n(a)}a \tag{3.2}$$

Since ϑ is a zero-power valued mapping on A and $\ker\alpha \subset A$, there is a positive integer $n(u) > 1$ such that $\vartheta^{n(u)}(u) = 0$, for all $u \in \ker\alpha$. In addition, it holds that $\vartheta(\ker\alpha) \subset \ker\alpha$ from (3.1) which means that ϑ is a zero-power valued mapping on $\ker\alpha$. Therefore, it follows from (3.2), that for all $u \in \ker\alpha$, a positive integer $n(u) > 1$

$$\vartheta^{n(u)}(u) = (-1)^{n(u)}u = 0$$

Then, it is obtained that $u = 0$, for all $u \in \ker\alpha$, which implies that $\ker\alpha = (0)$. Thus, α is an injective mapping of $\wp$. □

Corollary 3.3. Let ϑ be an additive mapping of $\wp$. If ϑ is a zero-power valued on $\wp$, then $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is a bijective mapping of $\wp$.

The proof is clear from Theorem 3.2.

Theorem 3.4. If $\vartheta : \wp \to \wp$ is a homoderivation, then ϑ is a semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$.

Proof. Let $\vartheta : \wp \to \wp$ be a homoderivation. If the definition of homoderivation is rearranged by using that $\alpha = \vartheta + 1$, it holds that, for all $u, v \in \wp$,

$$\begin{aligned}\vartheta(uv) &= \vartheta(u)\vartheta(v) + \vartheta(u)v + u\vartheta(v)\\ &= \vartheta(u)(\vartheta(v) + v) + u\vartheta(v)\\ &= \vartheta(u)\alpha(v) + u\vartheta(v)\end{aligned}$$

and

$$\begin{aligned}\vartheta(uv) &= \vartheta(u)\vartheta(v) + \vartheta(u)v + u\vartheta(v)\\ &= \vartheta(u)v + (\vartheta(u) + u)\vartheta(v)\\ &= \vartheta(u)v + \alpha(u)\vartheta(v)\end{aligned}$$

Thus, ϑ is written as, for all $u, v \in \wp$,

$$\vartheta(uv) = \vartheta(u)\alpha(v) + u\vartheta(v) = \vartheta(u)v + \alpha(u)\vartheta(v) \tag{3.3}$$

In addition, it implies that from Lemma 3.1,

$$(\alpha\vartheta)(u) = (\vartheta\alpha)(u), \text{ for all } u \in \wp \tag{3.4}$$

Thus, (3.3) and (3.4) mean that ϑ is a semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$. □

It is shown in the next corollary that Lemma 2.1 is satisfied without the condition of primeness of ring $\wp$.

Corollary 3.5. If $\vartheta : \wp \to \wp$ is a homoderivation, then ϑ is a semi-derivation associated with α where $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is a homomorphism of $\wp$.

The proof is clear from Lemma 2.2 and Theorem 3.4.

Corollary 3.6. Let ϑ be a zero-power valued homoderivation on $\wp$. Then, ϑ is a semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is an automorphism of $\wp$.

Proof. Assume that ϑ is a zero-power valued homoderivation on $\wp$. ϑ is a semi-derivation associated with α where $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is a homomorphism of $\wp$ from Corollary 3.5. Thus, α is a bijective mapping of $\wp$ from Theorem 3.2 which means that α is an automorphism of $\wp$. Hence, ϑ is a semi-derivation associated with $\alpha : \wp \to \wp$ such that $\alpha = \vartheta + 1$ is an automorphism of $\wp$. $\square$

Every homoderivation is a semi-derivation, but it is possible to find examples that are semi-derivation but not homoderivation:

Example 3.7. Let $\wp_1$ and $\wp_2$ be two rings. Let $\wp = \wp_1 \times \wp_2$ be a ring with the operations such that

$$(p_1, p_2) + (a_1, a_2) = (p_1 + a_1, p_2 + a_2)$$

and

$$(p_1, p_2)(a_1, a_2) = (p_1 a_1, p_2 a_2)$$

for all $(p_1, p_2), (a_1, a_2) \in \wp$. If $f : \wp \to \wp$ is defined as $f((u_1, u_2)) = (0, u_2)$ and $g : \wp \to \wp$ is defined as $g((u_1, u_2)) = (u_1, 0)$, for all $(u_1, u_2) \in \wp$, then f is a semi-derivation of $\wp$ associated with function g but f is not a homoderivation of $\wp$.

Let $p = (p_1, p_2)$ and $a = (a_1, a_2)$ be elements of $\wp$. It holds that

$$f(pa) = f((p_1, p_2)(a_1, a_2)) = f((p_1 a_1, p_2 a_2)) = (0, p_2 a_2) \tag{3.5}$$

Moreover,

$$\begin{aligned} f(p)g(a) + pf(a) &= f((p_1, p_2))g((a_1, a_2)) + (p_1, p_2)f((a_1, a_2)) \\ &= (0, p_2)(a_1, 0) + (p_1, p_2)(0, a_2) \\ &= (0, 0) + (0, p_2 a_2) \\ &= (0, p_2 a_2) \end{aligned}$$

and

$$\begin{aligned} f(p)a + g(p)f(a) &= f((p_1, p_2))(a_1, a_2) + g((p_1, p_2))f((a_1, a_2)) \\ &= (0, p_2)(a_1, a_2) + (p_1, 0)(0, a_2) \\ &= (0, p_2 a_2) + (0, 0) \\ &= (0, p_2 a_2) \end{aligned}$$

for all $p = (p_1, p_2), a = (a_1, a_2) \in \wp$. Thus, it implies that, for $p, a \in \wp$,

$$f(pa) = f(p)g(a) + pf(a) = f(p)a + g(p)f(a) \tag{3.6}$$

Besides, it holds that, for all $p = (p_1, p_2) \in \wp$,

$$(fg)(p) = (fg)((p_1, p_2)) = f(g((p_1, p_2))) = f((p_1, 0)) = (0, 0)$$

and

$$(gf)(p) = (gf)((p_1, p_2)) = g(f((p_1, p_2))) = g((0, p_2)) = (0, 0)$$

Therefore, it means that, for all $p \in \wp$,

$$(fg)(p) = (gf)(p)$$

Thus,

$$fg = gf \tag{3.7}$$

(3.6) and (3.7) imply that f is a semi-derivation associated with function g of $\wp$. But

$$\begin{aligned} f(p)f(a)+f(p)a+pf(a) &= f((p_1,p_2))f\left((a_1,a_2)\right)+f((p_1,p_2))(a_1,a_2)+(p_1,p_2)f((a_1,a_2)) \\ &= (0,p_2)(0,a_2)+(0,p_2)(a_1,a_2)+(p_1,p_2)(0,a_2) \\ &= (0,p_2a_2)+(0,p_2a_2)+(0,p_2a_2) \\ &= (0,3p_2a_2) \end{aligned}$$

for all $p=(p_1,p_2), a=(a_1,a_2)\in\wp$. Hence, it holds that, for $p=(p_1,p_2), a=(a_1,a_2)\in\wp$,

$$f(p)f(a)+f(p)a+pf(a)=(0,3p_2a_2) \tag{3.8}$$

(3.5) and (3.8) imply that, for all $p,a\in\wp$,

$$f(pa)\neq f(p)f(a)+f(p)a+pf(a)$$

which means that f is not a homoderivation of $\wp$.

Example 3.8. Let $\wp$ be a ring and $\vartheta:\wp\times\wp\to\wp\times\wp$ be a mapping such that $\vartheta\left((u,v)\right)=(u,0)$ and $\alpha:\wp\times\wp\to\wp\times\wp$ be a mapping such that $\alpha\left((u,v)\right)=(0,v)$, for all $(u,v)\in\wp\times\wp$. Then, ϑ is a semi-derivation associated with the function α but ϑ is not a homoderivation of $\wp\times\wp$.

Let $p=(p_1,p_2)$ and $a=(a_1,a_2)$ be elements of $\wp\times\wp$. It holds that

$$\vartheta(pa)=\vartheta\left((p_1,p_2)(a_1,a_2)\right)=\vartheta\left((p_1a_1,p_2a_2)\right)=(p_1a_1,0) \tag{3.9}$$

Besides,

$$\begin{aligned} \vartheta(p)\alpha(a)+p\vartheta(a) &= \vartheta((p_1,p_2))\alpha\left((a_1,a_2)\right)+(p_1,p_2)\vartheta((a_1,a_2)) \\ &= (p_1,0)(0,a_2)+(p_1,p_2)(a_1,0) \\ &= (0,0)+(p_1a_1,0) \\ &= (p_1a_1,0) \end{aligned}$$

and

$$\begin{aligned} \vartheta(p)a+\alpha(p)\vartheta(a) &= \vartheta((p_1,p_2))(a_1,a_2)+\alpha((p_1,p_2))\vartheta((a_1,a_2)) \\ &= (p_1,0)(a_1,a_2)+(0,p_2)(a_1,0) \\ &= (p_1a_1,0)+(0,0) \\ &= (p_1a_1,0) \end{aligned}$$

for all $p=(p_1,p_2)$ and $a=(a_1,a_2)\in\wp\times\wp$. Thus, it implies that, for $p,a\in\wp\times\wp$,

$$\vartheta(pa)=\vartheta(p)\alpha(a)+p\vartheta(a)=\vartheta(p)a+\alpha(p)\vartheta(a) \tag{3.10}$$

Moreover, it holds that, for all $p=(p_1,p_2)\in\wp\times\wp$,

$$(\vartheta\alpha)(p)=(\vartheta\alpha)\left((p_1,p_2)\right)=\vartheta\left(\alpha((p_1,p_2))\right)=\vartheta\left((0,p_2)\right)=(0,0)$$

and

$$(\alpha\vartheta)(p)=(\alpha\vartheta)((p_1,p_2))=\alpha\left(\vartheta((p_1,p_2))\right)=\alpha\left((p_1,0)\right)=(0,0)$$

Therefore, it means that, for all $p\in\wp\times\wp$,

$$(\vartheta\alpha)(p)=(\alpha\vartheta)(p)$$

Thus,

$$\vartheta\alpha = \alpha\vartheta \tag{3.11}$$

(3.10) and (3.11) imply that ϑ is a semi-derivation associated with function α of $\wp \times \wp$. But

$$\begin{aligned}\vartheta(p)\vartheta(a) + \vartheta(p)a + p\vartheta(a) &= \vartheta((p_1,p_2))\vartheta\left((a_1,a_2)\right) + \vartheta((p_1,p_2))(a_1,a_2) + (p_1,p_2)\vartheta((a_1,a_2))\\ &= (p_1,0)(a_1,0) + (p_1,0)(a_1,a_2) + (p_1,p_2)(a_1,0)\\ &= (p_1a_1,0) + (p_1a_1,0) + (p_1a_1,0)\\ &= (3p_1a_1,0)\end{aligned}$$

for all $p = (p_1,p_2), a = (a_1,a_2) \in \wp \times \wp$. Hence, it holds that, for $p = (p_1,p_2), a = (a_1,a_2) \in \wp \times \wp$,

$$\vartheta(p)\vartheta(a) + \vartheta(p)a + p\vartheta(a) = (3p_1a_1,0) \tag{3.12}$$

(3.9) and (3.12) imply that, for $p, a \in \wp \times \wp$,

$$\vartheta(pa) \neq \vartheta(p)\vartheta(a) + \vartheta(p)a + p\vartheta(a)$$

which means that ϑ is not a homoderivation of $\wp \times \wp$.

Example 3.9. Let $\wp$ be a ring and $d : \wp \to \wp$ be a derivation. Let $f : \wp \times \wp \to \wp \times \wp$ be a mapping such that $f\left((u,v)\right) = (d(u),0)$ and $\sigma : \wp \times \wp \to \wp \times \wp$ be a mapping such that $\sigma\left((u,v)\right) = (u,v)$, for all $(u,v) \in \wp \times \wp$. Then, f is a semi-derivation associated with function σ but f is not a homoderivation of $\wp \times \wp$.

Assume that d is a derivation of $\wp$. Let $p = (p_1,p_2), a = (a_1,a_2) \in \wp \times \wp$. It holds that

$$f(pa) = f\left((p_1,p_2)(a_1,a_2)\right) = f((p_1a_1,p_2a_2)) = (d(p_1a_1),0) \tag{3.13}$$

Further, using that d is a derivation

$$\begin{aligned}f(p)\sigma(a) + pf(a) &= f((p_1,p_2))\sigma\left((a_1,a_2)\right) + (p_1,p_2)f((a_1,a_2))\\ &= (d(p_1),0)(a_1,a_2) + (p_1,p_2)(d(a_1),0)\\ &= (d(p_1)a_1,0) + (p_1d(a_1),0)\\ &= (d(p_1)a_1 + p_1d(a_1),0)\\ &= (d(p_1a_1),0)\end{aligned}$$

and

$$\begin{aligned}f(p)a + \sigma(p)f(a) &= f((p_1,p_2))(a_1,a_2) + \sigma((p_1,p_2))f((a_1,a_2))\\ &= (d(p_1),0)(a_1,a_2) + (p_1,p_2)(d(a_1),0)\\ &= (d(p_1)a_1,0) + (p_1d(a_1),0)\\ &= (d(p_1)a_1 + p_1d(a_1),0)\\ &= (d(p_1a_1),0)\end{aligned}$$

for all $p = (p_1,p_2), a = (a_1,a_2) \in \wp \times \wp$. Thus, it implies that, for $p, a \in \wp \times \wp$,

$$f(pa) = f(p)\sigma(a) + pf(a) = f(p)a + \sigma(p)f(a) \tag{3.14}$$

Moreover, it holds that, for all $p = (p_1, p_2) \in \wp \times \wp$,

$$(f\sigma)(p) = (f\sigma)\left((p_1, p_2)\right) = f\left(\sigma((p_1, p_2))\right) = f\left((p_1, p_2)\right) = (d(p_1), 0)$$

and

$$(\sigma f)(p) = (\sigma f)((p_1, p_2)) = \sigma(f((p_1, p_2))) = \sigma\left((d(p_1), 0)\right) = (d(p_1), 0)$$

Therefore, it means that, for all $p \in \wp \times \wp$,

$$(f\sigma)(p) = (\sigma f)(p)$$

Thus,

$$f\sigma = \sigma f \tag{3.15}$$

(3.14) and (3.15) imply that f is a semi-derivation associated with function σ of $\wp \times \wp$. But

$$\begin{aligned} f(p)f(a) + f(p)a + pf(a) &= f((p_1, p_2))f\left((a_1, a_2)\right) + f((p_1, p_2))(a_1, a_2) + (p_1, p_2)f((a_1, a_2)) \\ &= (d(p_1), 0)(d(a_1), 0) + (d(p_1), 0)(a_1, a_2) + (p_1, p_2)(d(a_1), 0) \\ &= (d(p_1)d(a_1), 0) + (d(p_1)a_1, 0) + (p_1 d(a_1), 0) \\ &= (d(p_1)d(a_1) + d(p_1)a_1 + p_1 d(a_1), 0) \end{aligned}$$

for all $p = (p_1, p_2), a = (a_1, a_2) \in \wp \times \wp$. Thus, it holds that, for all $p = (p_1, p_2), a = (a_1, a_2) \in \wp \times \wp$,

$$f(p)f(a) + f(p)a + pf(a) = (d(p_1)d(a_1) + d(p_1)a_1 + p_1 d(a_1), 0) \tag{3.16}$$

(3.13) and (3.16) imply that, for all $p, a \in \wp \times \wp$,

$$f(pa) \neq f(p)f(a) + f(p)a + pf(a)$$

which means that f is not a homoderivation of $\wp \times \wp$.

These examples explain that the definition of semi-derivation is more general than homo-derivation on a ring.

Theorem 3.10. If $\vartheta : \wp \to \wp$ such that $\vartheta = \alpha - 1$ is a semi-derivation associated with α where $\alpha : \wp \to \wp$ is a homomorphism of $\wp$, then ϑ is a homoderivation of $\wp$.

PROOF. Let $\vartheta : \wp \to \wp$ such that $\vartheta = \alpha - 1$ be a semi-derivation associated with α where $\alpha : \wp \to \wp$ is a homomorphism of $\wp$. Then, it holds that, for all $u, v \in \wp$,

$$\vartheta(uv) = \vartheta(u)\alpha(v) + u\vartheta(v) = \vartheta(u)v + \alpha(u)\vartheta(v)$$

Since α satisfies that $\alpha = \vartheta + 1$, it is obtained that, for all $u, v \in \wp$,

$$\vartheta(uv) = \vartheta(u)\left(\vartheta + 1\right)(v) + u\vartheta(v) = \vartheta(u)v + \left(\vartheta + 1\right)(u)\vartheta(v)$$

which implies that, for all $u, v \in \wp$,

$$\vartheta(uv) = \vartheta(u)\vartheta(v) + \vartheta(u)v + u\vartheta(v)$$

That is, ϑ is a homoderivation of $\wp$. □

Corollary 3.11. ϑ is a homoderivation of $\wp$ if and only if $\vartheta : \wp \to \wp$ such that $\vartheta = \alpha - 1$ is a semi-derivation associated with α where $\alpha : \wp \to \wp$ is a homomorphism of $\wp$.

The proof is clear from Theorems 3.4 and 3.10.

4. Conclusion

The fact that every homoderivation ϑ of $\wp$ is a semi-derivation associated with $\alpha = \vartheta + 1$ and α is a homomorphism of an arbitrary ring $\wp$ shows that some results can be achieved without needing the primeness of the ring or being surjective of the associated function. Examples of this situation are provided in the introduction. Based on the examples, the reader may be advised to think about the relationship between homoderivation and semi-derivation ϑ such that $\vartheta = \alpha - 1$. Semi-derivation evidences typically require to be surjective of the ring's primeness or the related function. Although a function that is the form of $\vartheta = \alpha - 1$ is a semi-derivation associated with homomorphism α, there may be no need for being surjective of the associated function or the primeness of the ring while doing the proof. It is proved in this paper that the definition of homoderivation is equivalent to the definition of semi-derivation, the form of $\vartheta = \alpha - 1$ where $\alpha : \wp \to \wp$ such that $\alpha \neq 1$ is a homomorphism. Therefore, while generalizing the problems in the literature for homoderivation, using homoderivation is the same as using the semi-derivation, the form of $\vartheta = \alpha - 1$ associated with homomorphism α.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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