MSU Fen Bil. Dergi., Cilt 5 , Sayı 2, s 421-423 *Araştırma Makalesi/ Research Article*
MSU J. of Sci., Volume 5 , Issue 2, p 421-423 DOI : 10.18586/msufbd.309725

Approximation To Generalized Taylor Derivatives By Integral Operator Families

Lutfi AKIN[1], Yusuf ZEREN[2]
[1]Department of Business Administration, Faculty of Economics and Administrative Sciences, Mardin Artuklu University, Mardin, Turkey
[2]Department of Mathematics, Faculty of Sciences, Yildiz Technical University, Istanbul, Turkey
✉: lutfiakin@artuklu.edu.tr

Received (Geliş):29.04.2017 Revision (Düzeltme):14.06.2017 Accepted (Kabul): 25.08.2017

ABSTRACT

This theory has important applications of polynomial approximation in various areas of functional analysis, Fourier analysis, application mathematic, operator theory, in the field generalized derivatives and numerical solutions of differential and integral equations, etc. The study of approximation theory is a well-established area of research which deals with the problem of approximating a function f by means of a sequence L_n of positive linear operators. This theory is very important for mathematical world. Nowadays, many mathematicians are working in this field.

Keywords: Taylor derivative, Kernel function, Differentiable function, Operator theory

İntegral Operatör Aileleri ile Genelleştirilmiş Taylor Türevlerine Yaklaşım

ÖZ

Yaklaşımlar teorisi, diferansiyel ve integral denklemlerin nümerik çözümlerinde, fonksiyonel analizin çeşitli bölgelerindeki uygulamalarda, Fourier analizde, uygulamalı matematik alanında, operatör teoride, genelleştirilmiş türevler alanında ve polinomsal yaklaşımın uygulamalarında vs. çok önemli bir yere sahiptir. Bu çalışma L_n pozitif lineer operatör dizileri vasıtası ile bir f fonksiyonuna yaklaşımın araştırılması problemidir. Yaklaşımlar teorisi, çalışma alanı için iyi kurulmuş bir teoridir. Bu teori matematik dünyası için çok önemli bir yere sahiptir. Günümüzde birçok matematikçi yaklaşımlar teorisi alanında çalışmaktadır.

Anahtar Kelimeler: Taylor türev, Çekirdek fonksiyonu, Türevlenebilir fonksiyon, Operatör teori

INTRODUCTION

We can see in many problems of theory functions and differential equations to integral operators with positive kernel. For example, some collection methods of the Fourier series are expressed with such integrals. Furthermore, the solution of the Dirichlet problem and the solution of the problem the limit value is given through of positive kernel integrals. Hence, using positive-core integral operators examination the problem of approach to generalized derivatives in terms of both theoretical and practical have great importance (see [1-11]).

$L_\lambda : f(t) \to L_\lambda(f,x)$ integral operators families and $K_\lambda(t-x)$ is operator's kernel.

$$L_\lambda(f,x) = \int_a^b f(t) K_\lambda(t-x)\,dt \qquad (1.1)$$

The main problems related to the convergence of such integral operatör families, it is as follows.

1. Convergence of $L_\lambda(f,x_0) \to f(x_0)$

Examining for $\lambda \to \lambda_0$ at a certain x_0 point.

2. When X is a normative linear space and $f \in X, L_\lambda : X \to X$,

$$\lim_{\lambda\to\lambda_0} \left\| L_\lambda f - f \right\|_X = 0$$

3. Finding the convergence speeds of problems 1 and 2, that is, when $\lambda \to \lambda_0$ and

$\alpha_\lambda, \beta_\lambda$ are zero series

$$\left|L_\lambda(f, x_0) - f(x_0)\right| = o(\alpha_\lambda),$$

$$L_n(f;x) \underset{\rightarrow}{\rightarrow} f(x), \quad a \le x \le b$$

condition is satisfied (see [2]).

Theorem 2.3. Let $f(x)$ function differentiable from the $(n-1)$ th order neighbourhood at a certain point x_0 and let $f(x)$ function has left and right derivative from n th order at a certain point x_0, $f_+^{(n)}(x_0)$, $f_-^{(n)}(x_0)$.

$1 \le \varphi(x) < \infty$ for $x \in (-\infty, \infty)$, $\left|f(x)\right| \le \varphi(x)$ inequality is provide. $K_\lambda(t)$ is non-negative and even function.

At the same time

$$\int_{-\infty}^{\infty} K_\lambda(t)dt = 1$$

Get for $\forall \delta > 0$,

$$\mu(t) = \sup_{\substack{-\infty < x < \infty \\ |y| < t}} \frac{\varphi(x+y)}{\varphi(x)} < \infty$$

while $\lambda \to \infty$ if the following equality is satisfied

$$\int_{\delta}^{\infty} \mu(\alpha^* t)\Phi(t)K_\lambda(t)dt = o(\Delta_\lambda)$$

then,

$$\lim_{\lambda \to +\infty} \frac{L_\lambda(f; x_0) - f(x_0)}{R_{n,\lambda}.\Delta_\lambda} = \frac{f_+^n(x_0) \pm f_-^n(x_0)}{n!}$$

equality is provided (see [3],[4]).

Definition 2.4. f is defined in some neighbourhood of the point x_0 and if the $(r-1)$th ordinary derivative $f^{r-1}(x_0)$ exists, then we call

$$f^{(r)}(x_0) = \lim_{h \to 0} \frac{r!}{h^r}\left[f(x_0+h) - \sum_{k=0}^{r-1} \frac{h^k}{k!} f^k(x_0)\right]$$

the r th Taylor derivative of f at x_0 if the limit exists (see [5]).

Lemma 2.5. If the first-order ordinary derivative $f'(x)$ exists, so does the first-order Taylor derivative $f^{(1)}(x)$, and we have $f'(x) = f^{(1)}(x)$. At the same time, the opposite of this lemma is true **(**see [5]).

Proof. If $f(x)$ function there is a first-order derivative at point x_0,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

this derivative is also the first-order Taylor derivative

$$f^{(1)}(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Therefore,

$$f'(x_0) = f^{(1)}(x_0)$$

Lemma 2.6. If the second-order ordinary derivative $f''(x)$ exists, so does the second-order Taylor derivative $f^{(2)}(x)$, and we have $f''(x) = f^{(2)}(x)$. $f(x)$ has second-order Taylor derivative at $x = x_0$, but there exists no second-order ordinary derivative $x = x_0$ (see [5]).

Proof. Taylor expansion at x_0 point of $f(x)$ function is below

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + ...$$

If x is replaced by $x_0 + h$

$$f(x_0+h) = f(x_0) + \frac{f'(x_0)}{1!}h + \frac{(f''(x_0)+\alpha)}{2!}h^2$$

$$h \to 0 \;, \quad \alpha \to 0$$

is obtained. From here,

$$f(x_0+h) - f(x_0) - f'(x_0)h = \frac{(f''(x_0)+\alpha)}{2!}h^2$$

$$\frac{2}{h^2}\left[f(x_0+h)-f(x_0)-f'(x_0)h\right]=\lim_{h\to 0}\left(f''(x_0)+\alpha\right)$$

since,

$$f^{(2)}(x_0)=f''(x_0)$$

is obtained ($f(x)$ has second-order Taylor derivative at $x=x_0$, but there exists no second-order ordinary derivative $x=x_0$, for example see [5]).

MAIN RESULT

Theorem 3.1. Let $f(x)$ function differentiable from the $(n-1)$ th order neighbourhood at a certain point x_0. And let $f(x)$ function has derivative from n th order at a certain point x_0 meaning of Taylor.

And let integral operator families, $L_\lambda(f;x_0)=\int_{-\infty}^{\infty} f(x_0+t)K_\lambda(t)dt$.

$$\Delta_\lambda=\int_0^\infty t^2K_\lambda(t)dt\to 0 \quad , \ (\lambda\to\infty)$$

For $K_\lambda(t)$ kernel function,

i. $K_\lambda(t)>0$, $\lambda\geq 0$

ii. $K_\lambda(-t)=K_\lambda(t)$

iii. $\int_{-\infty}^{\infty}K_\lambda(t)dt=1$

$$\int_{-\infty}^{\infty}K_\lambda(t)dt=2\int_0^\infty K_\lambda(t)dt$$

$$\lim_{\lambda\to\infty}\frac{L_\lambda(f,x_0)-f(x_0)}{\Delta_\lambda}=f^{(2)}(x_0)$$

equality is provide..

Proof. Let

$$L_\lambda(f,x_0)-f(x_0)$$

$$=\int_0^\infty\left[f(x_0+t)-2f(x_0)\right]K_\lambda(t)dt=$$

$$=\int_0^\infty\left[f(x_0+t)-f(x_0)-tf'(x_0)+tf'(x_0)\right]K_\lambda(t)dt$$

$$-\int_0^\infty f(x_0)K_\lambda(t)dt$$

$$=\int_0^\infty\left[f(x_0+t)-f(x_0)-tf'(x_0)\right]K_\lambda(t)dt+$$

$$\int_0^\infty tf'(x_0)K_\lambda(t)dt-\int_0^\infty f(x_0)K_\lambda(t)dt \qquad (3.1)$$

Now, define a function as follows

$$\alpha_\lambda(t)=[f(x_0+t)-f(x_0)-tf'(x_0)]-t^2f^{(2)}(x_0)$$

(3.2)

From here, $\alpha_\lambda(t)\to 0$ while $t\to 0$

$$\left[f(x_0+t)-f(x_0)-tf'(x_0)\right]=\alpha_\lambda(t)+t^2f^{(2)}(x_0)$$

is obtained. If $f(x_0+t)-f(x_0)-tf'(x_0)$ is replaced by $\alpha_\lambda(t)+t^2f^{(2)}(x_0)$ above (3.1)

$$L_\lambda(f,x_0)-f(x_0)=\int_0^\infty\alpha_\lambda(t)K_\lambda(t)dt+f^{(2)}(x_0)\int_0^\infty t^2K_\lambda(t)dt+$$

$$+f'(x_0)\int_0^\infty tK_\lambda(t)dt-f(x_0)\int_0^\infty K_\lambda(t)dt$$

is obtained. Thus the proof is completed.

ACKNOWLEDGEMENT

The authors expressed their gratitude to the referee for helpful remarks. This work is supported by MAÜ-BAP-17-İİBF-07

REFERENCES

[1] Altomare F., Campiti M. Korovkin Type Approximation Theory and its Application, Walter de Gruyter, Berlin and NewYork. 1994.

[2] Korovkin P.P. Linear Positive Operators and Approximation Theory. Hindistan Publishing Corp. Hindustan. p. 222. 1960.

[3] Gadjiev A.D., Djifarov A.S., Labsker L.G. On Asymptotic Value of Approximation of Functions By Certain Families of Integral Operators, Izvestiya Acad. Sci. of The Azerbaijan., 3, 19-28, 1962.

[4] Gadjiev A.D. On Asymptotic Values of Approximation of Derivatives of functions By Derivatives of Families of Linear Operators. Izvestiya Acad. of Sci. of Azerbaijan, 6, 15-24, 1962.

[5] Butzer P.L., Nessel R.J. Fourier Analysis and Approximation, Academic Pres, New York and London. p. 555, 1971.

[6] Akin L. “Approximation to Generalized Derivatives by Integral Operator Families”, Harran University, Institute of Science, Master Thesis, Sanliurfa, p. 55, 2008.

[7] Butzer P.L. Representation and approximation of functions by general singular integrals. Ia. In: Indagationes Mathematicae (Proceedings). North-Holland, Nederland. 63, 1-2, 1960.

[8] Hacısalihoğlu H., Haciyev A. Convergence of Linear Positive Operator Sequences, Ankara, s. 100, 1995.

[9] Stein E.M. Singular Integrals and Differentiability Properties of Functions, Princeton Univ. The United States of America. p. 289, 1993.

[10] Zeren Y. Approximation of Schwartz Differentiable Functions of Several Variables By The Sequence of Integral Operators. Khazar Journal of mathematics, 1: 73-78. 2006.

[11] Zeren Y Asymptotic Value of Approximation of Derivatives Approximated by Integral Operator Family. Ankara University, Institute of Science, Ph.D. Tesis Ankara, p. 50, 2002.