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Fifth Order Predictor-Corrector Method for Quadratic Riccati Differential Equations

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Abstract

In this paper, fifth order predictor-corrector method is presented for solving quadratic Riccati differential equations. First, the interval is discretized and then the method is formulated by using the Newton's backward difference interpolation formula. The stability and convergence of the method have been investigated. To validate the applicability of the proposed method, three model examples with exact solutions have been considered and numerically solved by using MATLAB software. The numerical results are presented in tables and figures for different values of mesh size h. Pointwise absolute errors and maximum absolute errors are also estimated. Concisely, the present method gives better result than some existing numerical methods reported in the literature.

Keywords: Predictor-corrector method, Riccati differential equations, Stability analysis

1. Introduction

Problems arising in many physical phenomena, engineering and scientific applications are modeled with nonlinear differential equations. Riccati differential equation is applicable in engineering science applications such as stochastic realization theory, optimal control, robust stabilization, network synthesis, diffusion problems, and the newer applications include such areas as financial mathematics, Allahviranlooa and Behzadib [1]. The solution of Riccati differential equations has so much importance in numerical methods due to the fact that even higher order partial differential equations can be transformed into first order ordinary differential equation and solved, Baba [2]. Thus, methods of solution of such differential equations have attracted attention of researchers for a very long time.

Recently, some researchers have been tried to find the approximate solution of first order nonlinear ordinary differential equations. For example; Gashu and Habtamu [3] considered the comparison of higher order Taylor's method and fifth order Runge-Kutta method; Gemechis and Tesfaye [4] presented fourth order Runge-Kutta for solving quadratic Riccati differential equations; Vinod and Dimple [5] presented Newton-Raphson based modified Laplace Adomian decomposition method for solving quadratic Riccati differential equations;

Fateme and Esmaile [6] presented approximate solution for quadratic Riccati differential equations. Other authors [7-13] also discussed the quadratic Riccati differential equations. However, some of these methods are not more accurate. Thus in this paper, we present a stable and more accurate numerical method for solving quadratic Riccati differential equations.

2. Description of the Methods

Consider the quadratic Riccati differential equation of the form:

$$\frac{dy}{dx} = p(x) + q(x)\,y(x) + r(x)y^2(x) \tag{1}$$

with initial condition,

$$y(x_0) = \alpha \tag{2}$$

where, $p(x), q(x), r(x)$ are continuous functions, $r(x) \neq 0$ and α is arbitrary constant.
To describe the methods, denote Eq. (1) as:

$$\frac{dy}{dx} = f(x, y) \tag{3}$$

Now, divide the interval $[x_0, L]$ into N equal subintervals of mesh length h and the mesh points given by $x_i = x_0 + ih,\ i = 1, 2, \ldots, N.$ Then, $h = \frac{L - x_0}{N}$, where N is a positive integer. Integrating Eq. (3) on the interval $[x_i,\ x_{i+1}]$, we obtain:

$$\int_{x_i}^{x_{i+1}} \frac{dy}{dx}\,dx = \int_{x_i}^{x_{i+1}} f(x, y)\,dx$$

$$\Rightarrow y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(x, y)\,dx \tag{4}$$

To derive the methods, we approximate $f(x, y)$ by a suitable interpolation polynomials.

2.1. Description of Predictor Method

Let us take k data values (x_i, f_i), $(x_{i-1}, f_{i-1}), \ldots, (x_{i-k+1}, f_{i-k+1})$. For this data, we fit the Newton's backward difference interpolating polynomial of degree $k-1$ and we get:

$$p_{k-1}(x) = f(x_i + sh) = f(x_i) + s\nabla f(x_i) + \frac{s(s+1)}{2!}\nabla^2 f(x_i) + \ldots + \frac{s(s+1)(s+2)\ldots(s+k-2)}{(k-1)!}\nabla^{k-1} f(x_i) + T_k^p \quad (5)$$

where

$$s = \frac{x - x_i}{h} \text{ and } T_k^p = \frac{s(s+1)(s+2)\ldots(s+k-1)}{k!} h^k f^{(k)}(\xi) \quad (6)$$

is the error term, when ξ lies in some interval containing the points x_i, $x_{i-1}, \ldots, x_{i-k+1}$ and x. The limits of integration in Eq. (4) becomes:

$$x = x_i \Rightarrow s = 0,\ x = x_{i+1} \Rightarrow s = 1 \text{ and } dx = hds. \quad (7)$$

Thus, replacing $f(x, y)$ by $P_{k-1}(x)$ in Eq. (4) and using Eq. (5) in Eq. (4), we get:

$$y_{i+1} = y_i + h\int_0^1 \left\{ f_i + s\nabla f_i + \frac{1}{2}s(s+1)\nabla^2 f_i + \frac{1}{6}s(s+1)(s+2)\nabla^3 f_i + \ldots \right\} ds \quad (8)$$

Now, on integrating term by term in Eq. (8) with respect to s, we obtain:

$$\int_0^1 sds = \frac{1}{2}, \quad \int_0^1 s(s+1)ds = \frac{5}{6}, \quad \int_0^1 s(s+1)(s+2)ds = \frac{9}{4},$$

$$\int_0^1 s(s+1)(s+2)(s+3)ds = \frac{251}{30}, \quad \int_0^1 s(s+1)(s+2)(s+3)(s+4)ds = \frac{475}{12}$$

Hence, from Eq. (8), we get:

$$y_{i+1} = y_i + h\left\{ f_i + \frac{1}{2}\nabla f_i + \frac{5}{12}\nabla^2 f_i + \frac{3}{8}\nabla^3 f_i + \frac{251}{720}\nabla^4 f_i + \frac{475}{1440}\nabla^5 f_i + \ldots \right\} \quad (9)$$

We obtain the error term as:

$$T_k^p = h^{k+1}\int_0^1 \frac{s(s+1)\ldots(s+k-1)}{(k)!} f^{(k)}(\xi_k)ds = h^{k+1}\int_0^1 g_p(s)\, f^{(k)}(\xi_k)ds, \quad 0<\xi_k<1 \tag{10}$$

where

$$g_p(s) = \frac{1}{k!}\big(s(s+1)(s+2)\ldots(s+k-1)\big) \tag{11}$$

By choosing different values for *k*, we get different methods. But for this particular study, we choose the value for $k=5$ which is of order five method. Thus, we get:

$$\begin{aligned} y_{i+1} &= y_i + h\left[f_i + \frac{1}{2}\nabla f_i + \frac{5}{12}\nabla^2 f_i + \frac{3}{8}\nabla^3 f_i + \frac{251}{720}\nabla^4 f_i \right] + T_5 \\ &= y_i + h\Big\{ f_i + \frac{1}{2}(f_i - f_{i-1}) + \frac{5}{12}(f_i - 2f_{i-1} + f_{i-2}) + \frac{3}{8}(f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3}) \\ &\qquad + \frac{251}{720}(f_i - 4f_{i-1} + 6f_{i-2} - 4f_{i-3} + f_{i-4})\Big\} + T_5 \\ &= y_i + \frac{h}{720}\{1901 f_i - 2774 f_{i-1} + 2616 f_{i-2} - 1274 f_{i-3} + 251 f_{i-4}\} + T_5 \end{aligned} \tag{12}$$

where

$$T_5 = \frac{475}{1440} h^6 f^{(5)}(\xi_5) = \frac{475}{1440} h^6 y^{(6)}(\xi_5) \tag{13}$$

is the local truncation error. Hence, Eq. (12) is called fifth order predictor method.

2.2. Description of Corrector Method

Consider the $k+1$ data values, $(x_{i+1}, f_{i+1}), (x_i, f_i), (x_{i-1}, f_{i-1}), \ldots, (x_{i-k+1}, f_{i-k+1})$ which include the current data point. For this data, we fit the Newton's backward difference interpolating polynomial of degree *k* and we get:

$$\begin{aligned} p_k(x) = f(x_i + sh) &= f(x_{i+1}) + (s-1)\nabla f(x_{i+1}) + \frac{(s-1)s}{2!}\nabla^2 f(x_{i+1}) + \ldots \\ &\quad + \frac{(s-1)s(s+1)(s+2)\ldots(s+k-2)}{(k)!}\nabla^k f(x_{i+1}) + T_k^c \end{aligned} \tag{14}$$

where

$$s = \frac{x - x_i}{h}, \quad x - x_{i+1} = (x - x_i) - (x_{i+1} - x_i) = sh - h = h(s-1),$$

$$T_k^c = \frac{(s-1)s(s+1)(s+2)\ldots(s+k-1)}{(k+1)!} h^{k+1} f^{(k+1)}(\xi) \tag{15}$$

is the error term, when ξ lies in some interval containing the points $x_{i+1}, x_i, x_{i-1}, \ldots, x_{i-k+1}$ and x.

The limit of integration in Eq. (4) becomes:

$$x = x_i \Rightarrow s = 0, \; x = x_{i+1} \Rightarrow s = 1 \text{ and } dx = hds. \tag{16}$$

Thus, replacing $f(x, y)$ by $p_k(x)$ in Eq. (4) and using Eq. (14) in Eq. (4), we get:

$$y_{i+1} = y_i + h\int_0^1 \left\{ f_{i+1} + (s-1)\nabla f_{i+1} + \frac{(s-1)s}{2}\nabla^2 f_{i+1} + \frac{(s-1)s(s+1)}{6}\nabla^3 f_{i+1} + \ldots \right\} ds \tag{17}$$

Here,

$$\begin{gathered} \int_0^1 (s-1)\,ds = \frac{-1}{2}, \qquad \int_0^1 (s-1)s\,ds = \frac{-1}{6}, \qquad \int_0^1 (s-1)s(s+1)\,ds = \frac{-1}{4}, \\ \int_0^1 (s-1)s(s+1)(s+2)\,ds = \frac{-19}{30}, \quad \int_0^1 (s-1)s(s+1)(s+2)(s+3)\,ds = \frac{-9}{4} \end{gathered} \tag{18}$$

Hence, from Eq. (17), we get:

$$y_{i+1} = y_1 + h\left\{ f_{i+1} - \frac{1}{2}\nabla f_{i+1} - \frac{1}{12}\nabla^2 f_{i+1} - \frac{1}{24}\nabla^3 f_{i+1} - \frac{19}{720}\nabla^4 f_{i+1} - \frac{9}{480}\nabla^5 f_{i+1} - \ldots \right\} \tag{19}$$

Using Eq. (15), we obtain the error term as:

$$\begin{aligned} T_k^c &= h^{k+2}\int_0^1 \frac{(s-1)s(s+1)(s+2)\ldots(s+k-1)}{(k+1)!} f^{(k+1)}(\xi_k) \\ &= h^{k+2}\int_0^1 g_c(s) f^{(k+1)}(\xi_k)ds, \quad 0 < \xi_k < 1 \end{aligned} \tag{20}$$

where

$$g_c(s) = \frac{1}{(k+1)!}\{(s-1)s(s+1)\ldots(s+k-1)\} \tag{21}$$

By choosing different values for k, different corrector methods will be obtained. So, as we have discussed before we choose k for fifth order, which implies $k = 4$.

i.e, by choosing $k=4$, we get the method:

$$y_{i+1} = y_i + h\left\{ f_{i+1} - \frac{1}{2}\nabla f_{i+1} - \frac{1}{12}\nabla^2 f_{i+1} - \frac{1}{24}\nabla^3 f_{i+1} - \frac{19}{720}\nabla^4 f_{i+1} \right\} + T_4$$

$$= y_i + h\left\{ f_{i+1} - \frac{1}{2}(f_{i+1} - f_i) - \frac{1}{12}(f_{i+1} - 2f_i + f_{i-1}) - \frac{1}{24}(f_{i+1} - 3f_i + 3f_{i-1} - f_{i-2}) \right.$$

$$\left. - \frac{19}{720}(f_{i+1} - 4f_i + 6f_{i-1} - 4f_{i-2} + f_{i-3}) \right\} + T_4$$

$$= y_i + \frac{h}{720}\{251 f_{i+1} + 646 f_i - 264 f_{i-1} + 106 f_{i-2} - 19 f_{i-3}\} + T_4 \tag{22}$$

where

$$T_4 = \frac{-9}{480} h^6 f^{(5)}(\xi_4) = \frac{-9}{480} h^6 y^{(6)}(\xi_4) \tag{23}$$

is the local truncation error. Hence Eq. (22) is called fifth order corrector method. So that, we use Eqs. (12) and (22) for solving quadratic Riccati differential equation in Eq. (1) with Eq. (2).

Remarks:

1. For using the methods, we require the starting values $y_0,\ y_1,\ y_2,\ y_3$.
2. The required starting values for the application of predictor-corrector methods are obtained by using any single step method like Euler's method, Taylor series method or Runge-Kutta methods.

3. Stability and Convergence Analysis

Definition 1: Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the (not necessarily distinct) roots of the characteristic equation given by:

$$p(\lambda) = \lambda^k - a_{k-1}\lambda^{k-1} - \cdots - a_1\lambda - a_0 = 0 \tag{24}$$

associated with the multistep difference method of Eqs. (12) and (22) given as:

$$y_{i+1} = a_{k-1}y_i + a_{k-2}y_{i-1} + \ldots + a_0 y_{i+1-k} + hF\left(x_i, h,\ y_{i+1},\ y_i,\ \ldots,\ y_{i+1-k}\right),$$
$$y_0 = \alpha,\ \ y_1 = \alpha_1,\ \ \ldots,\ \ y_{k-1} = \alpha_{k-1}, \text{ for each } i = k-1,\ k, \ldots, N-1, \tag{25}$$

where $a_0, a_1, \ldots, a_{k+1}$ are constants.

If $|\lambda_i| \leq 1,$ for $i = 1, 2, \ldots, k,$ and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the root condition.

Definition 2 (Stability) [14]:

i. Methods that satisfy the roots condition in which λ = 1 is the only root of the characteristic equation with magnitude one are called strongly stable.
ii. Methods that satisfy the root condition and have more than one distinct root with magnitude one are called weakly stable.
iii. Methods that do not satisfy the root condition are called unstable.

Theorem 1: The fifth order predictor method in Eq. (12) is strongly stable.

Proof: The fifth order predictor method in Eq. (12) can be expressed as:

$$y_{i+1} = y_i + hF\left(x_i, h, y_{i+1}, y_i, \ldots, y_{i-3}, y_{i-4}\right) \tag{26}$$

where

$$F\left(x_i, h, y_{i+1}, y_i, \ldots, y_{i-3}, y_{i-4}\right) = \frac{1}{720}\Big\{1901 f\left(x_i, y_i\right) - 2774 f\left(x_{i-1}, y_{i-1}\right) + 2616 f\left(x_{i-2}, y_{i-2}\right) - 1274 f\left(x_{i-3}, y_{i-3}\right) + 251 f(x_{i-4}, y_{i-4})\Big\} \tag{27}$$

In this case, we have: $k = 5,$ $a_0 = a_1 = a_2 = a_3 = 0,$ and $a_4 = 1$.

The characteristic equation for the method becomes:

$$\begin{aligned} p(\lambda) &= \lambda^5 - \lambda^4 = \lambda^4\left(\lambda - 1\right) = 0 \\ &\Rightarrow \lambda_1 = 1,\ \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0 \end{aligned} \tag{28}$$

are the roots of the polynomial.

Therefore, it satisfies the root condition and is strongly stable by Definition 2 (i).

Theorem 2: The fifth order corrector method in Eq. (22) is also strongly stable.

Proof: The fifth order corrector method in Eq. (22) can be expressed as:

$$y_{i+1} = y_i + hF\left(x_i, h, y_{i+1}, y_i, \ldots, y_{i-3}\right) \tag{29}$$

where

$$F\left(x_i.h, y_{i+1}\ , y_i, \ldots, y_{i-3}\right) = \frac{1}{720}\left\{251 f\left(x_{i+1}, y_{i+1}\right) + 646 f\left(x_i, y_i\right) - 264 f\left(x_{i-1}, y_{i-1}\right) + 106 f\left(x_{i-2}, y_{i-2}\right) - 19 f\left(x_{i-3}, y_{i-3}\right)\right\} \quad (30)$$

Following the similar procedure as we have done in Theorem 1, here,

$$k = 4, \ \ a_0 = a_1 = a_2 = 0, \ \ a_3 = 1 \quad (31)$$

The characteristic equation for the method becomes:

$$p(\lambda) = \lambda^4 - \lambda^3 = \lambda^3(\lambda - 1) = 0 \quad (32)$$

Thus, this polynomial has roots

$$\lambda_1 = 1, \ \ \lambda_2 = 0 = \lambda_3 = \lambda_4. \quad (33)$$

Therefore, it satisfies the root condition and is strongly stable by Definition 2 (i).

Definition 3 (Consistency): The method is consistent, if the local truncation error $T_k(h) \to 0$ as $h \to 0$.

From Eqs. (13) and (23), we have:

$$T_4 = \frac{-9}{480} h^6 y^{(6)}(\xi_4) \ \text{ and } \ T_5 = \frac{475}{1440} h^6 y^{(6)}(\xi_5) \quad (34)$$

Thus, $T_k(h) \to 0$ as $h \to 0$ for $k = 4, 5$.

Therefore, the methods in Eq. (12) and (22) are consistent by Definition 3. Hence, they are convergent of fifth order. Since, consistency + stability $\Leftrightarrow$ convergence.

4. Numerical Examples and Results

To validate the applicability of the methods, three model examples of quadratic Riccati differential equations with initial conditions have been considered. Since all predictor-corrector methods are not a self-starter, we take the classical Runge-Kutta (RK4) method for the first four nodal points. For each number of nodal points N, the pointwise absolute errors are approximated by the formula, $\|E\| = |y(x_i) - y_i|$, for $i = 0, 1, 2, \cdots, N$, where $y(x_i)$ and y_i

are the exact and computed approximate solution of the given problems respectively, at the nodal point x_i. Numerical examples are given to illustrate the efficiency and convergence of the methods.

Example 1. Consider the following quadratic Riccati differential equation, Vinod and Dimple [5]; Khalid et al. [12],

$$\begin{aligned} &y'(x) = 1 + y^2(x), \quad 0 \le x \le 1, \\ &y(0) = 0 \end{aligned} \tag{35}$$

where the exact solution of this equation is $y(x) = \tan x$.

Table 1. Pointwise absolute errors for Example 1 with different values of *N*.

x	Vinod and Dimple [5]	Present Method $N=100$	$N=100$ Refinement	$N=200$	$N=300$	$N=400$	$N=500$
0.01	2.5001250e-07	8.3330e-13	8.3330e-13	5.2076e-14	1.0285e-14	3.2543e-15	1.0651e-15
0.02	2.0004001e-06	1.6657e-12	1.6657e-12	1.0409e-13	1.3538e-14	3.1954e-15	1.0443e-15
0.03	6.7530387e-06	2.4961e-12	2.4961e-12	1.0114e-13	1.3108e-14	3.0809e-15	1.0131e-15
0.04	1.6012809e-05	3.3232e-12	2.3825e-12	9.6610e-14	1.2476e-14	2.9421e-15	9.7145e-16
0.05	3.1289104e-05	3.1598e-12	3.1590e-12	9.0469e-14	1.1616e-14	2.7339e-15	9.0206e-16
0.06	5.4097349e-05	2.9452e-12	2.9438e-12	8.2712e-14	1.0547e-14	2.4772e-15	8.2573e-16
0.07	8.5960526e-05	2.6789e-12	2.6768e-12	7.3316e-14	9.2426e-15	2.1788e-15	7.3552e-16
0.08	1.2841072e-04	2.3600e-12	2.3571e-12	6.2270e-14	7.7161e-15	1.8319e-15	6.2450e-16
0.09	1.8299066e-04	1.9876e-12	1.9836e-12	4.9488e-14	5.9674e-15	1.4155e-15	4.8572e-16
0.1	2.5125533e-04	1.5604e-12	1.5553e-12	3.4958e-14	3.9829e-15	9.4369e-16	3.3307e-16

Table 2. The maximum absolute errors for Example 1 with different values of *N*.

$N=50$	$N=100$	$N=200$	$N=300$	$N=400$	$N=500$
1.6183e-07	6.6504e-09	2.3871e-10	3.2945e-11	8.0336e-12	2.6552e-12
Refinement					
1.7156e-07	6.7519e-09	2.3963e-10	3.3001e-11	8.0413e-12	2.6570e-12

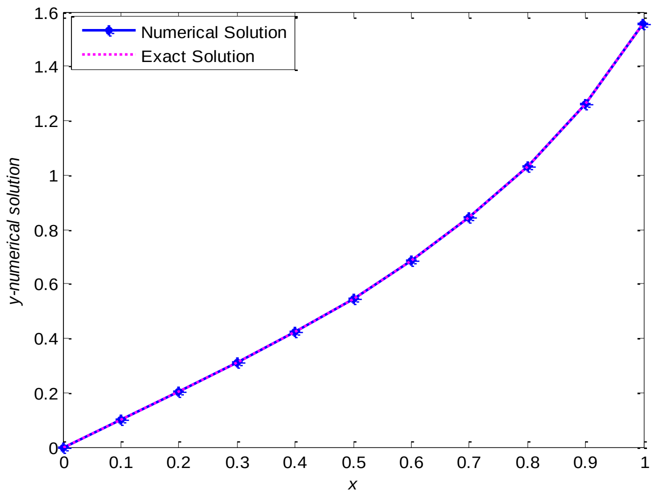

Fig. 1. Numerical solution versus exact solution for Example 1 when N = 10.

Example 2. Consider the following quadratic Riccati differential equation, Gemechis and Tesfaye [4]; Fateme and Esmaile [6],

$$\begin{aligned} &y'(x) = e^{x} - e^{3x} + 2e^{2x} y(x) - e^{x} y^{2}(x), \quad 0 \le x \le 1, \\ &y(0) = 1 \end{aligned} \tag{36}$$

The exact solution of the given problem is given by $y(x) = e^{x}$.

Table 3. Pointwise absolute errors for Example 2 with different values of N.

x	N = 10	N = 40	N = 70	N = 100	N = 200	N = 400
Present Method						
0.1	1.1153e-07	4.5427e-10	2.5490e-11	4.1438e-12	1.2457e-13	4.6629e-15
0.2	2.6297e-07	4.3364e-10	2.4216e-11	3.9286e-12	1.1813e-13	4.8850e-15
0.3	4.6838e-07	4.1083e-10	2.2808e-11	3.6910e-12	1.1036e-13	4.8850e-15
0.4	7.4674e-07	3.8563e-10	2.1253e-11	3.4284e-12	1.0236e-13	4.8850e-15
0.5	7.2073e-07	3.5778e-10	1.9533e-11	3.1379e-12	9.3259e-14	4.6629e-15
0.6	6.9197e-07	3.2699e-10	1.7632e-11	2.8164e-12	8.3489e-14	1.1102e-15
0.7	6.6020e-07	2.9297e-10	1.5530e-11	2.4616e-12	7.2387e-14	3.5527e-15
0.8	6.2508e-07	2.5537e-10	1.3208e-11	2.0699e-12	6.0396e-14	7.1054e-15
0.9	5.8627e-07	2.1382e-10	1.0642e-11	1.6369e-12	4.6185e-14	1.3323e-14
1.0	5.4337e-07	1.6790e-10	7.8062e-12	1.1582e-12	3.1530e-14	1.9096e-14
Gemechis and Tesfaye [4]						
0.1	3.8296e-07	4.5427e-10	4.8711e-11	1.1722e-11	7.3475e-13	4.5963e-14
0.2	2.6297e-07	1.0710e-09	1.1484e-10	2.7633e-11	1.7317e-12	1.0880e-13
0.3	4.6838e-07	1.9073e-09	2.0451e-10	4.9211e-11	3.0835e-12	1.9384e-13
0.4	7.4674e-07	3.0404e-09	3.2600e-10	7.8447e-11	4.9163e-12	3.0909e-13
0.5	1.1237e-06	4.5748e-09	4.9051e-10	1.1803e-10	7.3965e-12	4.6496e-13
0.6	1.6338e-06	6.6511e-09	7.1312e-10	1.7160e-10	1.0753e-11	6.7191e-13
0.7	2.3239e-06	9.4596e-09	1.0142e-09	2.4406e-10	1.5293e-11	9.5124e-13
0.8	3.2569e-06	1.3257e-08	1.4214e-09	3.4203e-10	2.1432e-11	1.3305e-12
0.9	4.5182e-06	1.8390e-08	1.9717e-09	4.7445e-10	2.9730e-11	1.8439e-12
1.0	6.2225e-06	2.5327e-08	2.7154e-09	6.5340e-10	4.0941e-11	2.5673e-12

Table 4. Pointwise absolute errors for Examples 2 with different values of N.

x	Fateme and Esmaile [6]	Present Method N = 10	Present Method N = 300
0.0	0.0000000000000	0.0	0.0
0.1	0.00034681435605	1.1153e-07	1.6209e-14
0.3	0.00067436472882	4.6838e-07	1.4655e-14
0.5	3.8747×10^{-10}	7.2073e-07	1.3989e-14
0.7	0.00067437189321	6.6020e-07	7.1054e-15
0.9	0.00034682221170	5.8627e-07	2.2204e-15
1.0	0.000000000000	5.4337e-07	6.2172e-15

Table 5. The maximum absolute errors for Examples 2 with different values of N.

N = 50	N = 100	N = 200	N = 300	N = 400	N = 500
1.4455e-10	4.2630e-12	1.2945e-13	1.9096e-14	1.6875e-14	4.4409e-15
Refinement					
1.4327e-10	4.2437e-12	1.2923e-13	1.9096e-14	1.6875e-14	4.4409e-15

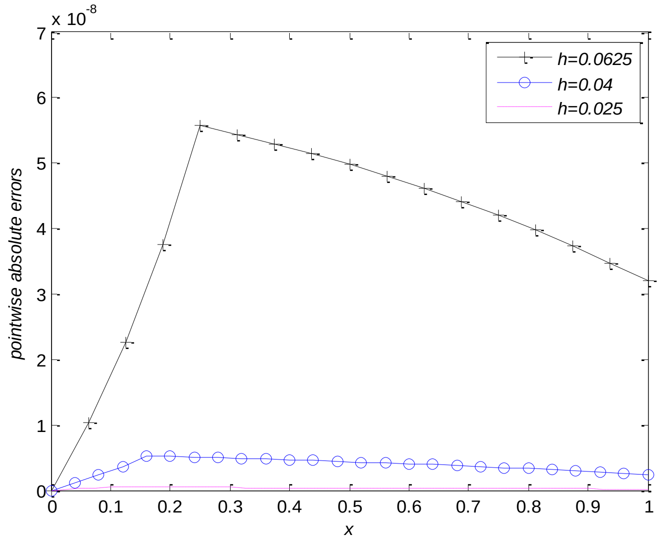

Fig. 2. Pointwise absolute errors of Example 2 for different values of h .

Example 3. Consider the following quadratic Riccati differential equation, Gemechis and Tesfaye [4]; Fateme and Esmaile [6],

$$\begin{aligned} &y'(x) = 1 + 2y(x) - y^2(x), \quad 0 \le x \le 1, \\ &y(0) = 0 \end{aligned} \tag{37}$$

where the exact solution of this equation is

$$y(x) = 1 + \sqrt{2}\tanh\left(\sqrt{2}x + 0.5\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) \tag{38}$$

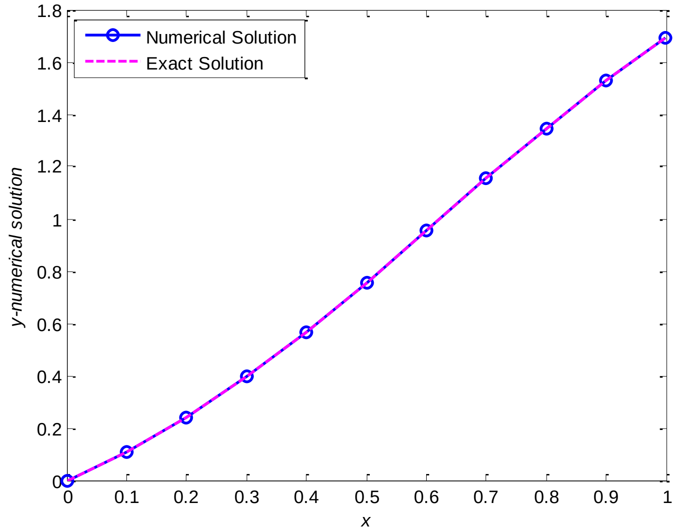

Fig. 3. Numerical solution versus exact solution of Example 3 when N = 10.

Table 6. Pointwise absolute errors for Example 3 with different values of N.

x	$N=10$	$N=40$	$N=70$	$N=100$	$N=200$	$N=400$
Present Method						
0.1	2.2551e-06	9.8491e-09	6.5613e-10	1.1501e-10	3.7895e-12	1.2192e-13
0.2	4.7763e-06	1.0576e-08	6.8253e-10	1.1788e-10	3.8127e-12	1.2154e-13
0.3	7.3083e-06	7.5309e-09	4.4688e-10	7.4441e-11	2.3045e-12	7.2109e-14
0.4	9.5635e-06	9.9895e-11	6.4516e-11	1.5673e-11	6.6713e-13	2.2982e-14
0.5	2.7825e-06	9.6406e-09	6.8248e-10	1.2102e-10	4.0088e-12	1.2823e-13
0.6	6.9705e-06	1.6943e-08	1.0936e-09	1.8762e-10	5.9928e-12	1.9307e-13
0.7	1.5393e-05	1.7084e-08	1.0377e-09	1.7402e-10	5.4168e-12	1.7675e-13
0.8	1.6632e-05	9.0960e-09	5.1689e-10	8.4664e-11	2.5677e-12	9.0150e-14
0.9	7.1404e-06	3.1027e-09	1.9362e-10	3.2240e-11	9.8810e-13	1.6209e-14
1.0	1.0444e-05	1.3554e-08	7.5227e-10	1.2112e-10	3.5889e-12	9.3703e-14
Gemechis and Tesfaye [4]						
0.1	2.2551e-06	9.8491e-09	1.0669e-09	2.8533e-10	1.6233e-11	1.0184e-12
0.2	4.7763e-06	2.0641e-08	2.2327e-09	5.3915e-10	3.3923e-11	2.1275e-12
0.3	7.3083e-06	3.1235e-08	3.3731e-09	8.1402e-10	5.1180e-11	3.2087e-12
0.4	9.5635e-06	4.0441e-08	4.3607e-09	1.0517e-09	6.6078e-11	4.1415e-12
0.5	1.1301e-05	4.7374e-08	5.1021e-09	1.2299e-09	7.7230e-11	4.8390e-12
0.6	1.2408e-05	5.1724e-08	5.5661e-09	1.3414e-09	8.4199e-11	5.2707e-12
0.7	1.2940e-05	5.3815e-08	5.7892e-09	1.3949e-09	8.7546e-11	5.4756e-12
0.8	1.3100e-05	5.4419e-08	5.8528e-09	1.4101e-09	8.8489e-11	5.5316e-12
0.9	1.3141e-05	5.4381e-08	5.8450e-09	1.4079e-09	8.8322e-11	5.5178e-12
1.0	1.3245e-05	5.4260e-08	5.8236e-09	1.4019e-09	8.7889e-11	5.5029e-12

Table 7. Pointwise absolute errors for Examples 3 with different values of N.

x	Fateme and Esmaile [6]	Present Method	
		$N=10$	$N=300$
0.0	9.6780×10^{-11}	0.0	0.0
0.1	0.000248944564121	2.2551e-06	5.0812e-13
0.3	0.0004481946615024	7.3083e-06	3.0248e-13
0.5	2.89441×10^{-10}	2.7825e-06	5.3524e-13
0.7	0.000374115023553	1.5393e-05	7.1365e-13
0.9	0.00017849475908	7.1404e-06	1.2368e-13
1.0	3.2516×10^{-10}	1.0444e-05	4.5808e-13

Table 8. The maximum absolute errors for Examples 3 with different values of N.

$N=50$	$N=100$	$N=200$	$N=300$	$N=400$	$N=500$
6.0022e-09	1.9281e-10	6.1084e-12	8.0802e-13	1.9695e-13	6.1950e-14
Refinement					
6.0007e-09	1.9281e-10	6.1084e-12	8.0802e-13	1.9695e-13	6.1950e-14

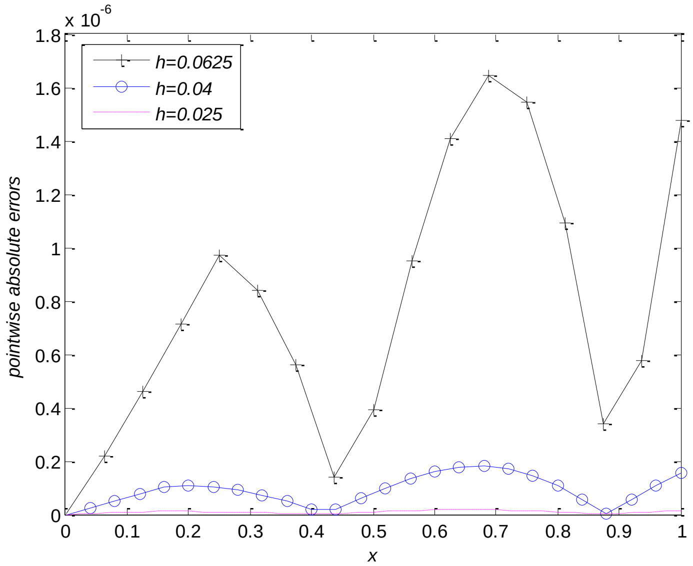

Fig. 4. Pointwise absolute errors of Example 3 for different values of *h*.

5. Discussion and Conclusion

In this paper, fifth order predictor-corrector method is presented for solving quadratic Riccati differential equations. The stability and convergence of the method have been investigated. The study is implemented on three model examples with exact solutions by taking different values for *N*, and the computational results are presented in the Tables (1–8). The results obtained by the present method are compared with the results of Gemechis and Tasfaye [4], Vinod and Dimple [5], Fateme and Esmaile [6] and shows betterment.

Furthermore, from the Tables (1–5) it is significant that all of the absolute errors decrease rapidly as *h* decreases, which in turn shows the convergence of the computed solution. Figs. (1, 3) shows that the present method approximates the exact solution very well. Figs. (2, 4) show that, as h decreases the absolute error goes to zero. This shows that the small step size provides the better approximation. Briefly, the present method is stable, more accurate and effective method for solving quadratic Riccati differential equations.

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