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Binomial Transforms of k-Narayana Sequences and Some Properties

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Abstract

The aim of the study is to obtain new binomial transforms for the $k-$Narayana sequence. The first of these is the binomial transform, which is its normal form, and in the first step, after finding the recurrence relation of this new binomial transform, the generating function and Binet formula were obtained. Finally, Pascal's triangle was calculated. In the rest of the article, $k-$binomial transform was performed for the $k-$Narayana sequence and the recurrence relation, generating function, Binet formula and Pascal's triangle were examined for the new sequence obtained. Then, by performing the falling binomial transform and the rising binomial transform, the features listed above were found again for these sequences.

1. Introduction

Some special sequences of numbers such as Fibonacci, Lucas, Horadam and Narayana have been of great interest to the scientific world in recent years. Generalizations of these number sequences in various ways abound in the literature, in particular you can look at [1]. One of the most popular transforms is the binomial transform and it is sufficiently available in the literature.

Authors [2] presented the $k-$Fibonacci sequence also the same authors for this sequences of numbers [3] introduced different binomial transforms, such as falling and rising binomial transforms. Binomial transforms and properties of $k-$Lucas sequences are presented in [4]. Spivey and Steil [5] gave various binomial transforms. In [6], they obtained some applications for the generalized (s,t) matrix sequences. In [7], authors obtained binomial transforms of Padovan and Perrin numbers from the third order.

The person who discovered the Narayana sequence is Narayana, an Indian mathematician, and is as follows

$$N_m = N_{m-1} + N_{m-3} \text{ with } m \geq 3 \tag{1.1}$$

where

$$N_0 = 0,\ N_1 = 1,\ N_2 = 1,$$

see [8]. The first few terms are $0,1,1,1,2,3,4,6,9,13,19,28,41,60,\cdots$.

The characteric equation of (1.1) is :

$$\Psi^3 - \Psi^2 - 1 = 0,$$

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and roots of the characteristic equation are :

$$\begin{aligned}
\Psi_1 &= \frac{1}{3}\left(\sqrt[3]{\frac{1}{2}(29-3\sqrt{93}}+\sqrt[3]{\frac{1}{2}(3\sqrt{93}+29)}+1\right),\\
\Psi_2 &= \frac{1}{3}-\frac{1}{3}(1-i\sqrt{3})\sqrt[3]{\frac{1}{2}(29-3\sqrt{93}}-\frac{1}{6}(1+i\sqrt{3})\sqrt[3]{\frac{1}{2}(3\sqrt{93}+29)},\\
\Psi_3 &= \frac{1}{3}-\frac{1}{3}(1+i\sqrt{3})\sqrt[3]{\frac{1}{2}(29-3\sqrt{93}}-\frac{1}{6}(1-i\sqrt{3})\sqrt[3]{\frac{1}{2}(3\sqrt{93}+29)}.
\end{aligned}$$

Hence, the Narayana sequence can be obtained by Binet's formula:

$$N_m = \frac{\Psi_1^2}{\Psi_1^3+2}\Psi_1^n + \frac{\Psi_2^2}{\Psi_2^3+2}\Psi_2^n + \frac{\Psi_3^2}{\Psi_3^3+2}\Psi_3^n.$$

Generating function found for Narayana equation is:

$$\frac{1}{1-\Psi-\Psi^3} = \sum_{n=0}^{\infty} N_{m+1}\Psi_1^n \text{ , for } n \geq 1,\ n \in \mathbb{Z}.$$

Narayana sequence which has attracted the attention of more mathematicians in recent years and its generalizations. Some of them are as follows:

Some basic properties of Fibonacci-Narayana numbers are proved in [9]. Bilgici in [10], defined a generalized order k Fibonacci-Narayana sequence and by using this generalization and some matrix properties, established some identities related to Fibonacci-Narayana numbers. Soykan studied on Narayana sequence in [11]. Ramirez and Sirvent in [12], introduced the $k-$Narayana sequence and found the identities between these numbers.

For any nonzero integer number k , $k-$Narayana sequence is defined by the following recurrence relation:

$$N_{k,m} = kN_{m-1} + N_{m-3} \text{ with } m \geq 3 \tag{1.2}$$

where

$$N_{k,0} = 0,\ N_{k,1} = 1,\ N_{k,2} = k,$$

see [12]. The first few terms are $0,1,k,k^2,k^3+1,k^4+2k,k^5+3k^2,k^6+4k^3+1,k^7+5k^4+3k\cdots$.

The characteric equation of (1.2) is :

$$\lambda^3 - k\lambda^2 - 1 = 0,$$

and the roots of characteristic equation are :

$$\begin{aligned}
\lambda_1 &= \frac{1}{3}\left(k+k^2\sqrt[3]{\frac{2}{27+2k^3+3\sqrt{81+12k^3}}}+\sqrt[3]{\frac{27+2k^3+3\sqrt{81+12k^3}}{2}}\right),\\
\lambda_2 &= \frac{1}{3}\left(k-\mu k^2\sqrt[3]{\frac{2}{27+2k^3+3\sqrt{81+12k^3}}}+\mu^2\sqrt[3]{\frac{27+2k^3+3\sqrt{81+12k^3}}{2}}\right),\\
\lambda_3 &= \frac{1}{3}\left(k+\mu^2k^2\sqrt[3]{\frac{2}{27+2k^3+3\sqrt{81+12k^3}}}-\mu\sqrt[3]{\frac{27+2k^3+3\sqrt{81+12k^3}}{2}}\right)
\end{aligned}$$

where $\mu = \frac{1+i\sqrt{3}}{2}$ is the primitive cube root of unity.

The generating function of the $k-$Narayana sequence is

$$\frac{1}{1-k\lambda-\lambda^3}.$$

Therefore the $k-$Narayana sequence can be obtained by Binet's formula:

$$N_{k,n} = \frac{\lambda_1^{n+1}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_2^{n+1}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\lambda_3^{n+1}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \text{ , } n \geq 0.$$

Other recent research ([13],[14],[15]) has also investigated various binomial transforms for various special sequences. These transforms are valuable because they bring a new approach. For details on the binomial transform, see ([16],[17]).

The focus of this paper is to apply binomial transforms and its generalization (like $k-$binomial transform, rising transform and falling transform) to the $k-$Narayana sequence. In addition to these, the recurrence relation, Binet's formula, generating function, Pascal triangle and matrix representation of related transforms were derived.

2. Binomial transform of $k-$Narayana sequences

The binomial transform of $k-$Narayana sequence $\{N_{k,n}\}_{n\in\mathbb{N}}$ is shown as $\{b_{k,n}\}_{n\in\mathbb{N}}$ where $b_{k,n}$ is dedicated by

$$b_{k,n} = \sum_{i=0}^{n}\binom{n}{i}N_{k,i}.$$

To find the recurrence relation of $\{b_{k,n}\}$, we first need a Lemma.

Lemma 2.1. *Let n is a positive integer greater than 1, then* $\{b_{k,n}\}$ *contents the next equation*

$$b_{k,n+1} = \sum_{i=0}^{n}\binom{n}{i}(N_{k,i}+N_{k,i+1}).$$

Proof. We have,

$$b_{k,n} = \sum_{i=0}^{n}\binom{n}{i}N_{k,i}.$$

If we bear in mind summation feature of binomial numbers $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$, also $\binom{n}{n+1} = 0$ for the proof. Then we find

$$\begin{aligned} b_{k,n+1} &= \sum_{i=0}^{n+1}\binom{n+1}{i}N_{k,i} \\ &= N_{k,0} + \sum_{i=1}^{n+1}\binom{n+1}{i}N_{k,i} \\ &= N_{k,0} + \sum_{i=1}^{n+1}\binom{n}{i}N_{k,i} + \sum_{i=1}^{n+1}\binom{n}{i-1}N_{k,i}. \end{aligned}$$

Also thanks to the operations performed on the sums

$$\begin{aligned} N_{k,0} + \sum_{i=1}^{n+1}\binom{n}{i}N_{k,i} + \sum_{i=1}^{n+1}\binom{n}{i-1}N_{k,i} &= N_{k,0} + \sum_{i=1}^{n}\binom{n}{i}N_{k,i} + \sum_{i=1}^{n}\binom{n}{i}N_{k,i+1} \\ &= \sum_{i=0}^{n}\binom{n}{i}N_{k,i} + \sum_{i=0}^{n}\binom{n}{i}N_{k,i+1} \\ &= \sum_{i=0}^{n}\binom{n}{i}(N_{k,i}+N_{k,i+1}). \end{aligned}$$

□

The next theorem presents recurrence relation for $\{b_{k,n}\}$.

Theorem 2.2. *The recurrence relation obtained for* $\{b_{k,n}\}$ *is as follows:*

$$b_{k,n+3} = (k+3)b_{k,n+2} - (2k+3)b_{k,n+1} + (k+2)b_{k,n} \tag{2.1}$$

where $b_{k,0} = 0, b_{k,1} = 1$, *and* $b_{k,2} = k+2$.

Proof. To find the coefficients in (2.1)

$$b_{k,n+3} = A_1 b_{k,n+2} + A_2 b_{k,n+1} + A_3 b_{k,n}.$$

If we take $n = 0, 1$ and 2, we have the system

$$\begin{aligned} b_{k,3} &= A_1 b_{k,2} + A_2 b_{k,1} + A_3 b_{k,0} = k^2 + 3k + 3 \\ b_{k,4} &= A_1 b_{k,3} + A_2 b_{k,2} + A_3 b_{k,1} = k^3 + 4k^2 + 6k + 5 \\ b_{k,5} &= A_1 b_{k,4} + A_2 b_{k,3} + A_3 b_{k,2} = k^4 + 5k^3 + 10k^2 + 12k + 10. \end{aligned}$$

By Cramer rule for the system, we get

$$A_1 = k+3,\ A_2 = -2k-3,\ \text{and } A_3 = k+2.$$

So which is completed the proof . □

The characteristic equation of sequences $b_{k,n}$ in (2.1) is

$$\alpha^3-(k+3)\alpha^2+(2k+3)\alpha-(k+2)=0,$$

whose solutions are

$$\begin{aligned}\alpha_1 &= T+S+\frac{k+3}{3}\\ \alpha_2 &= \frac{T(w-1)}{2}-\frac{S(w+1)}{2}+\frac{k+3}{3}\\ \alpha_3 &= \frac{S(w-1)}{2}-\frac{T(w+1)}{2}+\frac{k+3}{3}\end{aligned}$$

where

$$\begin{aligned}T &= \left(\frac{k^3}{27}+\sqrt{\Delta}\right)^{\frac{1}{3}},\\ S &= \left(\frac{k^3}{27}-\sqrt{\Delta}\right)^{\frac{1}{3}},\\ \Delta &= \frac{k^3}{27}+\frac{1}{4},\ \ w=i\sqrt{3}.\end{aligned}$$

Next we derive the Binet formula for $\{b_{k,n}\}$.

Theorem 2.3. *The Binet formula for the $k-$Narayana sequence is as follows:*

$$b_{k,n}=\frac{p_1\alpha_1^n}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}+\frac{p_2\alpha_2^n}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}+\frac{p_3\alpha_3^n}{(\alpha_3-\alpha_1)(\alpha_2-\alpha_3)}$$

where

$$\begin{aligned}p_1 &= b_{k,2}-(\alpha_2+\alpha_3)b_{k,1}+\alpha_2\alpha_3b_{k,0}=k+2-(\alpha_2+\alpha_3),\\ p_2 &= b_{k,2}-(\alpha_1+\alpha_3)b_{k,1}+\alpha_1\alpha_3b_{k,0}=k+2-(\alpha_1+\alpha_3),\\ p_3 &= b_{k,2}-(\alpha_1+\alpha_2)b_{k,1}+\alpha_1\alpha_2b_{k,0}=k+2-(\alpha_1+\alpha_2).\end{aligned}$$

Proof. To obtain Binet formula let us write

$$b_{k,n}=B_1\alpha_1^n+B_2\alpha_2^n+B_3\alpha_3^n$$

If we take $n=0,1$ and 2, we have the system

$$\begin{aligned}b_{k,0} &= B_1+B_2+B_3=0\\ b_{k,1} &= B_1\alpha_1+B_2\alpha_2+B_3\alpha_3=1\\ b_{k,2} &= B_1\alpha_1^2+B_2\alpha_2^2+B_3\alpha_3^2=k+2\end{aligned}$$

By Cramer rule for the system, we get

$$\begin{aligned}B_1 &= \frac{b_{k,2}-(\alpha_2+\alpha_3)b_{k,1}+\alpha_2\alpha_3b_{k,0}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}\\ B_2 &= \frac{b_{k,2}-(\alpha_1+\alpha_3)b_{k,1}+\alpha_1\alpha_3b_{k,0}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}\\ B_2 &= \frac{b_{k,2}-(\alpha_1+\alpha_2)b_{k,1}+\alpha_1\alpha_2b_{k,0}}{(\alpha_3-\alpha_1)(\alpha_2-\alpha_3)}\end{aligned}$$

So which is completed the proof . □

Now let's obtain the generating function for the $k-$Narayana binomial transform.

Theorem 2.4. *The generating function of $\{b_{k,n}\}$ is:*

$$b_k(x)=\frac{(1-kx-3x+2kx^2+3x^2)b_{k,0}+(x-kx^2-3x^2)b_{k,1}+x^2b_{k,2}}{1-(k+3)x-(2k+3)x^2-(k+2)x^3}.$$

Proof. We have, $b_k(x) = b_{k,0} + b_{k,1}x + b_{k,2}x^2 + b_{k,3}x^3 + \cdots + b_{k,n}x^n + \cdots$. After doing simple operations we obtain

$$\begin{aligned} b_k(x) &= b_{k,0} + b_{k,1}x + b_{k,2}x^2 + b_{k,3}x^3 + \cdots \\ -(k+3)xb_k(x) &= -b_{k,0}(k+3)x - b_{k,1}(k+3)x^2 - b_{k,3}(k+3)x^3 + \cdots \\ -(2k+3)x^2b_k(x) &= -b_{k,0}(2k+3)x^2 - b_{k,1}(2k+3)x^3 - b_{k,3}(2k+3)x^4 + \cdots \\ -(k+2)x^3b_k(x) &= -b_{k,0}(k+2)x^3 - b_{k,1}(k+2)x^4 - b_{k,3}(k+2)x^5 + \cdots . \end{aligned}$$

From these equations and (2.1), we get

$$\left[1-(k+3)x-(2k+3)x^2-(k+2)x^3\right] b_k(x) = (1-kx-3x+2kx^2+3x^2)b_{k,0} + (x-kx^2-3x^2)b_{k,1} + x^2b_{k,2}.$$

So the generating function for the binomial transform of the $k-$Narayana sequence is

$$b_k(x) = \frac{(1-kx-3x+2kx^2+3x^2)b_{k,0} + (x-kx^2-3x^2)b_{k,1} + x^2b_{k,2}}{1-(k+3)x-(2k+3)x^2-(k+2)x^3}.$$

□

Let's give a new triangle $\{b_{k,n}\}$ for each k to help with the next rules:

1. The part forming the left corner of the triangle consists of the elements of $k-$ Narayana numbers,
2. When we take any number and think that it is chosen outside the left diagonal, it is considered to be the sum of the number to the left of this number and also the number above its diagonal on the left side.
3. On the right diagonal is $\{b_{k,n}\}$.

The next triangle is an example of the $1-$Narayana sequence:

0
1 1
1 2 3
1 2 4 7
2 3 5 9 16

Figure 1: $1-$Narayana sequence

3. The $k-$Binomial transform of the $k-$Narayana sequence

The $k-$binomial transform of the $k-$Narayana sequence $\{N_{k,n}\}_{n\in\mathbb{N}}$ is denoted by $\{w_{k,n}\}_{n\in\mathbb{N}}$ where

$$w_{k,n} = \sum_{i=0}^{n} \binom{n}{i} k^n N_{k,i}.$$

Lemma 3.1. *Let n is an integer greater than and equal to 1, and $k-$binomial transform of $k-$Narayana sequence satisfies the following relation*

$$w_{k,n+1} = \sum_{i=0}^{n} \binom{n}{i} k^{n+1}(N_{k,i} + N_{k,i+1}).$$

Proof. We know that

$$w_{k,n} = \sum_{i=0}^{n} \binom{n}{i} N_{k,i}.$$

If we take $n+1$ instead of n and consider the binomial properties then we have

$$\begin{aligned} w_{k,n+1} &= \sum_{i=0}^{n+1}\binom{n+1}{i}N_{k,i} \\ &= k^{n+1}N_{k,0}+\sum_{i=1}^{n+1}\binom{n+1}{i}k^{n+1}N_{k,i} \\ &= k^{n+1}N_{k,0}+\sum_{i=1}^{n+1}\binom{n}{i}\binom{n}{i-1}k^{n+1}N_{k,i} \\ &= k^{n+1}N_{k,0}+\sum_{i=1}^{n+1}\binom{n}{i}N_{k,i}+\sum_{i=1}^{n+1}\binom{n}{i-1}N_{k,i} \\ &= k^{n+1}N_{k,0}+\sum_{i=1}^{n}\binom{n}{i}k^{n+1}N_{k,i}+\sum_{i=1}^{n}\binom{n}{i}k^{n+1}N_{k,i+1} \end{aligned}$$

so we get

$$w_{k,n}=\sum_{i=0}^{n}\binom{n}{i}k^{n+1}(N_{k,i}+N_{k,i+1}).$$

□

The next theorem will provides the recurrence relation for $\{w_{k,n}\}$.

Theorem 3.2. *The recurrence relation obtained for $\{w_{k,n}\}$ is as follows:*

$$w_{k,n+3}=(k^2+3k)w_{k,n+2}-(2k^3+3k^2)w_{k,n+1}+(k^4+2k^3)w_{k,n}. \tag{3.1}$$

Proof. From the recurrence relation of the corresponding transform, there is a general solution as follows

$$w_{k,n+3}=C_1w_{k,n+2}+C_2w_{k,n+1}+C_3w_{k,n}.$$

If $n=0,1$ and 2, the following system is obtained

$$\begin{aligned} w_{k,3} &= C_1w_{k,2}+C_2w_{k,1}+C_3w_{k,0}=k^5+3k^4+3k^3 \\ w_{k,4} &= C_1w_{k,3}+C_2w_{k,2}+C_3w_{k,1}=k^7+4k^6+6k^5+5k^4 \\ w_{k,5} &= C_1w_{k,4}+C_2w_{k,3}+C_3w_{k,2}=k^9+5k^8+10k^7+12k^6+10k^5 \end{aligned}$$

By Cramer rule for the system, we get

$$C_1=k^2+3k,\ C_2=-2k^3-3k^2,\ \text{and } C_3=k^4+2k^3.$$

so that the evidence is completed. □

The characteristic equation of sequences $w_{k,n}$ in (3.1) is

$$\beta^3-(k^2+3k)\beta^2+(2k^3+3k^2)\beta-(k^4+2k^3)=0,$$

whose solutions are β_1, β_2, and β_3.
Now we construct the Binet formula for $\{w_{k,n}\}$.

Theorem 3.3. *Whichever term of $\{w_{k,n}\}$ can be computed using the Binet formula. It is indicated by*

$$w_{k,n}=\frac{q_1\beta_1^n}{(\beta_1-\beta_2)(\beta_1-\beta_3)}+\frac{q_2\beta_2^n}{(\beta_2-\beta_1)(\beta_2-\beta_3)}+\frac{q_3\beta_3^n}{(\beta_3-\beta_1)(\beta_2-\beta_3)}$$

where

$$\begin{aligned} q_1 &= w_{k,2}-(\beta_2+\beta_3)w_{k,1}+\beta_2\beta_3w_{k,0}=k\left[k^2+2k-(\beta_2+\beta_3)\right] \\ q_2 &= w_{k,2}-(\beta_1+\beta_3)w_{k,1}+\beta_1\beta_3w_{k,0}=k\left[k^2+2k-(\beta_1+\beta_3)\right] \\ q_3 &= w_{k,2}-(\beta_1+\beta_2)w_{k,1}+\beta_1\beta_2w_{k,0}=k\left[k^2+2k-(\beta_1+\beta_2)\right] \end{aligned}$$

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 2 & & & \\ & & 2 & & 6 & & 16 & & \\ & 4 & & 12 & & 36 & & 104 & \\ 9 & & 26 & & 76 & & 224 & & 656 \end{array}$$

Figure 2: 2−Narayana sequence

Proof. To obtain Binet formula let us write

$$w_{k,n} = D_1\alpha_1^n + D_2\alpha_2^n + D_3\alpha_3^n$$

If we take $n = 0, 1$and 2, we have the system

$$\begin{aligned} w_{k,0} &= D_1 + D_2 + D_3 = 0 \\ w_{k,1} &= D_1\beta_1 + D_2\beta_2 + D_3\beta_3 = k \\ w_{k,2} &= D_1\beta_1^2 + D_2\beta_2^2 + D_3\beta_3^2 = k^3 + 2k^2 \end{aligned}$$

By Cramer rule for the system, we get

$$\begin{aligned} D_1 &= \frac{w_{k,2} - (\beta_2+\beta_3)w_{k,1} + \beta_2\beta_3 w_{k,0}}{(\beta_1-\beta_2)(\beta_1-\beta_3)}, \\ D_2 &= \frac{w_{k,2} - (\beta_1+\beta_3)w_{k,1} + \beta_1\beta_3 w_{k,0}}{(\beta_2-\beta_1)(\beta_2-\beta_3)}, \\ D_2 &= \frac{w_{k,2} - (\beta_1+\beta_2)w_{k,1} + \beta_1\beta_2 w_{k,0}}{(\beta_3-\beta_1)(\beta_2-\beta_3)}. \end{aligned}$$

So which is completed the proof . □

Theorem 3.4. *The generating function of* $\{w_{k,n}\}$ *is:*

$$w_k(x) = \frac{(1-k^2x-3kx+2k^3x^2+3k^2x^2)w_{k,0} + (x-k^2x^2-3kx^2)w_{k,1} + x^2 w_{k,2}}{1-k^2x-3kx+2k^3x^2+3k^2x^2-k^4x^3-2k^3x^3}.$$

Proof. We have $w_k(x) = w_{k,0} + w_{k,1}x + w_{k,2}x^2 + w_{k,3}x^3 \cdots + w_{k,n}x^n + \cdots$
Then, if multiplication is done $-(k^2+3k)x$, $(2k^3+3k^2)x^2$, and $-(k^4+2k^3)x^3$, we obtain

$$\begin{aligned} w_k(x) &= w_{k,0} + w_{k,1}x + w_{k,2}x^2 + w_{k,3}x^3 + \cdots \\ -(k^2+3k)xw_k(x) &= -w_{k,0}(k^2+3k)x - w_{k,1}(k^2+3k)x^2 - w_{k,3}(k^2+3k)x^3 + \cdots \\ (2k^3+3k^2)x^2w_k(x) &= w_{k,0}(2k^3+3k^2)x^2 + w_{k,1}(2k^3+3k^2)x^3 + w_{k,3}(2k^3+3k^2)x^4 + \cdots \\ -(k^4+2k^3)x^3w_k(x) &= -w_{k,0}(k^4+2k^3)x^3 - w_{k,1}(k^4+2k^3)x^4 - w_{k,3}(k^4+2k^3)x^5 + \cdots \end{aligned}$$

from these equations and (3.1), we get

$$\left[1-k^2x-3kx+2k^3x^2+3k^2x^2-k^4x^3-2k^3x^3\right] w_k(x) = (1-k^2x-3kx+2k^3x^2+3k^2x^2)w_{k,0} + (x-k^2x^2-3kx^2)w_{k,1} + x^2w_{k,2}$$

and so the generating function for the $k-$binomial transform of the $k-$Narayana sequence is

$$w_k(x) = \frac{(1-k^2x-3kx+2k^3x^2+3k^2x^2)w_{k,0} + (x-k^2x^2-3kx^2)w_{k,1} + x^2 w_{k,2}}{1-k^2x-3kx+2k^3x^2+3k^2x^2-k^4x^3-2k^3x^3}.$$

□

Now, we present a new triangle of the $k-$binomial transform of the $k-$Narayana sequence for each k. The next triangle is an example of the $2-$Narayana sequence:
Since the proofs in this section are similar to the proof steps in the previous section, the theorems are given without proofs.

4. The rising $k-$binomial transform of the $k-$Narayana sequence

The rising $k-$binomial transform of the $k-$Narayana sequence $\{N_{k,n}\}_{n\in\mathbb{N}}$ is denoted by $\{r_{k,n}\}_{n\in\mathbb{N}}$ where

$$r_{k,n}=\sum_{i=0}^{n}\binom{n}{i}k^{i}N_{k,i}.$$

Theorem 4.1. *The recurrence relation obtained for $\{r_{k,n}\}$ is as follows:*

$$r_{k,n+3}=(k^2+3)r_{k,n+2}-(2k^2+3)r_{k,n+1}+(k^3+k^2+1)r_{k,n}. \tag{4.1}$$

The characteristic equation of $\{b_{k,n}\}$ in (4.1) is

$$\gamma^3-(k^2+3)\gamma^2+(2k^2+3)\gamma-(k^3+k^2+1)=0,$$

whose solutions are γ_1, γ_2, and γ_3.
Next we derive the Binet formula for the rising $k-$binomial transform of the $k-$Narayana sequence.

Theorem 4.2. *Whichever term of $\{r_{k,n}\}$ can be computed using the Binet formula. It is indicated by*

$$r_{k,n}=\frac{u_1\gamma_1^n}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)}+\frac{u_2\gamma_2^n}{(\gamma_2-\gamma_1)(\gamma_2-\gamma_3)}+\frac{u_3\gamma_3^n}{(\gamma_3-\gamma_1)(\gamma_2-\gamma_3)}$$

where

$$\begin{aligned}u_1&=k^3-\gamma_2k-\gamma_3k+2k\\u_2&=k^3-\gamma_1k-\gamma_3k+2k\\u_3&=k^3-\gamma_1k-\gamma_2k+2k\end{aligned}$$

Theorem 4.3. *The generating function of $\{r_{k,n}\}$ is:*

$$r_k(x)=\frac{(1-kx^2-3+2k^2x^2+3x^2)r_{k,0}+(1-k^2x^2-3x^2)r_{k,1}+x^2r_{k,2}}{1-kx^2-3+2k^2x^2+3x^2-k^3x^3-k^2x^3-x^3}.$$

Now, we present a new triangle of $\{r_{k,n}\}$ for each k. The next triangle is an example of the 2$-$Narayana sequence:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 2 & & & \\ & & 2 & & 5 & & 12 & & \\ & 4 & & 10 & & 25 & & 62 & \\ 9 & & 22 & & 54 & & 133 & & 328\end{array}$$

Figure 3: 2$-$Narayana sequence and its rising 2$-$binomial transform

5. The falling $k-$Binomial transform of the $k-$Narayana sequence

The falling $k-$binomial transform of the $k-$Narayana sequence $\{N_{k,n}\}_{n\in\mathbb{N}}$ is denoted by $\{f_{k,n}\}_{n\in\mathbb{N}}$ where

$$f_{k,n}=\sum_{i=0}^{n}\binom{n}{i}k^{n-i}N_{k,i}.$$

Theorem 5.1. *The recurrence relation obtained for $\{f_{k,n}\}$ is as follows:*

$$f_{k,n+3}=4kf_{k,n+2}-5k^2f_{k,n+1}+(2k^3+1)f_{k,n}. \tag{5.1}$$

The characteristic equation of sequences $\{b_{k,n}\}$ in (5.1) is

$$\theta^3-4k\theta^2+5k^2\theta-(2k^3+1)=0,$$

whose solutions are θ_1, θ_2, and θ_3.
Next we derive the Binet formula for $\{f_{k,n}\}$.

Theorem 5.2. *Whichever term of* $\{f_{k,n}\}$ *can be computed using the Binet formula. It is indicated by*

$$f_{k,n}=\frac{t_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)}+\frac{t_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)}+\frac{t_3\theta_3^n}{(\theta_3-\theta_1)(\theta_2-\theta_3)}$$

where

$$\begin{aligned} t_1 &= 3k-\theta_2-\theta_3 \\ t_2 &= 3k-\theta_1-\theta_3 \\ t_3 &= 3k-\theta_1-\theta_2 \end{aligned}$$

Theorem 5.3. *The generating function of* $\{f_{k,n}\}$ *is:*

$$f_k(x)=\frac{(1-4kx+5k^2x^2)f_{k,0}+(x-4kx^2)f_{k,1}+x^2f_{k,2}}{1-4kx+5k^2x^2-2k^3x^3-x^3}.$$

Now, we present a new triangle of $\{f_{k,n}\}$ for each k. For example following triangle is for $2-$Narayana sequence and its falling $2-$binomial transform

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 2 & & 4 & & 6 & & \\ & 4 & & 8 & & 16 & & 28 & \\ 9 & & 17 & & 33 & & 65 & & 121 \end{array}$$

Figure 4: $2-$Narayana sequence and its falling $2-$binomial transform

Declarations

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