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Research Article

Some classifications for Gauss map of Tubular hypersurfaces in $\mathbb{E}_1^4$ concerning linearized operators $\mathcal{L}_k$

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Abstract

In this study, we deal with the Gauss map of tubular hypersurfaces in 4-dimensional Lorentz-Minkowski space concerning the linearized operators $\mathcal{L}_1$ (Cheng-Yau) and $\mathcal{L}_2$. We obtain the $\mathcal{L}_1$ (Cheng-Yau) operator of the Gauss map of tubular hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on timelike or spacelike curves with non-null Frenet vectors in $\mathbb{E}_1^4$ and give some classifications for these hypersurfaces which have generalized $\mathcal{L}_k$ 1-type Gauss map, first kind $\mathcal{L}_k$-pointwise 1-type Gauss map, second kind $\mathcal{L}_k$-pointwise 1-type Gauss map and $\mathcal{L}_k$-harmonic Gauss map, $k \in \{1, 2\}$.

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1. Introduction

Let (M, g) be a hypersurface of $(n+1)$-dimensional Minkowski space $\mathbb{E}_1^{n+1}$, Δ denote its Laplace operator. A smooth mapping $\phi : M \to \mathbb{E}_1^{n+1}$ is said to be *finite type* if it can be expressed as

$$\phi = \phi_0 + \phi_1 + \cdots + \phi_k,$$

where ϕ_0 is a constant vector and ϕ_i is an eigenvector of Δ corresponding to the eigenvector λ_i for $i = 1, 2, \ldots, k$. More precisely, if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, then ψ is said to be *k-type* ([4, 6, 7]). Several results on the study of finite type mappings were summed up in a report by B.-Y. Chen in [5] (See also [8, 24]).

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Let N denote the Gauss map of M. From the definition above, one can conclude that N is of *1-type* if and only if it satisfies the equation

$$\Delta N = \lambda(N + C) \tag{1.1}$$

for a constant $\lambda \in \mathbb{R}$ and a constant vector C. However, Gauss map of some important submanifolds such as catenoid and helicoid of the Euclidean 3-space $\mathbb{E}^3$ satisfies

$$\Delta N = f(N + C) \tag{1.2}$$

which is weaker than (1.1), where $f \in C^\infty(M)$ is a smooth function, [10]. These submanifolds whose Gauss map N satisfying (1.2) are said to have *pointwise 1-type Gauss map*. Submanifolds with pointwise 1-type Gauss map have been worked in several papers (cf. [10, 20, 23, 24]).

On the other hand, the Gauss map of some hypersurfaces of semi-Euclidean spaces satisfies the equation

$$\Delta N = f_1 N + f_2 C \tag{1.3}$$

for some smooth functions f_1, f_2 and a constant vector C. A submanifold is said to have *generalized 1-type Gauss map* if its Gauss map satisfies the condition (1.3), [25]. After this definition was given, hypersurfaces of pseudo-Euclidean spaces have been considered in terms of having generalized 1-type Gauss map, [17, 19, 25, 26].

In the recent years, the definition of $\mathcal{L}_k$-finite type maps has been obtained by replacing Δ in the definition above with the sequence of operators $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{n-1}$, [1,2]. Note that, by the definition of these operators, one can obtain $\mathcal{L}_0 = -\Delta$ and $\mathcal{L}_1 = \Box$ is called as the Cheng-Yau operator introduced in [9]. By motivating this idea, notion of $\mathcal{L}_k$-pointwise 1-type Gauss map and generalized $\mathcal{L}_k$ 1-type Gauss map was presented in [14] and [18], respectively (see Definition 2.1). After the case $k = 1$ is studied in these papers, many result obtained on hypersurfaces with certain type of Gauss map, [11–13, 19, 22, 25, 26].

On the other hand, in [3], the general expression of the canal hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres, pseudo hyperbolic hyperspheres and null hypercones whose centers lie on a non-null curve with non-null Frenet vector fields in $\mathbb{E}^4_1$ has been given and their some geometric invariants such as unit normal vector fields, Gaussian curvatures, mean curvatures and principal curvatures have been obtained. Also, tubular hypersurfaces in $\mathbb{E}^4_1$ by taking constant radius function have been studied in [3].

In this paper, we study the tubular hypersurfaces in Lorentz-Minkowski 4-space $\mathbb{E}^4_1$ with the aid of $\mathcal{L}_k$ operators, $k \in \{1, 2\}$. In Sect. 2, we give basic notation, facts and definitions about hypersurfaces of Minkowski spaces. In Sect. 3 and Sect 4, we consider some classifications of tubular hypersurfaces by considering their Gauss maps in terms of their types with respect to the operators $\mathcal{L}_1$ and $\mathcal{L}_2$.

2. Preliminaries

Let $\mathbb{E}^{n+1}_1$ be the $(n+1)$-dimensional Lorentz-Minkowski space with the canonical pseudo-Euclidean metric $\langle\ ,\rangle$ of index 1 and signature $(-, +, +, ..., +)$ given by

$$\langle\ ,\rangle = -dx_1^2 + dx_2^2 + dx_3^2 + ... + dx_{n+1}^2$$

where $(x_1, x_2, ..., x_{n+1})$ is a rectangular coordinate system in $\mathbb{E}^{n+1}_1$.

If $\Gamma : M \longrightarrow \mathbb{E}^{n+1}_1$ is an isometric immersion from an n-dimensional orientable manifold M to $\mathbb{E}^{n+1}_1$, then the induced metric on M by the immersion Γ can be Riemannian or Lorentzian. Let N denotes a unit normal vector field and put $\langle N, N\rangle = \varepsilon = \pm 1$, so that $\varepsilon = 1$ or $\varepsilon = -1$ according to M is endowed with a Lorentzian or Riemannian metric, respectively.

The operator $\mathcal{L}_k$ acting on the coordinate functions of the Gauss map N of the hypersurface M in $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{E}_1^{n+1}$ is

$$\mathcal{L}_k N = -\varepsilon \mathfrak{C}_k \left(\nabla H_{k+1} + \left(n H_1 H_{k+1} - (n-k-1) H_{k+2} \right) N \right). \tag{2.1}$$

Here,

$$\binom{n}{k} H_k = (-\varepsilon)^k a_k, \quad \left(\binom{n}{k} = \frac{n!}{k!(n-k)!} \right), \tag{2.2}$$

such that

$$\left. \begin{array}{l} a_1 = -\sum_{i=1}^{n} \kappa_i, \\ \\ a_k = (-1)^k \sum_{i_1<i_2<...<i_k}^{n} \kappa_{i_1}\kappa_{i_2}...\kappa_{i_k}, \ \ k = 2,3,...,n \end{array} \right\} \tag{2.3}$$

and H_k is called the k-th mean curvature of order k of M.

Also, the constant $\mathfrak{C}_k$ is given by

$$\mathfrak{C}_k = \binom{n}{k+1} (-\varepsilon)^k. \tag{2.4}$$

(For more details about the linearized operator $\mathcal{L}_k$, one can see [16].)

Definition 2.1. Let $\mathfrak{m}$ and $\mathfrak{n}$ be non-zero smooth functions on M, $C \in \mathbb{E}_1^{n+1}$ be a non-zero constant vector and $k \in \{0,1,2,...,n\}$.

If the Gauss map N of an oriented submanifold M in $\mathbb{E}_1^4$ satisfies

i: $\mathcal{L}_k N = \mathfrak{m} N + \mathfrak{n} C$, then M has generalized $\mathcal{L}_k$ 1-type Gauss map;
ii: $\mathcal{L}_k N = \mathfrak{m} N$, then M has first kind $\mathcal{L}_k$-pointwise 1-type Gauss map;
iii: $\mathcal{L}_k N = \mathfrak{m}(N + C)$, then M has second kind $\mathcal{L}_k$-pointwise 1-type Gauss map;
iv: $\mathcal{L}_k N = 0$, then N is called $\mathcal{L}_k$-harmonic.

In this study, we will deal with Gauss maps of tubular hypersurfaces in 4-dimensional Lorentz-Minkowski space $\mathbb{E}_1^4$ concerning linearized operators $\mathcal{L}_1$ and $\mathcal{L}_2$. So, let us give some notions in $\mathbb{E}_1^4$.

Let $\overrightarrow{u} = (u_1, u_2, u_3, u_4)$, $\overrightarrow{v} = (v_1, v_2, v_3, v_4)$ and $\overrightarrow{w} = (w_1, w_2, w_3, w_4)$ be three vectors in $\mathbb{E}_1^4$. The inner product and vector product are defined by

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle = -u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 \tag{2.5}$$

and

$$\overrightarrow{u} \times \overrightarrow{v} \times \overrightarrow{w} = \det \begin{bmatrix} -e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix}, \tag{2.6}$$

respectively. Here $e_i, (i = 1,2,3,4)$ are standard basis vectors.

A vector $\overrightarrow{u} \in \mathbb{E}_1^4$ is called spacelike, timelike or lightlike (null) if $\langle \overrightarrow{u}, \overrightarrow{u} \rangle > 0$ (or $\overrightarrow{u} = 0$), $\langle \overrightarrow{u}, \overrightarrow{u} \rangle < 0$ or $\langle \overrightarrow{u}, \overrightarrow{u} \rangle = 0$, respectively. A curve $\beta(s)$ in $\mathbb{E}_1^4$ is spacelike, timelike or lightlike (null), if all its velocity vectors $\beta'(s)$ are spacelike, timelike or lightlike, respectively and a non-null (i.e. timelike or spacelike) curve has unit speed if $\langle \beta', \beta' \rangle = \mp 1$. Also, the norm of the vector $\overrightarrow{u}$ is $\| \overrightarrow{u} \| = \sqrt{|\langle \overrightarrow{u}, \overrightarrow{u} \rangle|}$ [15].

Let F_1, F_2, F_3, F_4 be unit tangent vector field, principal normal vector field, binormal vector field, trinormal vector field of a timelike or spacelike curve $\beta(s)$, respectively and $\{F_1, F_2, F_3, F_4\}$ be the moving Frenet frame along $\beta(s)$ in $\mathbb{E}_1^4$. The Frenet equations can be given according to the causal characters of non-null Frenet vector fields F_1, F_2, F_3 and F_4 as follows [21]:

If the curve $\beta(s)$ is timelike, i.e. $\langle F_1, F_1\rangle = -1$, $\langle F_i, F_i\rangle = 1$ $(i = 2, 3, 4)$, then

$$\left.\begin{array}{l} F_1' = k_1 F_2, \\ F_2' = k_1 F_1 + k_2 F_3, \\ F_3' = -k_2 F_2 + k_3 F_4, \\ F_4' = -k_3 F_3; \end{array}\right\} \tag{2.7}$$

if the curve $\beta(s)$ is spacelike with timelike principal normal vector field F_2, i.e. $\langle F_2, F_2\rangle = -1$, $\langle F_i, F_i\rangle = 1$ $(i = 1, 3, 4)$, then

$$\left.\begin{array}{l} F_1' = k_1 F_2, \\ F_2' = k_1 F_1 + k_2 F_3, \\ F_3' = k_2 F_2 + k_3 F_4, \\ F_4' = -k_3 F_3; \end{array}\right\} \tag{2.8}$$

if the curve $\beta(s)$ is spacelike with timelike binormal vector field F_3, i.e. $\langle F_3, F_3\rangle = -1$, $\langle F_i, F_i\rangle = 1$ $(i = 1, 2, 4)$, then

$$\left.\begin{array}{l} F_1' = k_1 F_2, \\ F_2' = -k_1 F_1 + k_2 F_3, \\ F_3' = k_2 F_2 + k_3 F_4, \\ F_4' = k_3 F_3; \end{array}\right\} \tag{2.9}$$

if the curve $\beta(s)$ is spacelike with timelike trinormal vector field F_4, i.e. $\langle F_4, F_4\rangle = -1$, $\langle F_i, F_i\rangle = 1$ $(i = 1, 2, 3)$, then

$$\left.\begin{array}{l} F_1' = k_1 F_2, \\ F_2' = -k_1 F_1 + k_2 F_3, \\ F_3' = -k_2 F_2 + k_3 F_4, \\ F_4' = k_3 F_3. \end{array}\right\} \tag{2.10}$$

Here k_1, k_2, k_3 are the first, second and third curvatures of the non-null curve $\beta(s)$.

Also, if p is a fixed point in $\mathbb{E}^4_1$ and r is a positive constant, then the pseudo-Riemannian hypersphere and the pseudo-Riemannian hyperbolic space are defined by

$$S^3_1(p, r) = \{x \in \mathbb{E}^4_1 : \langle x - p, x - p\rangle = r^2\}$$

and

$$H^3_0(p, r) = \{x \in \mathbb{E}^4_1 : \langle x - p, x - p\rangle = -r^2\},$$

respectively.

If M is an oriented hypersurface in $\mathbb{E}^4_1$, then the gradient of a smooth function $f(s, t, w)$, defined on M, can be obtained by

$$\nabla f = \frac{1}{\mathfrak{g}} \left(\begin{array}{c} \left(\left(g_{23}^2 - g_{22}g_{33} \right) f_s + \left(-g_{13}g_{23} + g_{12}g_{33} \right) f_t + \left(g_{13}g_{22} - g_{12}g_{23} \right) f_w \right) \partial s \\ + \left(\left(-g_{13}g_{23} + g_{12}g_{33} \right) f_s + \left(g_{13}^2 - g_{11}g_{33} \right) f_t + \left(-g_{12}g_{13} + g_{11}g_{23} \right) f_w \right) \partial t \\ + \left(\left(g_{13}g_{22} - g_{12}g_{23} \right) f_s + \left(-g_{12}g_{13} + g_{11}g_{23} \right) f_t + \left(g_{12}^2 - g_{11}g_{22} \right) f_w \right) \partial w \end{array} \right), \tag{2.11}$$

where

$$\mathfrak{g} = g_{13}^2 g_{22} - 2g_{12}g_{13}g_{23} + g_{11}g_{23}^2 + g_{12}^2 g_{33} - g_{11}g_{22}g_{33};$$

$\{s, t, w\}$ is a local coordinat system of M; f_s, f_t, f_w are the partial derivatives of f and $g_{11} = \langle \partial s, \partial s\rangle$, $g_{12} = \langle \partial s, \partial t\rangle$, $g_{13} = \langle \partial s, \partial w\rangle$, $g_{22} = \langle \partial t, \partial t\rangle$, $g_{23} = \langle \partial t, \partial w\rangle$, $g_{33} = \langle \partial w, \partial w\rangle$.

3. Some classifications for Tubular hypersurfaces generated by timelike curves with L_k operators in $\mathbb{E}_1^4$

In this section, we obtain the $\mathcal{L}_1$ (Cheng-Yau) and $\mathcal{L}_2$ operators of the Gauss map of the tubular hypersurfaces $\mathcal{T}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres whose centers lie on a timelike curve with non-null Frenet vectors in $\mathbb{E}_1^4$ and give some classifications for these hypersurfaces which have generalized $\mathcal{L}_k$ 1-type Gauss map, first kind $\mathcal{L}_k$-pointwise 1-type Gauss map and second kind $\mathcal{L}_k$-pointwise 1-type Gauss map and $\mathcal{L}_k$-harmonic Gauss map, $k\in\{1,2\}$.

The tubular hypersurfaces $\mathcal{T}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres whose centers lie on a timelike curve with non-null Frenet vectors in $\mathbb{E}_1^4$ can be parametrized by

$$\mathcal{T}(s,t,w)=\beta(s)+r(\cos t\cos wF_2(s)+\sin t\cos wF_3(s)+\sin wF_4(s)). \tag{3.1}$$

The unit normal vector field of (3.1) is

$$N=-\left(\cos t\cos wF_2+\sin t\cos wF_3+\sin wF_4\right) \tag{3.2}$$

and so,

$$\langle N,N\rangle=1. \tag{3.3}$$

The coefficients of the first fundamental form of (3.1) are

$$\left.\begin{aligned}&g_{11}=(rk_2\cos t\cos w-rk_3\sin w)^2+r^2\left(k_2^2+k_3^2\right)\sin^2 t\cos^2 w-(1+rk_1\cos t\cos w)^2,\\&g_{12}=g_{21}=r^2(k_2\cos w-k_3\cos t\sin w)\cos w,\ \ g_{22}=r^2\cos^2 w,\\&g_{13}=g_{31}=r^2k_3\sin t,\ \ g_{23}=g_{32}=0,\ \ g_{33}=r^2.\end{aligned}\right\} \tag{3.4}$$

The principal curvatures of (3.1) are

$$\kappa_1=\kappa_2=\frac{1}{r},\ \ \kappa_3=\frac{k_1\cos t\cos w}{1+rk_1\cos t\cos w}. \tag{3.5}$$

For more details about these hypersurfaces, one can see [3].

3.1. Some classifications for Tubular hypersurfaces generated by timelike curves with $\mathcal{L}_1$ (Cheng-Yau) operator in $\mathbb{E}_1^4$

The functions a_k of the tubular hypersurfaces (3.1) in $\mathbb{E}_1^4$ are obtained from (2.3) and (3.5) by

$$a_1=\frac{-2-3rk_1\cos t\cos w}{r(1+rk_1\cos t\cos w)},\ \ a_2=\frac{1+3rk_1\cos t\cos w}{r^2(1+rk_1\cos t\cos w)},\ \ a_3=-\frac{k_1}{r^2(rk_1+\sec t\sec w)}. \tag{3.6}$$

Also, from (2.11), (3.4) and (3.6), we have

$$\begin{aligned}\nabla a_2=&-\frac{2\left(k_1'\cos t+k_1k_2\sin t\right)\cos w}{r(1+rk_1\cos t\cos w)^3}F_1-\frac{k_1\left(2\cos^2 t\cos(2w)+\cos(2t)-3\right)}{2r^2(1+rk_1\cos t\cos w)^2}F_2\\&-\frac{2k_1\sin t\cos t\cos^2 w}{r^2(1+rk_1\cos t\cos w)^2}F_3-\frac{2k_1\cos t\sin w\cos w}{r^2(1+rk_1\cos t\cos w)^2}F_4.\end{aligned} \tag{3.7}$$

So, from (2.1), (2.2), (3.2), (3.3), (3.6) and (3.7), we reach that

$$\begin{aligned}\mathcal{L}_1N=&-\frac{2\left(k_1k_2\sin t+k_1'\cos t\right)\cos w}{r(1+rk_1\cos t\cos w)^3}F_1\\&-\frac{2\left(rk_1\left(3rk_1\cos^3 t\cos^3 w+2\cos^2 t\cos(2w)+\cos(2t)\right)+\cos t\cos w\right)}{r^3(1+rk_1\cos t\cos w)^2}F_2\\&-\frac{2(1+3rk_1\cos t\cos w)\sin t\cos w}{r^3(1+rk_1\cos t\cos w)}F_3-\frac{2(3rk_1\cos t\cos w+1)\sin w}{r^3(1+rk_1\cos t\cos w)}F_4.\end{aligned} \tag{3.8}$$

Now, let us give some classifications for the tubular hypersurfaces (3.1) which have generalized $\mathcal{L}_1$ 1-type Gauss map, first kind $\mathcal{L}_1$-pointwise 1-type Gauss map, second kind $\mathcal{L}_1$-pointwise 1-type Gauss map and $\mathcal{L}_1$-harmonic Gauss map.

Let the tubular hypersurfaces $\mathcal{T}(s,t,w)$ have generalized $\mathcal{L}_1$ (Cheng-Yau) 1-type Gauss map, i.e., $\mathcal{L}_1 N = \mathfrak{m}N + \mathfrak{n}C$, where $C = C_1F_1 + C_2F_2 + C_3F_3 + C_4F_4$ is a constant vector. Here, by taking derivatives of the constant vector C with respect to s, from (2.7) we obtain that

$$\left.\begin{array}{l} C_1' + C_2k_1 = 0, \\ C_2' + C_1k_1 - C_3k_2 = 0, \\ C_3' + C_2k_2 - C_4k_3 = 0, \\ C_4' + C_3k_3 = 0. \end{array}\right\} \tag{3.9}$$

Also, by taking derivatives the constant vector C with respect to t and w separately, one can see that the functions C_i depend only on s.

Firstly, let us classify the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have generalized $\mathcal{L}_1$ (Cheng-Yau) 1-type Gauss map.

From (3.2) and (3.8), we get

$$\left.\begin{array}{l} -\frac{2\left(k_1'\cos t + k_1k_2\sin t\right)\cos w}{r(1+rk_1\cos t\cos w)^3} = \mathfrak{n}C_1, \\ -\frac{2\left(rk_1\left(3rk_1\cos^3 t\cos^3 w + 2\cos^2 t\cos(2w) + \cos(2t)\right) + \cos t\cos w\right)}{r^3(1+rk_1\cos t\cos w)^2} = \mathfrak{m}\left(-\cos t\cos w\right) + \mathfrak{n}C_2, \\ -\frac{2(1+3rk_1\cos t\cos w)\sin t\cos w}{r^3(1+rk_1\cos t\cos w)} = \mathfrak{m}\left(-\sin t\cos w\right) + \mathfrak{n}C_3, \\ -\frac{2(1+3rk_1\cos t\cos w)\sin w}{r^3(1+rk_1\cos t\cos w)} = \mathfrak{m}\left(-\sin w\right) + \mathfrak{n}C_4. \end{array}\right\} \tag{3.10}$$

Now, let us investigate the non-zero functions $\mathfrak{m}(s,t,w)$ and $\mathfrak{n}(s,t,w)$ from the above four equations.

Firstly, let us assume that $C_1 \neq 0$.

In this case, from the first equation of (3.10) it's easy to see that

$$\mathfrak{n}(s,t,w) = -\frac{2\left(k_1'\cos t + k_1k_2\sin t\right)\cos w}{r(1+rk_1\cos t\cos w)^3 C_1}. \tag{3.11}$$

Here, when the equation (3.11) is successively substituted into the second, third and fourth equations of (3.10), we obtain

$$\mathfrak{m}(s,t,w) = \frac{2\left(\begin{array}{l}\left(rk_1\left(\begin{array}{l}3rk_1\cos^3 t\cos^3 w\\ +2\cos^2 t\cos(2w) + \cos(2t)\end{array}\right) + \cos t\cos w\right) C_1(1+rk_1\cos t\cos w) \\ -C_2r^2\left(k_1'\cos t + k_1k_2\sin t\right)\cos w\end{array}\right)}{C_1r^3(1+rk_1\cos t\cos w)^3\cos t\cos w},$$

$$\mathfrak{m}(s,t,w) = \frac{2\left(C_1(1+rk_1\cos t\cos w)^2(1+3rk_1\cos t\cos w) - C_3r^2\left(k_1'\cot t + k_1k_2\right)\right)}{C_1r^3(1+rk_1\cos t\cos w)^3},$$

$$\mathfrak{m}(s,t,w) = \frac{2\left(C_1(1+rk_1\cos t\cos w)^2(1+3rk_1\cos t\cos w) - C_4r^2\left(k_1'\cos t + k_1k_2\sin t\right)\cot w\right)}{C_1r^3(1+rk_1\cos t\cos w)^3}.$$

When we equate the functions $\mathfrak{m}(s,t,w)$ found above to each other, we arrive at the following equations:

$$(C_3 - C_4 \sin t \cot w)\left(k_1' \cot t + k_1 k_2\right) = 0, \tag{3.12}$$

$$k_1(C_1 \sec t \sec w + k_2 r(C_2 \tan t - C_4 \sin t \cot w)) + r(C_1 k_1^2 - k_1'(C_4 \cos t \cot w - C_2)) = 0, \tag{3.13}$$

$$k_1(rk_2(C_3 - C_2 \tan t) - C_1 \sec t \sec w) - C_1 r k_1^2 + r k_1'(C_3 \cot t - C_2) = 0. \tag{3.14}$$

In the equation (3.12), it holds that $k_1' \cot t + k_1 k_2 \neq 0$. This is because, when $k_1' \cot t + k_1 k_2 = 0$, the function $\mathfrak{n}(s,t,w)$ in the first equation of (3.10) becomes zero. This, in turn, contradicts the definition of the function $\mathfrak{n}(s,t,w)$ in our classification as $\mathcal{L}_1 N = \mathfrak{m}N + \mathfrak{n}C$. So, from the equation (3.12) and $k_1' \cot t + k_1 k_2 \neq 0$, we have $C_3 = C_4 = 0$. When $C_3 = C_4 = 0$, substituting this into the equation (3.14) yields

$$\left(C_1 r k_1^2 + C_2 r k_1'\right) \cos t + C_1 k_1 \sec w + C_2 r k_1 k_2 \sin t = 0.$$

Thus, we have

$$C_1 k_1^2 + C_2 k_1' = C_1 k_1 = C_2 k_1 k_2 = 0$$

and so $C_1 = C_2 = 0$. This is a contradiction.

Secondly, let us assume that $C_1 = 0$.

In this case, from the first equation of the set of equations (3.9) it's easy to see that

$$C_2 k_1 = 0. \tag{3.15}$$

If $k_1 = 0$ in (3.15), then from the second, third and fourth equations of (3.10), it is calculated as

$$\left.\begin{aligned} C_2 r^3 \mathfrak{n}(s,t,w) &= \left(\mathfrak{m}(s,t,w) r^3 - 2\right) \cos t \cos w, \\ C_3 r^3 \mathfrak{n}(s,t,w) &= \left(\mathfrak{m}(s,t,w) r^3 - 2\right) \sin t \cos w, \\ C_4 r^3 \mathfrak{n}(s,t,w) &= \left(\mathfrak{m}(s,t,w) r^3 - 2\right) \sin w, \end{aligned}\right\} \tag{3.16}$$

respectively. Since the functions C_i depend only on s, there is no solution for functions $\mathfrak{n}(s,t,w)$ in (3.16).

Now, let us assume that $C_2 = 0$ in (3.15). In this case, from the second equation of (3.10), it's easy to see that

$$\mathfrak{m}(s,t,w) = \frac{2\left(\cos t \cos w + r k_1 \left(\cos(2t) + 3 r k_1 \cos^3 t \cos^3 w + 2 \cos^2 t \cos(2w)\right)\right)}{r^3 (1 + r k_1 \cos t \cos w)^2 \cos t \cos w}. \tag{3.17}$$

Here, when the equation (3.17) is successively substituted into the third and fourth equations of (3.10), we obtain

$$\mathfrak{n}(s,t,w) C_3 = \frac{-2k_1}{r^2 (1 + r \cos t \cos w k_1)^2} \tan t,$$

$$\mathfrak{n}(s,t,w) C_4 = \frac{-2k_1}{r^2 (1 + r k_1 \cos t \cos w)^2} \sec t \tan w.$$

Here, there is no solution for functions $\mathfrak{n}(s,t,w)$.

Hence, we can state the following theorem:

Theorem 3.1. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with generalized $\mathcal{L}_1$ 1-type Gauss map.*

Now, let us classify the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have second kind $\mathcal{L}_1$-pointwise 1-type Gauss map, i.e., $\mathcal{L}_1 N = \mathfrak{m}\left(N + C\right)$.

From (3.2) and (3.8), we get

$$\left.\begin{aligned}
&-\tfrac{2\left(k_1'\cos t+k_1k_2\sin t\right)\cos w}{r(1+rk_1\cos t\cos w)^3}=\mathfrak{m}C_1,\\
&-\tfrac{2\left(rk_1\left(3rk_1\cos^3 t\cos^3 w+2\cos^2 t\cos(2w)+\cos(2t)\right)+\cos t\cos w\right)}{r^3(1+rk_1\cos t\cos w)^2}=\mathfrak{m}\left(-\cos t\cos w+C_2\right),\\
&-\tfrac{2(1+3rk_1\cos t\cos w)\sin t\cos w}{r^3(1+rk_1\cos t\cos w)}=\mathfrak{m}\left(-\sin t\cos w+C_3\right),\\
&-\tfrac{2(1+3rk_1\cos t\cos w)\sin w}{r^3(1+rk_1\cos t\cos w)}=\mathfrak{m}\left(-\sin w+C_4\right).
\end{aligned}\right\} \quad (3.18)$$

Here, from the fourth equation of (3.18) it's easy to see that

$$\mathfrak{m}(s,t,w)=-\frac{2\left(1+3rk_1\cos t\cos w\right)\sin w}{r^3(1+rk_1\cos t\cos w)\left(-\sin w+C_4\right)}. \quad (3.19)$$

When the equation (3.19) is successively substituted into the second and third equations of (3.18), we obtain

$$\begin{aligned}
&3C_2r^2k_1^2\cos^2 t\cos^2 w+4C_2rk_1\cos t\cos w+C_2-rk_1=0,\\
&C_4\sin t\cos w-C_3\sin w=0.
\end{aligned}$$

So, we have

$$k_1=C_2=C_3=C_4=0. \quad (3.20)$$

Now, when the components of the equation (3.20) is substituted into the second, third or fourth equations of (3.18), we calculated

$$\mathfrak{m}(s,t,w)=\frac{2}{r^3}. \quad (3.21)$$

Also, from the first equation of (3.18) and (3.21), we have $C_1=0$.

From the calculations made above for classify the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have second kind $\mathcal{L}_1$-pointwise 1-type Gauss map, i.e., $\mathcal{L}_1N=\mathfrak{m}\left(N+C\right)$, we can give the following theorem:

Theorem 3.2. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in* $\mathbb{E}_1^4$*, with second kind* $\mathcal{L}_1$*-pointwise 1-type Gauss map.*

Moreover, if the function m is constant in Definition 2.1 (**ii** or **iii**), then we say M has first or second kind $\mathcal{L}_k$-(global) pointwise 1-type Gauss map. Thus, we can state the following theorem:

Theorem 3.3. *The tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in* $\mathbb{E}_1^4$*, have first kind* $\mathcal{L}_1$*-(global) pointwise 1-type Gauss map, i.e.,* $\mathcal{L}_1N=\mathfrak{m}N$ *if and only if* $k_1=0$*, where* $\mathfrak{m}(s,t,w)=\frac{2}{r^3}$.

Finally, in the equation (3.8), since all of the coefficients of $F_1,\ F_2,\ F_3$ and F_4 cannot be zero, we can give the following theorem:

Theorem 3.4. *The tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in* $\mathbb{E}_1^4$*, cannot have* $\mathcal{L}_1$*-harmonic Gauss map.*

3.2. Some classifications for Tubular hypersurfaces generated by timelike curves with $\mathcal{L}_2$ operator in $\mathbb{E}_1^4$

Firstly, it is calculated from (2.11), (3.4) and (3.6) as

$$\begin{aligned}\nabla a_3 =& \frac{(k_1' \cos t + k_1 k_2 \sin t)\cos w}{r^2(1+rk_1\cos t\cos w)^3}F_1 + \frac{k_1\left(2\cos^2 t\cos(2w)+\cos(2t)-3\right)}{4r^3(1+rk_1\cos t\cos w)^2}F_2 \\ &+ \frac{k_1\sin t\cos t\cos^2 w}{r^3(1+rk_1\cos t\cos w)^2}F_3 + \frac{k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)^2}F_4. \end{aligned} \tag{3.22}$$

So, from (2.1), (2.2), (3.2), (3.3), (3.6) and (3.22), we have

$$\begin{aligned}\mathcal{L}_2 N =& \frac{(k_1' \cos t + k_1 k_2 \sin t)\cos w}{r^2(1+rk_1\cos t\cos w)^3}F_1 \\ &+ \frac{k_1\left(\begin{array}{l}\frac{rk_1}{2}\left(\begin{array}{l}24rk_1\cos^4 t\cos^4 w + 12\cos^3 t\cos(3w)\\ +19\cos t\cos w + 9\cos(3t)\cos w\end{array}\right)\\ +6\cos^2 t\cos(2w)+3\cos(2t)-1\end{array}\right)}{4r^3(1+rk_1\cos t\cos w)^3}F_2 \\ &+ \frac{3k_1\sin t\cos t\cos^2 w}{r^3(1+rk_1\cos t\cos w)}F_3 + \frac{3k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)}F_4. \end{aligned} \tag{3.23}$$

Now, let us give some classifications for the tubular hypersurfaces (3.1) which have generalized $\mathcal{L}_2$ 1-type Gauss map, first kind $\mathcal{L}_2$-pointwise 1-type Gauss map, second kind $\mathcal{L}_2$-pointwise 1-type Gauss map and $\mathcal{L}_2$-harmonic Gauss map.

Now, let us classify the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have generalized $\mathcal{L}_2$ 1-type Gauss map. From (3.2) and (3.23), we get

$$\left.\begin{aligned} &\frac{\left(k_1'\cos t + k_1k_2\sin t\right)\cos w}{r^2(1+rk_1\cos t\cos w)^3} = \mathfrak{n}C_1, \\ &\frac{k_1\left(\begin{array}{l}\frac{rk_1}{2}\left(\begin{array}{l}24rk_1\cos^4 t\cos^4 w + 12\cos^3 t\cos(3w)\\ +19\cos t\cos w + 9\cos(3t)\cos w\end{array}\right)\\ +6\cos^2 t\cos(2w)+3\cos(2t)-1\end{array}\right)}{4r^3(1+rk_1\cos t\cos w)^3} = \mathfrak{m}\left(-\cos t\cos w\right) + \mathfrak{n}C_2, \\ &\frac{3k_1\sin t\cos t\cos^2 w}{r^3(1+rk_1\cos t\cos w)} = \mathfrak{m}\left(-\sin t\cos w\right) + \mathfrak{n}C_3, \\ &\frac{3k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)} = \mathfrak{m}\left(-\sin w\right) + \mathfrak{n}C_4. \end{aligned}\right\} \tag{3.24}$$

Firstly, let us assume that $C_1 \neq 0$.

In this case, from the first equation of (3.24) it's easy to see that

$$\mathfrak{n}(s,t,w) = \frac{(k_1'\cos t + k_1k_2\sin t)\cos w}{r^2(1+rk_1\cos t\cos w)^3 C_1}. \tag{3.25}$$

Here, when the equation (3.25) is successively substituted into the second, third and fourth equations of (3.24), we obtain

$$\mathfrak{m}(s,t,w) = \frac{\left(\begin{array}{l}2k_1(-3C_1\cos t\cos w + C_1\sec t\sec w + C_2rk_2\tan t) + 2C_2rk_1'\\ -6C_1r^2k_1^3\cos^3 t\cos^3 w - C_1rk_1^2\left(6\cos^2 t\cos(2w)+3\cos(2t)+1\right)\end{array}\right)}{2C_1r^3(1+rk_1\cos t\cos w)^3},$$

$$\mathfrak{m}(s,t,w) = \frac{C_3r\left(k_1'\cot t + k_1k_2\right) - 3C_1k_1(1+rk_1\cos t\cos w)^2\cos t\cos w}{C_1r^3(1+rk_1\cos t\cos w)^3},$$

$$\mathfrak{m}(s,t,w) = \frac{C_4r\left(k_1'\cos t + k_1k_2\sin t\right)\cot w - 3C_1k_1(1+rk_1\cos t\cos w)^2\cos t\cos w}{C_1r^3(1+rk_1\cos t\cos w)^3}.$$

When we equate the functions $\mathfrak{m}(s,t,w)$ found above to each other, we arrive at the following equations:

$$(C_4 \sin t \cos w - C_3 \sin w)\left(k_1' \cos t + k_1 k_2 \sin t\right) = 0, \tag{3.26}$$

$$k_1(C_1 \sec t \sec w + rk_2(C_2 \tan t - C_4 \sin t \cot w)) + C_1 r k_1^2 + r k_1'(C_2 - C_4 \cos t \cot w) = 0, \tag{3.27}$$

$$k_1(C_1 \sec t \sec w + rk_2(C_2 \tan t - C_3)) + C_1 r k_1^2 + r k_1'(C_2 - C_3 \cot t) = 0. \tag{3.28}$$

In the equation (3.26), it holds that $k_1' \cos t + k_1 k_2 \sin t \neq 0$. This is because when $k_1' \cos t + k_1 k_2 \sin t = 0$, the function $\mathfrak{n}(s,t,w)$ in the first equation of (3.24) becomes zero. This, in turn, contradicts the definition of the function $\mathfrak{n}(s,t,w)$ in our classification as $\mathcal{L}_2 N = \mathfrak{m}N + \mathfrak{n}C$. So, from the equation (3.26) and $k_1' \cos t + k_1 k_2 \sin t \neq 0$, we have $C_3 = C_4 = 0$. When $C_3 = C_4 = 0$, substituting this into the equation (3.28) yields

$$r\left(C_1 k_1^2 + C_2 k_1'\right)\cos t + C_2 r k_1 k_2 \sin t + C_1 k_1 \sec w = 0.$$

Thus, we have

$$C_1 k_1^2 + C_2 k_1' = C_1 k_1 = C_2 k_1 k_2 = 0$$

and so $C_1 = C_2 = 0$. This is a contradiction.

Secondly, let us assume that $C_1 = 0$.

In this case, from the first equation of the set of equations (3.9) it's easy to see that

$$C_2 k_1 = 0. \tag{3.29}$$

If $k_1 = 0$ in (3.29), then from the second, third and fourth equations of (3.24), it is calculated as

$$\left.\begin{array}{l} \mathfrak{m}(s,t,w)\cos t \cos w = \mathfrak{n}(s,t,w)C_2, \\ \mathfrak{m}(s,t,w)\sin t \cos w = \mathfrak{n}(s,t,w)C_3, \\ \mathfrak{m}(s,t,w)\sin w = \mathfrak{n}(s,t,w)C_4, \end{array}\right\} \tag{3.30}$$

respectively. Since the functions C_i depend only on s, there is no solution for functions $\mathfrak{m}(s,t,w)$ and $\mathfrak{n}(s,t,w)$ in (3.30).

Now, let us assume that $C_2 = 0$ in (3.29). In this case, from the second equation of (3.24), it's easy to see that

$$\mathfrak{m}(s,t,w) = -\frac{k_1\left(\begin{array}{c} rk_1\left(\begin{array}{c} 24rk_1\cos^4 t\cos^4 w + 12\cos^3 t\cos(3w) \\ +19\cos t\cos w + 9\cos(3t)\cos w \end{array}\right) \\ +12\cos^2 t\cos(2w) + 6\cos(2t) - 2 \end{array}\right)}{8r^3(1 + rk_1\cos t\cos w)^3\cos t\cos w}. \tag{3.31}$$

Here, when the equation (3.31) is successively substituted into the third and fourth equations of (3.24), we obtain

$$\mathfrak{n}(s,t,w)C_3 = \frac{k_1(rk_1 \sin t \cos w + \tan t)}{r^3(1 + rk_1 \cos t \cos w)^3},$$

$$\mathfrak{n}(s,t,w)C_4 = \frac{k_1(rk_1 \sin w + \sec t \tan w)}{r^3(1 + rk_1 \cos t \cos w)^3}.$$

Here, there is no solution for functions $\mathfrak{n}(s,t,w)$.

Therefore, we can give the following theorem:

Theorem 3.5. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with generalized $\mathcal{L}_2$ 1-type Gauss map.*

Now, let us classify the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have second kind $\mathcal{L}_2$-pointwise 1-type Gauss map, i.e., $\mathcal{L}_2 N = \mathfrak{m}\,(N + C)$.

From (3.2) and (3.23), we get

$$\left.\begin{array}{l}
\frac{\left(k_1' \cos t + k_1 k_2 \sin t\right)\cos w}{r^2(1+rk_1\cos t\cos w)^3} = \mathfrak{m} C_1, \\[2ex]
\frac{k_1\left(\begin{array}{l}\frac{rk_1}{2}\left(\begin{array}{l}24rk_1\cos^4 t\cos^4 w + 12\cos^3 t\cos(3w)\\ +19\cos t\cos w + 9\cos(3t)\cos w\end{array}\right)\\ +6\cos^2 t\cos(2w) + 3\cos(2t) - 1\end{array}\right)}{4r^3(1+rk_1\cos t\cos w)^3} = \mathfrak{m}\left(-\cos t\cos w + C_2\right), \\[2ex]
\frac{3k_1\sin t\cos t\cos^2 w}{r^3(1+rk_1\cos t\cos w)} = \mathfrak{m}\left(-\sin t\cos w + C_3\right), \\[2ex]
\frac{3k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)} = \mathfrak{m}\left(-\sin w + C_4\right).
\end{array}\right\} \tag{3.32}$$

Here, from the last equation of (3.32) it's easy to see that

$$\mathfrak{m}(s,t,w) = \frac{3k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)\left(-\sin w + C_4\right)}. \tag{3.33}$$

Here, when the equation (3.33) is substituted into the second equation of (3.32), we obtain

$$-1 + 3C_2\cos t\cos w + 3C_2 rk_1\cos^2 t\cos^2 w = 0.$$

Since this is not possible, we can give the following theorem:

Theorem 3.6. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with second kind $\mathcal{L}_2$-pointwise 1-type Gauss map.*

Now, let us classify the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have first kind $\mathcal{L}_2$-pointwise 1-type Gauss map, i.e., $\mathcal{L}_2 N = \mathfrak{m} N$.

From (3.2) and (3.23), we get

$$\left.\begin{array}{l}
\frac{\cos w\left(\cos t k_1' + k_1 k_2 \sin t\right)}{r^2(1+rk_1\cos t\cos w)^3} = 0, \\[2ex]
\frac{k_1\left(\begin{array}{l}\frac{1}{2}rk_1\left(\begin{array}{l}24rk_1\cos^4 t\cos^4 w + 12\cos^3 t\cos(3w)\\ +19\cos t\cos w + 9\cos(3t)\cos w\end{array}\right)\\ +6\cos^2 t\cos(2w) + 3\cos(2t) - 1\end{array}\right)}{4r^3(1+rk_1\cos t\cos w)^3} = \mathfrak{m}\left(-\cos t\cos w\right), \\[2ex]
\frac{3k_1\sin t\cos t\cos^2 w}{r^3(1+rk_1\cos t\cos w)} = \mathfrak{m}\left(-\sin t\cos w\right), \\[2ex]
\frac{3k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)} = \mathfrak{m}\left(-\sin w\right).
\end{array}\right\} \tag{3.34}$$

Here, from the last equation of (3.34) it's easy to see that

$$\mathfrak{m}(s,t,w) = \frac{-3k_1\cos t\cos w}{r^3(1+rk_1\cos t\cos w)}. \tag{3.35}$$

Here, when the equation (3.35) is substituted into the second equation of (3.32), we obtain

$$k_1(rk_1 + \sec t\sec w) = 0.$$

Since this is not possible, we can give the following theorem:

Theorem 3.7. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with first kind $\mathcal{L}_2$-pointwise 1-type Gauss map.*

Finally, since the coefficients F_1, F_2, F_3 and F_4 in equation (3.23) are all zero only for $k_1 = 0$, we can give the following theorem:

Theorem 3.8. *The tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, have $\mathcal{L}_2$-harmonic Gauss map if and only if $k_1 = 0$.*

4. Some classifications for Tubular hypersurfaces generated by spacelike curves with $\mathcal{L}_k$ operators in $\mathbb{E}_1^4$

In this section, we give the general formulas for $\mathcal{L}_1$ (Cheng-Yau) and $\mathcal{L}_2$ operators of the Gauss maps of the six types of tubular hypersurfaces $\mathcal{T}^{\{j,\lambda\}}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on spacelike curves $\beta(s)$ with non-null Frenet vectors in $\mathbb{E}_1^4$ and give some classifications for these hypersurfaces which have generalized $\mathcal{L}_k$ 1-type Gauss map, first kind $\mathcal{L}_k$-pointwise 1-type Gauss map and second kind $\mathcal{L}_k$-pointwise 1-type Gauss map and $\mathcal{L}_k$-harmonic Gauss map, $k \in \{1,2\}$.

The tubular hypersurfaces $\mathcal{T}^{\{j,\lambda\}}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vectors F_i in $\mathbb{E}_1^4$ can be parametrized by

$$\left.\begin{aligned}
\mathcal{T}^{\{2,1\}}(s,t,w) &= \beta(s) + r(\cosh t \sinh w F_2(s) + \cosh w F_3(s) + \sinh t \sinh w F_4(s)),\\
\mathcal{T}^{\{2,-1\}}(s,t,w) &= \beta(s) + r(\cosh t \cosh w F_2(s) + \sinh w F_3(s) + \sinh t \cosh w F_4(s)),\\
\mathcal{T}^{\{3,1\}}(s,t,w) &= \beta(s) + r(\sinh t \sinh w F_2(s) + \cosh t \sinh w F_3(s) + \cosh w F_4(s)),\\
\mathcal{T}^{\{3,-1\}}(s,t,w) &= \beta(s) + r(\sinh t \cosh w F_2(s) + \cosh t \cosh w F_3(s) + \sinh w F_4(s)),\\
\mathcal{T}^{\{4,1\}}(s,t,w) &= \beta(s) + r(\cosh w F_2(s) + \sinh t \sinh w F_3(s) + \cosh t \sinh w F_4(s)),\\
\mathcal{T}^{\{4,-1\}}(s,t,w) &= \beta(s) + r(\sinh w F_2(s) + \sinh t \cosh w F_3(s) + \cosh t \cosh w F_4(s)),
\end{aligned}\right\} \tag{4.1}$$

respectively. Here, we suppose for $\mathcal{T}^{\{j;\lambda\}}(s,t,w)$ that

i) $\langle F_j, F_j\rangle = -1 = \varepsilon_j$ and for $i \neq j$, $\langle F_i, F_i\rangle = 1 = \varepsilon_i$, $i,j \in \{1,2,3,4\}$,

ii) if the tubular hypersurface is foliated by pseudo hyperspheres or pseudo hyperbolic hyperspheres, then $\lambda = 1$ *or* $\lambda = -1$, respectively (for more details, one can see [3]).

Now, let us write the following lemma which states the general parametric expressions of 6 different types of tubular hypersurfaces given by (4.1) and obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields in $\mathbb{E}_1^4$.

Lemma 4.1. *The general expression of the tubular hypersurfaces $\mathcal{T}^{\{j,\lambda\}}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve $\beta(s)$ with non-null Frenet vectors $F_i(s)$ in $\mathbb{E}_1^4$ can be given by*

$$\mathcal{T}^{\{j,\lambda\}}(s,t,w) = \beta(s) + r\left(\textstyle\sum_{i=2}^{4} \mu_i^\lambda(s,t,w) F_i(s)\right), \tag{4.2}$$

where

$$\mu_5^\lambda(s,t,w) = \mu_2^\lambda(s,t,w),\ \ \mu_6^\lambda(s,t,w) = \mu_3^\lambda(s,t,w)$$

and for $j = 2,3,4$

$$\mu_j^\lambda(s,t,w) = (\sinh w)^{\frac{1+\lambda}{2}} (\cosh w)^{\frac{1-\lambda}{2}} \cosh t,$$

$$\mu_{j+1}^\lambda(s,t,w) = (\sinh w)^{\frac{1-\lambda}{2}} (\cosh w)^{\frac{1+\lambda}{2}},$$

$$\mu_{j+2}^\lambda(s,t,w) = (\sinh w)^{\frac{1+\lambda}{2}} (\cosh w)^{\frac{1-\lambda}{2}} \sinh t.$$

Here, if the canal hypersurface is foliated by pseudo hyperspheres or pseudo hyperbolic hyperspheres, then $\lambda = 1$ or $\lambda = -1$, respectively.

Here, we can give the general parametric expressions of the unit normal vector fields, the coefficients of the first fundamental forms and the principal curvatures of the tubular hypersurfaces $\mathcal{T}^{\{j,\lambda\}}$ parametrized by (4.2).

The unit normal vector fields $N^{\{j,\lambda\}}$ $(j = 2, 3, 4)$ of (4.2) are

$$N^{\{j,\lambda\}} = -(-1)^{(4-j)!}\lambda^j \textstyle\sum_{i=2}^4 \mu_i^\lambda F_i. \tag{4.3}$$

The coefficients of the first fundamental forms $g_{ik}^{\{j,\lambda\}}$ $(j = 2, 3, 4)$ of (4.2) are

$$\left.\begin{aligned}
g_{11}^{\{j,\lambda\}} &= 1 + r^2(k_2)^2\left(-(-1)^{(4-j)!}\left(\mu_3^\lambda\right)^2 + (-1)^j\left(\mu_2^\lambda\right)^2\right)\\
&\quad + r^2(k_3)^2\left((-1)^{(5-j)!}\left(\mu_3^\lambda\right)^2 + (-1)^j\left(\mu_4^\lambda\right)^2\right)\\
&\quad + 2(-1)^{(4-j)!}rk_1\mu_2^\lambda + r^2(k_1)^2\left(\mu_2^\lambda\right)^2 - 2(-1)^{(5-j)!}r^2k_2k_3\mu_2^\lambda\mu_4^\lambda,\\
g_{12}^{\{j,\lambda\}} &= g_{21}^{\{j,\lambda\}} = r^2(\mu_{j+1}^\lambda)_w\left((-1)^jk_3\left(\mu_2^\lambda\right)_w - (-1)^{(4-j)!}k_2\left(\mu_4^\lambda\right)_w\right),\\
g_{22}^{\{j,\lambda\}} &= r^2\left((\mu_{j+1}^\lambda)_w\right)^2,\\
g_{13}^{\{2,\lambda\}} &= g_{31}^{\{2,\lambda\}} = \lambda r^2\left(-k_2\cosh t + k_3\sinh t\right),\\
g_{13}^{\{3,\lambda\}} &= g_{31}^{\{3,\lambda\}} = -\lambda r^2k_3\cosh t,\\
g_{13}^{\{4,\lambda\}} &= g_{31}^{\{4,\lambda\}} = \lambda r^2k_2\sinh t,\\
g_{23}^{\{j,\lambda\}} &= g_{32}^{\{j,\lambda\}} = 0,\\
g_{33}^{\{j,\lambda\}} &= -\lambda r^2.
\end{aligned}\right\} \tag{4.4}$$

The principal curvatures $\kappa_i^{\{j,\lambda\}}$ $(j = 2, 3, 4)$ of (4.2) are

$$\left.\begin{aligned}
\kappa_1^{\{j,\lambda\}} &= \kappa_2^{\{j,\lambda\}} = \tfrac{(-1)^{(4-j)!}\lambda^j}{r},\\
\kappa_3^{\{j,\lambda\}} &= \tfrac{k_1\mu_2^\lambda}{\lambda^j\left(1+(-1)^{(4-j)!}rk_1\mu_2^\lambda\right)}.
\end{aligned}\right\} \tag{4.5}$$

From Lemma 4.1, (4.3), (4.4) and (4.5), we get

$$\begin{aligned}
\mathcal{L}_1N^{\{j,\lambda\}} &= \frac{2\left(-(-1)^{(5-j)!}k_1k_2\mu_3^\lambda + k_1'\mu_2^\lambda\right)}{r\left((-1)^{(4-j)!} + rk_1\mu_2^\lambda\right)^3}F_1\\
&+ \frac{-2\lambda\left(\mu_2^\lambda + 3r^2(\mu_2^\lambda)^3(k_1)^2 + k_1\left(\lambda r + 4(-1)^{(4-j)!}r(\mu_2^\lambda)^2\right)\right)}{r^3\left((-1)^{(4-j)!} + rk_1\mu_2^\lambda\right)^2}F_2\\
&+ \frac{-2\lambda(-1)^{(4-j)!}\mu_3^\lambda\left(1 + 3(-1)^{(4-j)!}rk_1\mu_2^\lambda\right)}{r^3\left((-1)^{(4-j)!} + rk_1\mu_2^\lambda\right)}F_3 + \frac{-2\lambda\mu_4^\lambda\left((-1)^{(4-j)!} + 3rk_1\mu_2^\lambda\right)}{r^3\left((-1)^{(4-j)!} + rk_1\mu_2^\lambda\right)}F_4.
\end{aligned} \tag{4.6}$$

Let $\mathcal{T}^{\{j,\lambda\}}(s,t,w)$ have generalized L_1 (Cheng-Yau) 1-type Gauss map, i.e., $\mathcal{L}_1N^{\{j,\lambda\}} = \mathfrak{m}N^{\{j,\lambda\}} + \mathfrak{n}C$, where $C = C_1F_1 + C_2F_2 + C_3F_3 + C_4F_4$ is a constant vector. Here, by taking derivatives of the constant vector C with respect to s, from (2.8)-(2.10) we obtain for $\mathcal{T}^{\{j,\lambda\}}$ that

$$\left.\begin{aligned}
&C_1' + (-1)^{(4-j)!}C_2k_1 = 0,\\
&C_2' + C_1k_1 + (-1)^{(5-j)!}C_3k_2 = 0,\\
&C_3' + C_2k_2 - (-1)^{(4-j)!}C_4k_3 = 0,\\
&C_4' + C_3k_3 = 0.
\end{aligned}\right\} \tag{4.7}$$

Also, by taking derivatives the constant vector C with respect to $t,\ w$ separately, one can see that the functions C_i depend only on s.

So, with similar procedure in Subsection 3.1, we obtain the following theorems:

Theorem 4.2. *There are no tubular hypersurfaces (4.2), obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$, with generalized $\mathcal{L}_1$ 1-type Gauss map in $\mathbb{E}^4_1$.*

Theorem 4.3. *There are no tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ with second kind $\mathcal{L}_1$-pointwise 1-type Gauss map in $\mathbb{E}^4_1$.*

Theorem 4.4. *The tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ have first kind $\mathcal{L}_1$-(global) pointwise 1-type Gauss map, i.e., $\mathcal{L}_1 N^{\{j,\lambda\}} = \mathfrak{m} N^{\{j,\lambda\}}$ in $\mathbb{E}^4_1$ if and only if $k_1 = 0$, where $\mathfrak{m}(s,t,w) = \frac{2\lambda^{j+1}(-1)^{(4-j)!}}{r^3}$.*

Theorem 4.5. *The tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ cannot have $\mathcal{L}_1$-harmonic Gauss map.*

Also, from Lemma 4.1, (4.3), (4.4) and (4.5), we get

$$\begin{aligned}\mathcal{L}_2 N^{\{j,\lambda\}} &= \frac{\lambda^j\left((-1)^j\mu_3^\lambda k_1k_2 - (-1)^{(4-j)!}\mu_2^\lambda k_1'\right)}{r^2\left((-1)^{(4-j)!} + rk_1\mu_2^\lambda\right)^3}F_1 \\ &+ \frac{-\lambda^{j+1}k_1\left(2\lambda(-1)^{(4-j)!} - 3(-1)^j\left(\mu_4^\lambda\right)^2 - 3(-1)^{(5-j)!}\left(\mu_3^\lambda\right)^2 - 3(-1)^{(4-j)!}rk_1\left(\mu_2^\lambda\right)^3\right)}{r^3\left((-1)^{(4-j)!} + rk_1\mu_2^\lambda\right)^2}F_2 \\ &+ \frac{\lambda^{j+1}\mu_3^\lambda\left(3(-1)^{(5-j)!}rk_1\mu_2^\lambda\right)}{r^4\left((-1)^{(5-j)!} + (-1)^j rk_1\mu_2^\lambda\right)}F_3 + \frac{\lambda^{j+1}\mu_4^\lambda\left(3(-1)^{(5-j)!}rk_1\mu_2^\lambda\right)}{r^4\left((-1)^{(5-j)!} + (-1)^j rk_1\mu_2^\lambda\right)}F_4. \end{aligned} \tag{4.8}$$

Thus, with similar procedure in Subsection 3.2, we can give the following theorems:

Theorem 4.6. *There are no tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ with generalized $\mathcal{L}_2$ 1-type Gauss map in $\mathbb{E}^4_1$.*

Theorem 4.7. *There are no tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ with second kind $\mathcal{L}_2$-pointwise 1-type Gauss map in $\mathbb{E}^4_1$.*

Theorem 4.8. *There are no tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ with first kind $\mathcal{L}_2$-pointwise 1-type Gauss map in $\mathbb{E}^4_1$.*

Theorem 4.9. *The tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ have $\mathcal{L}_2$-harmonic Gauss map if and only if $k_1 = 0$.*

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