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EXISTENCE AND UNIQUENESS OF A WEAK SOLUTION FOR SINGULAR WEIGHTED ROBIN PROBLEM INVOLVING $p(.)$-BIHARMONIC OPERATOR

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Abstract. The aim of this paper is to find the existence of solutions for the following class of singular fourth order equation involving the weighted $p(.)$-biharmonic operator:

$$\begin{cases} \Delta\left(a(x)\left|\Delta u\right|^{p(x)-2}\Delta u\right) = \lambda b(x)\left|u\right|^{q(x)-2}u + V(x)\left|u\right|^{-\gamma(x)}, & x \in \Omega, \\ a(x)\left|\Delta u\right|^{p(x)-2}\frac{\partial u}{\partial \upsilon} + \beta(x)\left|u\right|^{p(x)-2}u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$. Using variational methods, we prove the existence at least one nontrivial weak solution of such a Robin problem in weighted variable exponent second order Sobolev spaces $W_a^{2,p(.)}(\Omega)$ under some appropriate conditions. Finally, we deduce some uniqueness results.

1. Introduction

In this paper, the weighted singular Robin problem

$$\begin{cases} \Delta\left(a(x)\left|\Delta u\right|^{p(x)-2}\Delta u\right) = \lambda b(x)\left|u\right|^{q(x)-2}u + V(x)\left|u\right|^{-\gamma(x)}, & x \in \Omega, \\ a(x)\left|\Delta u\right|^{p(x)-2}\frac{\partial u}{\partial \upsilon} + \beta(x)\left|u\right|^{p(x)-2}u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

is investigated with respect to some suitable assumptions, where a and b are weight functions and nonnegative, $\frac{\partial u}{\partial \upsilon}$ is the outer unit normal derivative of u on $\partial\Omega$, p, q are continuous functions on $\overline{\Omega}$, i.e. $p, q \in C\left(\overline{\Omega}\right)$ with $1 < p^- = \inf_{x\in\Omega} p(x) \leq p(x) \leq p^+ = \sup_{x\in\Omega} p(x) < \frac{N}{2}$, $\beta \in L^\infty(\partial\Omega)$ such that $\beta^- = \inf_{x\in\partial\Omega} \beta(x) > 0$, and $\Omega \subset \mathbb{R}^N$ $(N > 2)$ is a bounded smooth domain, λ is a positive parameter,

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$\gamma:\Omega\to(0,1)$ is a continuous function, $1-\gamma^{-}<p^{-}$, $q^{+}<p^{-}$, $V\in L_{a}^{\frac{p^{*}(.)}{p^{*}(.)+\gamma(.)-1}}(\Omega)$, $V>0$ and $p^{*}(x)=\frac{Np(x)}{N-2p(x)}$.

In 2018, Chung [12] consider the $p(x)$-Laplacian Robin eigenvalue problem

$$\begin{cases} -\Delta_{p(x)}u=\lambda V(x)\left|u\right|^{q(x)-2}u, & x\in\Omega,\\ \left|\nabla u\right|^{p(x)-2}\frac{\partial u}{\partial \upsilon}+\beta(x)\left|u\right|^{p(x)-2}u=0, & x\in\partial\Omega,\end{cases}$$

and prove the existence of a continuous family of eigenvalues in a neighborhood of the origin using variational methods under some suitable conditions on the functions q and V.

In 2024, Chung and Ho [14] use a concentration-compactness principle to solve the lack of compactness of the critical Sobolev imbedding, and obtain the existence of solutions to the following problem involving critical growth

$$\begin{cases} \Delta^{2}_{p(x)}u-M\left(\int\limits_{\Omega}\frac{1}{p(x)}\left|\nabla u\right|^{p(x)}dx\right)\Delta_{p(x)}u=\lambda f(x,u)+\left|u\right|^{q(x)-2}u, & x\in\Omega,\\ u=\Delta u=0, & x\in\partial\Omega.\end{cases}$$

In 2011, Ayoujil and Amrouss [8] investigate the following problem:

$$\begin{cases} \Delta\left(\left|\Delta u\right|^{p(x)-2}\Delta u\right)=\lambda\left|u\right|^{q(x)-2}u, & x\in\Omega,\\ u=\Delta u=0, & \text{on }\partial\Omega,\end{cases} \tag{2}$$

and obtained that the energy functional associated to the problem (2) has a nontrivial minimum for any positive λ for $\max_{x\in\Omega}q(x)<\min_{x\in\Omega}p(x)$ (see Theorem 3.1 in [8]). When $p(x)=q(x)$, the problem (2) is considered by Ayoujil and Amrouss [7].

In 2015, Ge, Zhou and Wu [20] discuss the following problem:

$$\begin{cases} \Delta\left(\left|\Delta u\right|^{p(x)-2}\Delta u\right)=\lambda V(x)\left|u\right|^{q(x)-2}u, & \text{in }\Omega,\\ u=\Delta u=0, & \text{on }\partial\Omega,\end{cases} \tag{3}$$

where V is an indefinite weight and λ is a positive real number. They obtained several situations concerning the growth rates, and they showed, using the mountain pass lemma and Ekeland's principle, the existence of a continuous family of eigenvalues.

In 2019, Kefi and Saoudi [25] search the existence of solutions for the following inhomogeneous singular equation involving the $p(x)$-biharmonic operator:

$$\begin{cases} \Delta\left(\left|\Delta u\right|^{p(x)-2}\Delta u\right)=g(x)u^{-\gamma(x)}\mp\lambda f(x,u), & \text{in }\Omega,\\ u=\Delta u=0, & \text{on }\partial\Omega.\end{cases} \tag{4}$$

They study the problem (4), which contains a singular term and indefinite many more general terms than the equation (3), and prove the existence of a weak solution for problem (4).

In 2022, using variational techniques combined with the theory of the generalized Lebesgue-Sobolev spaces Alsaedi, Ali and Ghanmi [1] studied weak solutions for the following class of singular fourth order elliptic equations:

$$\begin{cases} \Delta\left(|x|^{p(x)}|\Delta u|^{p(x)-2}\Delta u\right)=a(x)u^{-\gamma(x)}+\lambda f(x,u), & \text{in } \Omega, \\ u=\Delta u=0, & \text{on } \partial\Omega, \end{cases} \tag{5}$$

and prove the existence at least one nontrivial weak solution in $W_0^{2,p(.)}(\Omega)$.

In 2022, Mbarki [32] discuss the existence of solutions for a class of singular $p(x)$-biharmonic Laplacian problem with Navier boundary conditions:

$$\begin{cases} \Delta\left(|x|^{p(x)}|\Delta u|^{p(x)-2}\Delta u\right)=\lambda V(x)|u|^{q(x)-2}u+a(x)u^{-\gamma(x)}, & \text{in } \Omega, \\ u=\Delta u=0, & \text{on } \partial\Omega. \end{cases} \tag{6}$$

In 2022, Kulak, Aydın and Unal [28] consider the existence of weak solutions of weighted Robin problem involving $p(.)$-biharmonic operator:

$$\begin{cases} \Delta\left(a(x)|\Delta u|^{p(x)-2}\Delta u\right)=\lambda b(x)|u|^{q(x)-2}u, & \text{in } \Omega, \\ a(x)|\Delta u|^{p(x)-2}\frac{\partial u}{\partial \upsilon}+\beta(x)|u|^{p(x)-2}u=0, & \text{on } \partial\Omega. \end{cases} \tag{7}$$

under some conditions in $W_{a,b}^{2,p(.)}(\Omega)$. We refer for instance to see ([2], [13], [22], [24], [26]).

Inspired by the articles mentioned above, we show the existence and uniqueness of nontrivial solutions of problem (1) using compact embedding theorems in $W_a^{2,p(.)}(\Omega)$ and variational methods. Therefore, we will obtain more general results than the problems (4), (5), (6).

2. Abstract setting

Let Ω be a bounded open subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Put

$$C_+\left(\overline{\Omega}\right)=\left\{h\in C\left(\overline{\Omega}\right):\inf_{x\in\overline{\Omega}}h(x)>1\right\},$$

For any $p\in C_+\left(\overline{\Omega}\right)$, we set

$$p^-=\inf_{x\in\Omega}p(x) \text{ and } p^+=\sup_{x\in\Omega}p(x)$$

such that $1<p^-\leq p^+<\infty$ and

$$L^{p(\cdot)}(\Omega)=\left\{u\,\middle|\,u:\Omega\to\mathbb{R}\text{ is measurable and }\int_\Omega|u(x)|^{p(x)}\,dx<\infty\right\}$$

with the (Luxemburg) norm

$$\|u\|_{p(.)}=\inf\left\{\lambda>0:\varrho_{p(.)}\left(\frac{u}{\lambda}\right)\leq 1\right\},$$

where

$$\varrho_{p(.)}(u)=\int\limits_{\Omega}|u(x)|^{p(x)}\,dx.$$

Moreover, the space $\left(L^{p(.)}(\Omega),\|.\|_{p(.)}\right)$ is a reflexive Banach space [27]. The weighted Lebesgue space $L_a^{p(.)}(\Omega)$ is defined by

$$L_a^{p(.)}(\Omega)=\left\{u\left|u:\Omega\longrightarrow\mathbb{R}\text{ measurable and }\int\limits_{\Omega}|u(x)|^{p(x)}\,a(x)dx<\infty\right.\right\}$$

such that $\|u\|_{p(.),a}=\left\|ua^{\frac{1}{p(.)}}\right\|_{p(.)}<\infty$ for $u\in L_a^{p(.)}(\Omega)$, where a is a weight function from Ω to $(0,\infty)$. Moreover, $u\in L_a^{p(.)}(\Omega)$ if and only if $|u|^{p(.)}\,a\in L^1(\Omega)$ [34].

We can define the space $L_a^{p(.)}(\partial\Omega)$ similarly by

$$L_a^{p(.)}(\partial\Omega)=\left\{u\left|u:\partial\Omega\longrightarrow\mathbb{R}\text{ measurable and }\int\limits_{\partial\Omega}|u(x)|^{p(x)}\,a(x)d\sigma<+\infty\right.\right\}$$

with the norm

$$\|u\|_{p(.),a,\partial\Omega}=\inf\left\{\tau>0:\int\limits_{\partial\Omega}\left|\frac{u(x)}{\tau}\right|^{p(x)}a(x)d\sigma\leq 1\right\}$$

for $u\in L_a^{p(.)}(\partial\Omega)$, where $d\sigma$ is the measure on the boundary of Ω. Then $\left(L_a^{p(.)}(\partial\Omega),\|.\|_{p(.),a,\partial\Omega}\right)$ is a a reflexive Banach space. If $a\in L^{\infty}(\Omega)$, then $L_a^{p(.)}=L^{p(.)}$ [15].

Proposition 1. *(see [3], [5], [6], [19], [21], [30], [31]) For all $u,v\in L_a^{p(.)}(\Omega)$, we have*

(i) $\|u\|_{p(.),a}<1$ *(resp.*$=1,>1$*) if and only if* $\varrho_{p(.),a}(u)<1$ *(resp.*$=1,>1$*),*

(ii) $\|u\|_{p(.),a}^{p^-}\leq\varrho_{p(.),a}(u)\leq\|u\|_{p(.),a}^{p^+}$ *with* $\|u\|_{p(.),a}>1$,

(iii) $\|u\|_{p(.),a}^{p^+}\leq\varrho_{p(.),a}(u)\leq\|u\|_{p(.),a}^{p^-}$ *with* $\|u\|_{p(.),a}<1$

(iv) $\min\left\{\|u\|_{p(.),a}^{p^-},\|u\|_{p(.),a}^{p^+}\right\}\leq\varrho_{p(.),a}(u)\leq\max\left\{\|u\|_{p(.),a}^{p^-},\|u\|_{p(.),a}^{p^+}\right\}$,

(v) $\min\left\{\varrho_{p(.),a}(u)^{\frac{1}{p^-}},\varrho_{p(.),a}(u)^{\frac{1}{p^+}}\right\}\leq\|u\|_{p(.),a}\leq\max\left\{\varrho_{p(.),a}(u)^{\frac{1}{p^-}},\varrho_{p(.),a}(u)^{\frac{1}{p^+}}\right\}$,

(vi) $\varrho_{p(.),a}(u-v)\to 0$ *if and only if* $\|u-v\|_{p(.),a}\to 0$.

Proposition 2. *(see [17])Let p and q be two measurable functions such that $p\in L^{\infty}(\Omega)$ and $1\leq p(x)q(x)\leq\infty$ for a.e. $x\in\Omega$. Let $u\in L^{q(.)}(\Omega)$, $u\neq 0$. Then*

$$\min\left\{\|u\|_{p(.)q(.)}^{p^+},\|u\|_{p(.)q(.)}^{p^-}\right\}\leq\left\||u|^{p(.)}\right\|_{q(.)}\leq\max\left\{\|u\|_{p(.)q(.)}^{p^+},\|u\|_{p(.)q(.)}^{p^-}\right\}.$$

Let $a^{-\frac{1}{p(.)-1}} \in L^1_{loc}(\Omega)$ and $k \in \mathbb{Z}^+$. Hence we define the weighted variable exponent Sobolev space $W_a^{k,p(.)}(\Omega)$ is defined by

$$W_a^{k,p(.)}(\Omega) = \left\{ u \in L_a^{p(.)}(\Omega) : D^\alpha u \in L_a^{p(.)}(\Omega), 0 \leq |\alpha| \leq k \right\},$$

where $\alpha \in \mathbb{N}_0^N$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_N$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} ... \partial_{x_N}^{\alpha_N}}$. Then $W_a^{k,p(.)}(\Omega)$ is a separable and reflexive Banach space equipped with the norm

$$\|u\|_{W_a^{k,p(.)}} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_{p(.),a}.$$

Alternatively, the space $W_a^{k,p(.)}(\Omega)$ could also be introduced as

$$W_a^{k,p(.)}(\Omega) = \left\{ u \in W_a^{k-1,p(.)}(\Omega) : D_i u = \frac{\partial u}{\partial x_i} \in W_a^{k-1,p(.)}(\Omega), \forall i = 1, 2, ...N \right\}.$$

To find out solutions of the problem (1), we need some essential theories on the space $W_a^{2,p(.)}(\Omega)$. The space $X = W_a^{2,p(.)}(\Omega)$ consists of all measurable functions $u \in L_a^{p(.)}(\Omega)$ such that $D^\alpha u \in L_a^{p(.)}(\Omega)$ for $0 \leq |\alpha| \leq 2$. Hence for any $u \in X$,

$$\|u\|_X = \|u\|_{p(.),a} + \|\nabla u\|_{p(.),a} + \sum_{|\alpha|=2} \|D^\alpha u\|_{p(.),a}$$

Let

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)}, & \text{if } p(x) < \frac{N}{2}, \\ +\infty, & \text{if } p(x) \geq \frac{N}{2}, \end{cases}$$

for every $x \in \overline{\Omega}$. For $p, q \in C_+\left(\overline{\Omega}\right)$ in which $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous and compact embedding $W^{2,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ (non-weighted). It is obvious that $p(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

Remark 1. *There is a continuous embedding $X \hookrightarrow L_a^{p^*(.)}(\Omega)$ under some conditions.*

Proof. Firstly, we show by induction on k that $W_a^{k,p(.)}(\Omega) \hookrightarrow L_a^{p^*(.)}(\Omega)$. Let $k = 1$. If $0 < a_1 \leq a(x) < a_2 < \infty$ for a.e. $x \in \Omega$, then it is well known that the embedding $W_a^{1,p(.)}(\Omega) \cong W^{1,p(.)}(\Omega) \hookrightarrow L^{p^*(.)}(\Omega)$ for non-weighted case. Moreover, the embedding $W_a^{1,p(.)}(\Omega) \hookrightarrow L_a^{p^*(.)}(\Omega)$ is also valid for weighted case (see [18], [25], [27]). Suppose that the embedding $W_a^{k-1,p(.)}(\Omega) \hookrightarrow L_a^{r(.)}(\Omega)$ is satisfied for $r(x) = Np(x)/(N - ((k-1)p(x)))$ when $p(x) < \frac{N}{k-1}$. Since $u \in W_a^{k,p(.)}(\Omega)$, then u and $D_j u$ $(1 \leq j \leq N)$ belong to $W_a^{k-1,p(.)}(\Omega)$, where $p(x) < \frac{N}{k}$. So it is easy to see that $u \in W_a^{1,r(.)}(\Omega)$ and

$$\|u\|_{W_a^{1,r(.)}} \leq C_1 \|u\|_{W_a^{k,p(.)}}.$$

Due to $kp(x) < N$, we get $r(x) < N$ and $W_a^{1,r(.)}(\Omega) \hookrightarrow L_a^{p^*(.)}(\Omega)$, where $p^*(x) = Nr(x)/N - r(x) = Np(x)/N - kp(x)$ and

$$\|u\|_{p^*,a} \leq C_2 \|u\|_{W_a^{1,r(.)}} \leq C_3 \|u\|_{W_a^{k,p(.)}},$$

i.e. the embedding $W_a^{k,p(.)}(\Omega) \hookrightarrow L_a^{p^*(.)}(\Omega)$ is continuous. So $X \hookrightarrow L_a^{p^*(.)}(\Omega)$. $\square$

For $A \subset \overline{\Omega}$, denote by $p^-(A) = \inf_{x\in A} p(x)$ and $p^+(A) = \sup_{x\in A} p(x)$. Define

$$p^\partial(x) = (p(x))^\partial = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N, \end{cases}$$

and

$$p^\partial_{r(x)}(x) = \frac{r(x)-1}{r(x)} p^\partial(x)$$

for any $x \in \partial\Omega$ and $r \in C(\partial\Omega, \mathbb{R})$ with $r^- = \inf_{x\in\partial\Omega} r(x) > 1$.

Theorem 1. *(see [15])Assume that the set $\partial\Omega$ possesses the cone property and $p \in C(\overline{\Omega})$ with $p^- > 1$. If $q \in C(\partial\Omega)$ and the inequality $1 \leq q(x) < p^\partial_{r(x)}(x)$ is valid for all $x \in \partial\Omega$, then there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L_a^{q(.)}(\partial\Omega)$ for $a \in L^{r(.)}(\partial\Omega)$, $r \in C(\partial\Omega)$ with $r(x) > \frac{p^\partial(x)}{p^\partial(x)-1}$ for all $x \in \partial\Omega$. In particular, there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega)$, where $1 \leq q(x) < p^\partial(x)$, $\forall x \in \partial\Omega$.*

It is easy to see that $p^\partial_{r(x)}(x) < p^\partial(x)$ and $p(x) < p^\partial(x)$. So we have the following Corollary under conditions in Theorem 1.

Corollary 1. *(see [15])*

(i) There is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\partial\Omega)$, where $1 \leq p(x) < p^\partial(x)$, $\forall x \in \partial\Omega$.

(ii) There is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L_a^{p(.)}(\partial\Omega)$, where $1 \leq p(x) < p^\partial_{r(x)}(x) < p^\partial(x)$, $\forall x \in \partial\Omega$.

Theorem 2. *([5])Let $a^{-\alpha(.)} \in L^1(\Omega)$ with $\alpha(x) \in \left(\frac{N}{p(x)}, \infty\right) \cap \left[\frac{1}{p(x)-1}, \infty\right)$. Then we have the compact embedding $W_a^{1,p(.)}(\Omega) \hookrightarrow W^{1,p_*(.)}(\Omega)$, where $p_*(x) = \frac{\alpha(x)p(x)}{\alpha(x)+1}$.*

Corollary 2. *If the inequality $p(x) < p^\partial_{*,r(x)}(x) < p^\partial_*(x)$ is valid for all $x \in \partial\Omega$, then there exists a compact embedding between $W_a^{1,p(.)}(\Omega)$ and $L_a^{p(.)}(\partial\Omega)$.*

Corollary 3. $X \hookrightarrow W_a^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L_a^{p(.)}(\partial\Omega)$.

Theorem 3. *(see [19])Assume that the set $\partial\Omega$ possesses the cone property and $p \in C(\overline{\Omega})$. Suppose that $b \in L^{r(.)}(\Omega)$, $b(x) > 0$ for $x \in \Omega$, $r \in C(\overline{\Omega})$ and $r^- > 1$. If $q \in C(\overline{\Omega})$ and*

$$1 \leq q(x) < \frac{r(x)-1}{r(x)} p^\blacklozenge(x)$$

for all $x \in \overline{\Omega}$, *then there is a compact embedding* $W^{1,p(.)}(\Omega) \hookrightarrow L_b^{q(.)}(\Omega)$, *where*

$$p^{\blacklozenge}(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

Corollary 4. *If the inequality* $1 \leq q(x) < \frac{r(x)-1}{r(x)} (p_*)^{\blacklozenge}(x)$ *is true for all* $x \in \overline{\Omega}$, *then there exists a compact embedding between* $W_a^{1,p(.)}(\Omega)$ *and* $L_b^{q(.)}(\Omega)$. *So* $X \hookrightarrow\hookrightarrow L_b^{q(.)}(\Omega)$.

If we use the method in Theorem 2.1 in [16] and [4], then we obtain the following theorem. In addition, this theorem plays an important role for the existence of weak solutions of the problem (1).

Theorem 4. *(see Theorem 3 in [28])Let* $u \in X$. *Then the norms* $\|u\|_\partial$ *and* $\|u\|_X$ *are equivalent on* X, *where*

$$\|u\|_\partial = \|\Delta u\|_{p(.),a} + \|u\|_{p(.),a,\partial\Omega}.$$

Let $\beta \in L^\infty(\partial\Omega)$ such that $\beta^- = \inf_{x\in\partial\Omega} \beta(x) > 0$. Then, the norm $\|u\|_{\beta(x)}$ is defined by

$$\|u\|_{\beta(x)} = \inf\left\{\tau > 0 : \int_\Omega a(x) \left|\frac{\Delta u(x)}{\tau}\right|^{p(x)} dx + \int_{\partial\Omega} \beta(x) \left|\frac{u(x)}{\tau}\right|^{p(x)} d\sigma \leq 1\right\}$$

for any $u \in X$. Moreover, $\|.\|_{\beta(x)}$ and $\|.\|_X$ are equivalent on X by Theorem 4.

Proposition 3. *(see [6], [21], [30], [31]) Let* $I_{\beta(x)}(u) = \int_\Omega a(x) |\Delta u(x)|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |u(x)|^{p(x)} d\sigma$ *with* $\beta^- > 0$. *For any* $u, u_k \in X$ $(k = 1, 2, ...)$, *we have*

- *(i)* $\|u\|_{\beta(x)}^{p^-} \leq I_{\beta(x)}(u) \leq \|u\|_{\beta(x)}^{p^+}$ *with* $\|u\|_{\beta(x)} \geq 1$,
- *(ii)* $\|u\|_{\beta(x)}^{p^+} \leq I_{\beta(x)}(u) \leq \|u\|_{\beta(x)}^{p^-}$ *with* $\|u\|_{\beta(x)} \leq 1$,
- *(iii)* $\min\left\{\|u\|_{\beta(x)}^{p^-}, \|u\|_{\beta(x)}^{p^+}\right\} \leq I_{\beta(x)}(u) \leq \max\left\{\|u\|_{\beta(x)}^{p^-}, \|u\|_{\beta(x)}^{p^+}\right\}$,
- *(iv)* $\|u - u_k\|_{\beta(x)} \to 0$ *if and only if* $I_{\beta(x)}(u - u_k) \to 0$ *as* $k \to \infty$,
- *(v)* $\|u_k\|_{\beta(x)} \to \infty$ *if and only if* $I_{\beta(x)}(u_k) \to \infty$ *as* $k \to \infty$.

Definition 1. *We say that* $u \in X$ *is a weak solution of (1) if*

$$\int_\Omega a(x) |\Delta u|^{p(x)-2} \Delta u \Delta v dx + \int_{\partial\Omega} \beta(x) |u(x)|^{p(x)-2} uv d\sigma$$
$$-\lambda \int_\Omega b(x) |u|^{q(x)-2} uv dx - \int_\Omega V(x) |u|^{-\gamma(x)} v dx = 0$$

for all $v \in X$. *We point out that if* $\lambda \in \mathbb{R}$ *is an eigenvalue of the problem (1), then the corresponding* $u \in X - \{0\}$ *is a weak solution of (1).*

To obtain a weak solution to (1), let us introduce the functional $E_\lambda : X \to \mathbb{R}$ defined by

$$E_\lambda(u) = \phi(u) - \lambda \int_\Omega \frac{b(x)}{q(x)} |u|^{q(x)} dx - \Phi_\lambda(u),$$

for any $\lambda > 0$, where

$$\phi(u) = \int_\Omega \frac{a(x)}{p(x)} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u(x)|^{p(x)} d\sigma$$

and

$$\Phi_\lambda(u) = \int_\Omega \frac{V(x)}{1-\gamma(x)} |u|^{1-\gamma(x)} dx.$$

Due to the singular term $V(x)|u|^{-\gamma(x)}$, E_λ is not of class C^1 functional in X, and classical variational methods (e.g Mountain-Pass Lemma of Ambrosetti-Robinowitz) are not applicable. It is easy to see that

$$\begin{aligned} < \quad E'_\lambda(u), u >= & \int_\Omega a(x) |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |u(x)|^{p(x)} d\sigma \\ & -\lambda \int_\Omega b(x) |u|^{q(x)} dx - \int_\Omega V(x) |u|^{-\gamma(x)} dx \end{aligned}$$

for all $u \in X$.

3. Main Results

In this section, we will show that the problem (1) has at least one nontrivial weak solution. Throughout this paper, assume that $1 < p^- \leq p^+ < \frac{N}{2}$, $\beta \in L^\infty(\partial\Omega)$, $V \in L_a^{\frac{p^*(.)}{p^*(.)+\gamma(.)-1}}(\Omega)$, $V > 0$ and $a, b > 0$.

Theorem 5 (Vitali's Theorem)**.** *(see p. 60 in [29])Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions with finite integrals over a measurable set $\Omega \subset \mathbb{R}^N$. Suppose that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for almost all $x \in \Omega$ and let f be an almost everywhere finite function. Suppose that the following condition (P) is satisfied:

(P) (Equi-absolutely-continuous) For every $\varepsilon > 0$ there exists a $\delta > 0$ with the property: if $B \subset \Omega$, $\mu(B) < \delta$, then

$$\int_\Omega |f_n(x)| dx < \varepsilon$$

for all $n \in \mathbb{N}$. Hence, the function f has a finite integral over Ω and

$$\lim_{n\to\infty} \int_{\Omega} |f_n(x)|\, dx = \int_{\Omega} |f(x)|\, dx.$$

Theorem 6 (Absolute Continuity of the Lebesgue Integral)**.** *(see Theorem 12.34 in [23]) Let $f \in L^1(\Omega)$. For every $\varepsilon > 0$ there exists a $\delta > 0$ depending only on ε and f such that for all $A \subset \mathbb{R}^N$ satisfying $\mu(A) < \delta$, we have*

$$\int_{A} |f(x)|\, dx < \varepsilon.$$

Lemma 1. *Let $V \in L_a^{\frac{p^*(.)}{p^*(.)+\gamma(.)-1}}(\Omega)$ and $0 < r < a(x)$ for a.e $x \in \Omega$ and some $r > 0$. Then E_λ is weakly lower semi-continuous.*

Proof. The proof consists of three steps.

Step 1: The functional $\phi : X \to \mathbb{R}$ is convex. Indeed, since the function $t \to t^\theta$ is convex on $[0, \infty)$ for any $\theta > 1$, so for each $x \in \Omega$ (or $x \in \partial\Omega$)

$$\left|\frac{\xi+\mu}{2}\right|^{p(x)} \leq \left(\frac{|\xi|+|\mu|}{2}\right)^{p(x)} \leq \frac{1}{2}\,|\xi|^{p(x)} + \frac{1}{2}\,|\mu|^{p(x)}$$

for all $\xi, \mu \in \mathbb{R}^N$. Hence, we have

$$\left|\frac{\Delta u+\Delta v}{2}\right|^{p(x)} \leq \left(\frac{|\Delta u|+|\Delta v|}{2}\right)^{p(x)} \leq \frac{1}{2}\,|\Delta u|^{p(x)} + \frac{1}{2}\,|\Delta v|^{p(x)} \tag{8}$$

and

$$\left|\frac{u+v}{2}\right|^{p(x)} \leq \left(\frac{|u|+|v|}{2}\right)^{p(x)} \leq \frac{1}{2}\,|u|^{p(x)} + \frac{1}{2}\,|v|^{p(x)}. \tag{9}$$

Multiplying (8) and (9) by $\frac{a(x)}{p(x)}$, $\frac{\beta(x)}{p(x)}$ and integrating over Ω and $\partial\Omega$ respectively, we obtain

$$\phi(\frac{u+v}{2}) \leq \frac{1}{2}\phi(u) + \frac{1}{2}\phi(v)$$

for any $u, v \in X$. So ϕ is convex.

Step 2: ϕ is weakly lower semi continuous on X. From Step 1 and Corollary 3.8 in [10] it is enough to show that ϕ is strongly lower semi continuous on X. Let $\varepsilon > 0$, $u, v \in X$ such that

$$\|u-v\|_X < \frac{\varepsilon}{\left\|a^{\frac{p(x)-1}{p(x)}}\,|\Delta u|^{p(x)-1}\right\|_{\frac{p(x)}{p(x)-1}}} < \frac{\varepsilon}{C_6+C_7}. \tag{10}$$

Since the functional ϕ is convex, variable Hölder inequality and Proposition 2, we obtain

$$\phi(v) \quad \geq \quad \phi(u) + \left\langle \phi'(u), v-u \right\rangle$$

$$\begin{aligned}
&\geq \phi(u)-\int\limits_{\Omega} a(x)\left|\Delta u\right|^{p(x)-1}\left|\Delta\left(v-u\right)\right|dx-\int\limits_{\partial\Omega}\beta(x)\left|u(x)\right|^{p(x)-1}\left|u-v\right|d\sigma \\
&\geq \phi(u)-C_4\left\|a^{\frac{p(.)-1}{p(.)}}\left|\Delta u\right|^{p(.)-1}\right\|_{\frac{p(.)}{p(.)-1}}\left\|a^{\frac{1}{p(.)}}\left|\Delta\left(v-u\right)\right|\right\|_{p(.)} \\
&\quad -C_5\left\|\beta^{\frac{p(.)-1}{p(.)}}\left|u\right|^{p(.)-1}\right\|_{\frac{p(.)}{p(.)-1},\partial\Omega}\left\|\beta^{\frac{1}{p(.)}}\left|u-v\right|\right\|_{p(.),\partial\Omega} \\
&\geq \phi(u)-C_4\max\left\{\left\|a^{\frac{1}{p(.)}}\left|\Delta u\right|\right\|_{p(.)}^{p^+-1},\left\|a^{\frac{1}{p(.)}}\left|\Delta u\right|\right\|_{p(.)}^{p^--1}\right\}\left\|\left|\Delta\left(v-u\right)\right|\right\|_{p(.),a} \\
&\quad -C_5\max\left\{\left\|\beta^{\frac{1}{p(.)}}\left|u\right|\right\|_{p(.),\partial\Omega}^{p^+-1},\left\|\beta^{\frac{1}{p(.)}}\left|u\right|\right\|_{p(.),\partial\Omega}^{p^--1}\right\}\left\|\left|u-v\right|\right\|_{p(.),\beta,\partial\Omega} \\
&= \phi(u)-C_4\max\left\{\left\|\Delta u\right\|_{p(.),a}^{p^+-1},\left\|\Delta u\right\|_{p(.),a}^{p^--1}\right\}\left\|\left|\Delta\left(v-u\right)\right|\right\|_{p(.),a} \\
&\quad -C_5\max\left\{\left\|u\right\|_{p(.),\beta,\partial\Omega}^{p^+-1},\left\|u\right\|_{p(.),\beta,\partial\Omega}^{p^--1}\right\}\left\|\left|u-v\right|\right\|_{p(.),\beta,\partial\Omega} \\
&\geq \phi(u)-C_6\left\|u-v\right\|_X-C_7\left\|u-v\right\|_X\geq\phi(u)-\varepsilon,
\end{aligned}$$

for some positive constants C_4, C_5, C_6 and C_7. It follows that ϕ is strongly lower semi continuous and convex, so we deduce that the functional I is weakly lower semi continuous.

Step 3: E_λ is weakly lower semi-continuous. Let $\{u_n\}$ be a sequence which is weakly converges to u in X. Then, from Step 2, we have

$$\phi(u)\leq\liminf_{n\to\infty}\phi(u_n). \tag{11}$$

By Corollary 4 we have the compact embedding $X\hookrightarrow\hookrightarrow L_b^{q(.)}(\Omega)$. Hence, the sequence $\{u_n\}$ converges strongly to u in $L_b^{q(.)}(\Omega)$ and

$$\lim_{n\to\infty}\int\limits_{\Omega}\frac{b\left(x\right)}{q\left(x\right)}\left|u_n\right|^{q(x)}dx=\liminf_{n\to\infty}\int\limits_{\Omega}\frac{b\left(x\right)}{q\left(x\right)}\left|u_n\right|^{q(x)}dx=\int\limits_{\Omega}\frac{b\left(x\right)}{q\left(x\right)}\left|u\right|^{q(x)}dx. \tag{12}$$

On the other hand, by Vitali's Theorem, we can claim that

$$\lim_{n\to\infty}\int\limits_{\Omega}V(x)\left|u_n\right|^{1-\gamma(x)}dx=\int\limits_{\Omega}V(x)\left|u\right|^{1-\gamma(x)}dx. \tag{13}$$

Indeed, we only need to prove that

$$\left\{\int\limits_{\Omega}V(x)\left|u_n\right|^{1-\gamma(x)}dx,n\in\mathbb{N}\right\} \tag{14}$$

is equi-absolutely-continuous. It is known that every weakly convergent sequence is bounded. So $(u_n)_{n\in\mathbb{N}}$ is bounded in X. In addition, using the continuous embedding $X\hookrightarrow L_a^{p^*(.)}(\Omega)$ by Remark 1, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L_a^{p^*(.)}(\Omega)$, and there exists a $C_8>0$ such that $\left\|u_n\right\|_{p^*(.),a}<C_8$ for all $n\in\mathbb{N}$. Now, let $\varepsilon>0$, then, using Proposition 1 and the absolutely-continuity of $\int_\Omega |V(x)|^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} a(x)dx$, there exist two positive constants ς and ξ such that

$$\|V\|^{\varsigma}_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \le \int_\Omega |V(x)|^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} a(x)dx < \varepsilon^{\xi} \tag{15}$$

for every $\Omega_2 \subset \Omega$. Consequently, by the Hölder inequality, Proposition 2 and (15) we have

$$\begin{aligned}
\int_\Omega |V(x)|\,|u_n|^{1-\gamma(x)}\,dx &\le \int_\Omega \left(|V(x)|\,a(x)^{\frac{p^*(x)+\gamma(x)-1}{p^*(x)}}\right)\left(|u_n|^{1-\gamma(x)}\,a(x)^{-\frac{p^*(x)+\gamma(x)-1}{p^*(x)}}\right)dx \\
&\le C_9 \left\| |V(x)|\,a(x)^{\frac{p^*(x)+\gamma(x)-1}{p^*(x)}} \right\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1}} \left\| |u_n|^{1-\gamma(x)}\,a(x)^{\frac{1-p^*(x)-\gamma(x)}{p^*(x)}} \right\|_{\frac{p^*(.)}{1-\gamma(.)}} \\
&= C_9 \|V\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \cdot \left\| |u_n|^{1-\gamma(x)}\,a(x)^{\frac{1-\gamma(x)}{p^*(x)}}\,a(x)^{-1} \right\|_{\frac{p^*(.)}{1-\gamma(.)}} \\
&\le C_{10} \|V\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \left\| \left(|u_n|\,a(x)^{\frac{1}{p^*(x)}}\right)^{1-\gamma(x)} \right\|_{\frac{p^*(.)}{1-\gamma(.)}} \\
&\le C_{10} \|V\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \max\left\{ \left\| |u_n|\,a(x)^{\frac{1}{p^*(x)}} \right\|^{1-\gamma^+}_{p^*(.)}, \left\| |u_n|\,a(x)^{\frac{1}{p^*(x)}} \right\|^{1-\gamma^-}_{p^*(.)} \right\} \\
&= C_{10} \|V\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \max\left\{ \|u_n\|^{1-\gamma^+}_{p^*(.),a}, \|u_n\|^{1-\gamma^-}_{p^*(.),a} \right\} \\
&\le C_{10} \|V\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \|u_n\|^{d}_{p^*(.),a} < C_{10}\varepsilon^{\xi} \|u_n\|^{d}_{p^*(.),a}
\end{aligned}$$

for $d > 0$. So the claim (13) is obtained because of the boundedness of the sequence $(u_n)_{n\in\mathbb{N}}$ in $L^{p^*(.)}_a(\Omega)$. So we have

$$E_\lambda(u) \le \liminf_{n\to\infty} E_\lambda(u_n)$$

by (11), (12) and (13). □

Lemma 2. *E_λ is bounded from below and coercive.*

Proof. It is clear that

$$\begin{aligned}
E_\lambda(u) &= \int_\Omega \frac{a(x)}{p(x)} |\Delta u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u(x)|^{p(x)}\,d\sigma - \lambda \int_\Omega \frac{b(x)}{q(x)} |u|^{q(x)}\,dx \\
&\quad - \int_\Omega \frac{V(x)}{1-\gamma(x)} |u|^{1-\gamma(x)}\,dx \\
&\ge \frac{1}{p^+} I_{\beta(x)} - \frac{\lambda}{q^-} \int_\Omega b(x)\,|u|^{q(x)}\,dx - \frac{1}{1-\gamma^+} \int_\Omega V(x)\,|u|^{1-\gamma(x)}\,dx
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{p^+} I_{\beta(x)}(u) - \frac{\lambda}{q^-} \max\left\{ \|u\|_{q(.),b}^{q^-}, \|u\|_{q(.),b}^{q^+} \right\} - \frac{1}{1-\gamma^+} \int_\Omega |V(x)| \, |u|^{1-\gamma(x)} \, dx \\
&\geq \frac{1}{p^+} I_{\beta(x)}(u) - \frac{\lambda}{q^-} \|u\|_{q(.),b}^{q^-} - \frac{1}{1-\gamma^+} \|V\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \max\left\{ \|u\|_{\beta(x)}^{1-\gamma^+}, \|u\|_{\beta(x)}^{1-\gamma^-} \right\} \\
&\geq \frac{1}{p^+} \|u\|_{\beta(x)}^{p^-} - \frac{\lambda C_{11}}{q^-} \|u\|_{\beta(x)}^{q^-} - \frac{1}{1-\gamma^+} \|V\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \|u\|_{\beta(x)}^{1-\gamma^-}.
\end{aligned}$$

Since $1-\gamma^- < p^-$ and $q^+ < p^-$, we infer that $E_\lambda(u) \to \infty$ as $u \to \infty$. So E_λ is is bounded from below and coercive. □

Lemma 3. *There exists a function $\varphi \in X$ such that $\varphi \neq 0$ and $E_\lambda(\varphi) < 0$.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$ such that $\Omega' \subset \operatorname{supp}\varphi \subset \Omega_1 \subset \Omega$ and $0 \leq \varphi \leq 1$ in Ω_1. Then we have

$$\begin{aligned}
E_\lambda(t\varphi) &= \int_\Omega \frac{a(x)t^{p(x)}}{p(x)} |\Delta\varphi|^{p(x)} \, dx + \int_{\partial\Omega} \frac{\beta(x)t^{p(x)}}{p(x)} |\varphi|^{p(x)} \, d\sigma - \lambda \int_\Omega \frac{b(x)\, t^{q(x)}}{q(x)} |\varphi|^{q(x)} \, dx \\
&\quad - \int_\Omega \frac{V(x)t^{1-\gamma(x)}}{1-\gamma(x)} |\varphi|^{1-\gamma(x)} \, dx \\
&\leq \frac{t^{p^-}}{p^-} I_{\beta(x)}(\varphi) - \frac{\lambda}{q^+} \int_\Omega t^{q(x)} |\varphi|^{q(x)} b(x) \, dx - \int_\Omega \frac{V(x)t^{1-\gamma(x)}}{1-\gamma^-} |\varphi|^{1-\gamma(x)} \, dx \\
&\leq \frac{t^{p^-}}{p^-} I_{\beta(x)}(\varphi) - \frac{t^{1-\gamma^-}}{1-\gamma^-} \int_\Omega V(x) \, |\varphi|^{1-\gamma(x)} \, dx
\end{aligned}$$

for any $t \in (0,1)$. Since $1-\gamma^- < p^-$, we obtain $E_\lambda(t\varphi) < 0$ for any $t < \delta^{\frac{1}{p^- - (1-\gamma^-)}}$ with $0 < \delta < \min\left\{ 1, \frac{\frac{p^-}{1-\gamma^-}}{I_{\beta(x)}(\varphi)} \int_\Omega V(x) \, |\varphi|^{1-\gamma(x)} \, dx \right\}$. Finally, we point out that $I_{\beta(x)}(\varphi) > 0$. In fact, if $I_{\beta(x)}(\varphi) = 0$, then $\|\varphi\|_{\beta(x)} = 0$ and consequently $\varphi = 0$ in Ω, which is a contradiction. □

Theorem 7. *The problem (1) has at least one nontrivial weak solution.*

Proof. From Lemma 2 we can define

$$m_\lambda = \inf_{u \in X} E_\lambda(u).$$

Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence, that is $E_\lambda(u_n) \to m_\lambda$ as $n \to \infty$. Assume that $(u_n)_{n\in\mathbb{N}}$ is not bounded. So $\|u_n\|_X \to \infty$ as $n \to \infty$. Since E_λ is coercive, we have

$$E_\lambda(u_n) \to +\infty \text{ as } \|u_n\|_X \to \infty.$$

This contradicts the fact that $(u_n)_{n\in\mathbb{N}}$ is a minimizing sequence, so $(u_n)_{n\in\mathbb{N}}$ is bounded in X. Since X is a reflexive Banach space, then there exists a subsequence still denoted by u_n and $u_\lambda \in X$ such that $u_n \rightharpoonup u_\lambda$ weakly in X. From Lemma 1

$$E_\lambda(u_\lambda) \leq \liminf_{n\to\infty} E_\lambda(u_n) = m_\lambda.$$

On the other hand, from the definition of m_λ, we have $m_\lambda \leq E_\lambda(u_\lambda)$. Therefore, u_λ is a global minimum for E_λ, which is a weak solution for the problem (1). Finally, Lemma 3 it follows that $u_\lambda \neq 0$. The proof of the Theorem is completed. □

4. Uniqueness of the Solution

We begin considering the following problem

$$\begin{cases} \Delta\left(a(x)\left|\Delta u_n\right|^{p(x)-2}\Delta u_n\right) = \frac{V(x)}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}, & x\in\Omega,\\ a(x)\left|\Delta u_n\right|^{p(x)-2}\frac{\partial u_n}{\partial v} + \beta(x)\left|u_n\right|^{p(x)-2}u_n = 0, & x\in\partial\Omega, \end{cases} \tag{16}$$

where $u_n = \min\{u,n\}$. By Theorem 7, the problem (16) has a solution $u_n \in X\cap L^\infty(\Omega)$ and $u_n > 0$ for each $n\in\mathbb{N}$ (see Lemma 4.1 in [11] and Lemma 3.1 in [9]). Now we recall the algebraic inequality from Lemma A.0.5 in [33].

Lemma 4. *Let $x, y \in \mathbb{R}^N$ and $\langle .,.\rangle$ the standard scalar product in $\mathbb{R}^N$. Then*

$$\left\langle |x|^{p-2}x - |y|^{p-2}y, x-y\right\rangle \geq c\,|x-y|^p$$

for $p \geq 2$.

Theorem 8. *The problem (16) has a unique solution in $X\cap L^\infty(\Omega)$.*

Proof. Let $n\in\mathbb{N}$ and $u_n, v_n \in X\cap L^\infty(\Omega)$ solves the problem (16). Then we can write

$$\int_\Omega a(x)\left|\Delta u_n\right|^{p(x)-2}\Delta u_n\Delta\varphi dx + \int_{\partial\Omega}\beta(x)\left|u_n\right|^{p(x)-2}u_n\varphi d\sigma = \int_\Omega \frac{V(x)\varphi}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}dx \tag{17}$$

and

$$\int_\Omega a(x)\left|\Delta v_n\right|^{p(x)-2}\Delta v_n\Delta\varphi dx + \int_{\partial\Omega}\beta(x)\left|v_n\right|^{p(x)-2}v_n\varphi d\sigma = \int_\Omega \frac{V(x)\varphi}{\left(v_n+\frac{1}{n}\right)^{\gamma(x)}}dx \tag{18}$$

for all $\varphi\in X$. By choosing $(u_n-v_n)^+ = \max\{u_n-v_n, 0\}$ as a test function for the weak solution, and subtracting (18) from (17) we obtain

$$\int_\Omega V(x)\left\{\frac{1}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}} - \frac{1}{\left(v_n+\frac{1}{n}\right)^{\gamma(x)}}\right\}(u_n-v_n)^+dx = \tag{19}$$

$$\int_\Omega a(x)\left\{\left|\Delta u_n\right|^{p(x)-2}\Delta u_n - \left|\Delta v_n\right|^{p(x)-2}\Delta v_n\right\}\Delta(u_n-v_n)^+dx$$

$$+\int_{\partial\Omega}\beta(x)\left\{|u_n|^{p(x)-2}u_n-|v_n|^{p(x)-2}v_n\right\}(u_n-v_n)^+\,d\sigma$$

$$\geq C_{12}\int_{\Omega}a(x)\left|\Delta(u_n-v_n)^+\right|^{p(x)-2}dx+C_{13}\int_{\partial\Omega}\beta(x)\left|(u_n-v_n)^+\right|^{p(x)-2}d\sigma\geq 0$$

by Lemma 4. On the other hand, we have

$$\begin{aligned}&\int_{\Omega}V(x)\left\{\frac{1}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}-\frac{1}{\left(v_n+\frac{1}{n}\right)^{\gamma(x)}}\right\}(u_n-v_n)^+\,dx\\ =\;&\int_{\Omega}V(x)\left\{\frac{\left(v_n+\frac{1}{n}\right)^{\gamma(x)}-\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}\left(v_n+\frac{1}{n}\right)^{\gamma(x)}}\right\}(u_n-v_n)^+\,dx\leq 0.\end{aligned}\tag{20}$$

Hence, we infer that $(u_n-v_n)^+=0$ a.e. in Ω and $u_n\leq v_n$ from (19) and (20). By symmetry, this also implies $u_n=v_n$. □

5. Conclusion

In this paper we obtain the existence of solutions for the class of singular fourth order equation (1) involving the weighted $p(.)$-biharmonic operator. Moreover, we find a unique solution for (16) in $X\cap L^{\infty}(\Omega)$. The existence of multiple weak solutions to the problem (1) can also be investigated in other studies in the future.

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References

[1] Alsaedi, R., Ali, K. Ben., Ghanmi, A., Existence results for singular $p(x)$-Laplacian equation, *Adv. in Pure and Appl. Math.*, 3(13) (2022), 62-71. https://doi.org/10.21494/ISTE.OP.2022.0840

[2] Allali, Z. E., Hamdani, M. K., Taarabti, S., Three solutions to a Neumann boundary value problem driven by $p(x)$-biharmonic operator, *J. Elliptic Parabol Equ.*, 10(1) (2024), 195-209. https://doi.org/10.1007/s41808-023-00257-1

[3] Aydın, I., Weighted variable Sobolev spaces and capacity, *J. Funct. Spaces Appl.*, 2012 (2012). https://doi.org/10.1155/2012/132690

[4] Aydin, I., Unal, C., Existence and multiplicity of weak solutions for eigenvalue Robin problem with weighted $p(.)$-Laplacian,. *Ric. Mat.*, 72 (2023), 511-528. https://doi.org/10.1007/s11587-021-00621-0

[5] Aydin, I., Unal, C., Three solutions to a Steklov problem involving the weighted $p(.)$-Laplacian, *Rocky Mountain J. Math.*, 51(1) (2021), 67-76. https://doi.org/10.1216/rmj.2021.51.67

[6] Aydın, I., Almost all weak solutions of the weighted $p(.)$-biharmonic problem, *J. Anal.*, 32 (2024), 171-190. https://doi.org/10.1007/s41478-023-00628-w

[7] Ayoujil, A., El Amrouss, A. R., On the spectrum of a fourth order elliptic equation with variable exponent, *Nonlinear Anal.*, 71(10) (2009), 4916-4926. https://doi.org/10.1016/j.na.2009.03.074

[8] Ayoujil, A., El Amrouss, A. R., Continuous spectrum of a fourth order nonhomogeneous elliptic equation with variable exponent, *Electron. J. Differential Equations*, 2011(24) (2011), 1-12. http://ejde.math.txstate.edu

[9] Bal, K., Garain, P., Mukherjee, T., On an anisotropic p-Laplace equation with variable singular exponent, A*dv. Differential Equations*, 26(11/12) (2021), 535-562. https://doi.org/10.57262/ade026-1112-535

[10] Brezis, H., Analise functional Theorie Methodes et Applications, Masson Paris, 1992.

[11] Canino, A., Sciunzi, B., Trombetta, A., Existence and uniqueness for p-Laplace equations involving singular nonlinearities, *Nonlinear Differ. Equ. Appl.*, 23(8) (2016), 1-18. https://doi.org/10.1007/s00030-016-0361-6

[12] Chung, N. T., Some remarks on a class of $p(x)$-Laplacian Robin eigenvalue problems, Mediterr. J. Math., 15(147) (2018), 1-14. https://doi.org/10.1007/s00009-018-1196-7

[13] Chung, N. T., On a class of $p(x)$-Kirhhoff type problems with robin boundary conditions and indefinite weights, *TWMS J. App. and Eng. Math.*, 10(2) (2020), 400-410. https://orcid.org/0000-0001-7345-620X.

[14] Chung, N. T., Ho, K., On a $p(\cdot)$-biharmonic problem of Kirchhoff type involving critical growth, *App. Analy.*, 101(16) (2022), 5700-5726. https://doi.org/10.1080 /00036811.2021.1903445

[15] Deng, S. G., Eigenvalues of the $p(x)$-Laplacian Steklov problem, *J. Math. Anal. Appl.*, 339(2) (2008), 925-937. https://doi.org/10.1016/j.jmaa.2007.07.028

[16] Deng, S. G., Positive solutions for Robin problem involving the $p(x)$-Laplacian, *J. Math. Anal. Appl.*, 360(2) (2009), 548-560. https://doi.org/10.1016/j.jmaa.2009.06.032

[17] Edmunds, D. E., Rákosník, J., Sobolev embeddings with variable exponent, *Studia Math.*, 143(3) (2000), 267-293.

[18] Fan, X., Zhao, D., On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl., 263(2) (2001), 424-446. https://doi.org/10.1006/jmaa.2000.7617

[19] Fan, X. L., Solutions for $p(x)$-Laplacian Dirichlet problems with singular coefficients, *J. Math. Anal. Appl.*, 312(2) (2005), 464-477. https://doi.org/10.1016/j.jmaa.2005.03.057

[20] Ge, B., Zhou, Q., Wu, Y., Eigenvalues of the $p(x)$-biharmonic operator with indefinite weight, *Z. Angew. Math. Phys.*, 66 (2015), 1007-1021. https://doi.org/10.1007/s00033-014-0465-y

[21] Ge, B., Zhou, Q. M., Multiple solutions for a Robin-type differential inclusion problem involving the $p(x)$-Laplacian, *Math. Methods Appl. Sci.*, 40(18) (2017), 6229-6238. https://doi.org/10.1002/mma.2760

[22] Hamdani, M. K., Harrabi, A., Mtiri, F., Repovš, D. D., Existence and multiplicity results for a new $p(x)$-Kirchhoff problem, *Nonlinear Anal.*, 190 (2020), 111598, 1-15. https://doi.org/10.1016/j.na.2019.111598

[23] Hewitt, E., Stromberg, K., Real and Abstract Analysis, Springer-Verlag, 1965.

[24] Kefi, K., On the Robin problem with indefinite weight in Sobolev spaces with variable exponents, *Z. Anal. Anwend.* , 37(1) (2018), 25-38. https://doi.org/10.4171/ZAA/1600

[25] Kefi, K., Saoudi, K., On the existence of a weak solution for some singular $p(x)$-biharmonic equation with Navier boundary conditions, *Adv. Nonlinear Anal.*, 8 (2019), 1171-1183. https://doi.org/10.1515/anona-2016-0260

[26] Kefi, K., Al-Shomrani, M.M., Variational approach for a Robin problem involving non standard growth conditions, *Mathematics*, 10(7) (2022), 1127. https://doi.org/10.3390/math10071127

[27] Kováčik, O., Rákosník, J., On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czech. Math. J.*, 41(4) (1991), 592-618. http://dml.cz/dmlcz/102493

[28] Kulak, O., Aydin, I., Unal, C., Existence of weak solutions for weighted Robin problem involving $p(.)$-biharmonic operator, *Differ. Equ. Dyn. Syst.*, 2(4) (2024), 1159–1174. https://doi.org/10.1007/s12591-022-00619-6.

[29] Kufner, A., John, O., Fučik, S., Function Spaces. Prague: Academia, 1977.

[30] Liu, Q., Compact trace in weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega;\nu_0,\nu_1)$, *J. Math. Anal. Appl.*, 348(2) (2008), 760-774. https://doi.org/10.1016/j.jmaa.2008.08.004

[31] Liu, Q., Liu, D., Existence and multiplicity of solutions to a $p(x)$-Laplacian equation with nonlinear boundary condition on unbounded domain, *Diff. Equa. Appl.*, 5(4) (2013), 595-611. https://doi.org/10.7153/dea-05-35

[32] Mbarki, L., The Nehari manifold approach involving a singular $p(x)$-biharmonic problem with Navier boundary conditions, *Acta Appl. Math.*, 182(3) (2022), https://doi.org/10.1007/s10440-022-00538-2.

[33] Peral, I., Multiplicity of Solutions for the p-Laplacian, Lecture Notes at the Second School on Nonlinear Functional Analysis and Applications to Differential Equations at ICTP of Trieste, ICTP Lecture Notes, 1997, 114 pages.

[34] Unal, C., Aydın, I., Compact embeddings of weighted variable exponent Sobolev spaces and existence of solutions for weighted $p(.)$-Laplacian, *Complex Variables and Elliptic Equations*, 66(10) (2021), 1755-1773. https://doi.org/10.1080/17476933.2020.1781831