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Contact CR-Warped Product Submanifolds of Nearly Lorentzian Para-Sasakian Manifolds

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Abstract. In the present paper we have investigated CR-submanifold of a nearly Lorentzian para-Sasakian manifolds, generalizing sharp inequality namely $||h||^2 \geq \frac{2}{9}s + 2s||\nabla \ln f||^2$, for contact CR-warped products of nearly Lorentzian para-Sasakian manifolds. It is also proved that the derive inequalities for contact CR-warped products either in any contact metric manifold. The equality case is also handled.

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1. Introduction

The notion of warped product manifolds was introduced by Bishop and O'Neill [5]. These manifolds appear in differential geometric studies in natural way and these are generalization of Riemannian product manifolds and then it was studied by many geometers in different known spaces [7, 10]. Chen, B.Y has introduced CR-warped product in Kaehler manifolds and showed many interesting results on the existence of warped product and proved general sharp inequalities for the second fundamental form in terms of the warping function f [8]. Later on, many articles have been appeared for the same inequalities in almost Hermitian as well as almost contact metric manifolds [4, 6, 13]. On the other hand, there is a class of almost paracontact metric manifolds, namely Lorentzian para-Sasakian manifolds. Matsumoto, K [12], introduced the idea of Lorentzian para-Sasakian manifold. Then Mihai, I et al. [14] introduced the same notion independently and they obtained several results on this manifolds. Lorentzian para-Sasakian manifolds have also been studied by Matsumoto, K et al. [7], De, U.C et al., [9], and others. Lorentzian para-Sasakian manifolds with different connection was studied by several authors in ([1, 2, 15–17]). In [18], the author and et al., studied semi-invariant submanifolds of a nearly Lorentzian para-Sasakian manifold.

In the present paper we study the warped product contact CR-submanifolds of nearly Lorentzian para-Sasakian manifolds. In the beginning, we prove some existence and nonexistence results and then obtain a general sharp inequality for the second fundamental form in terms of the warping function f and Sasakian and cosymplectic on nearly Lorentzian para-Sasakian manifolds. The inequality obtained in this paper is more general as it generalizes all inequalities obtained for contact CR-warped products in contact metric manifolds.

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2. Preliminaries

Let $\bar{M}$ be an n dimensional almost contact metric manifold with almost contact metric stucture (ϕ, ξ, η, g) such that

$$\phi^2 = I + \eta(X)\xi, \quad \eta(\xi) = -1, \quad g(X, \xi) = \eta(X), \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad g(\phi X, Y) = g(X, \phi Y) = \psi(X, Y) \tag{2.2}$$

for vector fields X,Y tangent to M. Then the structure (ϕ, ξ, η, g) is termed as Lorentzian para-contact structure. Also in a Lorentzian para-contact structure the following relations hold:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad and \quad rank(\phi) = n - 1.$$

A Lorentzian para-contact manifold $\bar{M}$ is called Lorentzian para-Sasakian (LP-Sasakian) manifold if

$$(\bar{\nabla}_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

$$\bar{\nabla}_X \xi = \phi X$$

for all vector fields X, Y tangent to $\bar{M}$ where $\bar{\nabla}$ is the Riemannian connection with respect to g. Further, an almost contact metric manifold $\bar{M}$ on (ϕ, ξ, η, g) is called nearly Lorentzian para-Sasakian if

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 2g(X, Y)\xi + 4\eta(X)\eta(Y)\xi + \eta(X)Y + \eta(Y)X. \tag{2.3}$$

The covariant derivative of the tensor filed ϕ is defined as

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y. \tag{2.4}$$

Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^{\perp}M$ be the Lie algebras of vector fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connection on M, then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.6}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N as

$$g(A_N X, Y) = g(h(X, Y), N).$$

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f be a positive differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where [5]

$$g = g_1 + f^2 g_2.$$

A warped product manifold $N_1 \times_f N_2$ is said to be trivial if the warping function f is constant.

We recall the following general result for later use.

Lemma 2.1 ([5]). *Let $M = N_1 \times_f N_2$ be a warped product manifold with the warping function f, then*
(i) $\nabla_X Y \in \Gamma(TN_1)$ is the lift of $\nabla_X Y$ on N_1,
(ii) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$,
(iii) $\nabla_Z \omega = \nabla^{N_2}_Z \omega - g(Z, \omega)\nabla \ln f$
for each $X, Y \in \Gamma(TN_1)$ and $Z, \omega \in \Gamma(TN_2)$, where $\nabla \ln f$ is the gradient of $\ln f$ and ∇ and ∇^{N_2} denote the Levi-Civita connections on M and N_2, respectively.

For a Riemannian manifold M of dimension n and a smooth function f on M, we recall ∇f, the gradient of f which is defined by

$$g(\nabla f, X) = X(f)$$

for any $X \in \Gamma(TM)$. As a consequence, we have

$$\|\nabla f\|^2 = \sum_{i=1}^{n} (e_i(f))^2 \tag{2.7}$$

for an orthonormal frame $\{e_1, ..., e_n\}$ on M.

3. Contact CR-Warped Product Submanifolds

In this section first we recall the invariant, anti-invariant and contact CR-submanifolds. For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds. We mention the following:

(i) A submanifold M tangent to ξ is an invariant submanifold if ϕ preserves any tangent space of M, that is, $\phi(T_pM) \subset T_pM$, for every $p \in M$.

(ii) A submanifold M tangent to ξ is an anti-invariant submanifold if ϕ maps any tangent space of M into the normal space, that is, $\phi(T_pM) \subset T_p^{\perp}M$, for every $p \in M$.

Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $\bar{M}$, then for every $p \in M$ there exists a maximal invariant subspace denoted by D_p of the tangent space T_pM of M. If the dimension of D_p is same for all values of $p \in M$, then D_p gives an invariant distribution D on M.

A submanifold M of an almost contact manifold $\bar{M}$ is said to be a contat CR submanifold if there exists on M a differentiable distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, that is;

(i) $TM = D \oplus D^{\perp} \oplus \langle\xi\rangle$,

(ii) D is an invariant distribution, i.e., $\phi D \subseteq TM$,

(iii) $D^{\perp}$ is an anti-invariant distribution, i.e., $\phi D^{\perp} \subseteq T^{\perp}M$.

A contact CR-submanifold is anti-invariant if $D_p = \{0\}$ and invariant if $D_p^{\perp} = \{0\}$ respectively, for every $p \in M$. It is a proper contact CR-submanifold if neither $D_p = \{0\}$ nor $D_p^{\perp} = \{0\}$, for each $p \in M$.

If ν is the ϕ-invariant subspace of the normal bundle $T^{\perp}M$, then in case of contact CR-submanifold, the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = \phi D^{\perp} \oplus \nu$$

where ν is the ϕ -invariant normal subbundle of $T^{\perp}M$.

In this section, we investigate the warped products $M = N_{\perp}\times_f N_T$ and $M = N_T\times_f N_{\perp}$ where N_T and $N_{\perp}$ are invariant and anti-invariant submanifolds of a nearly Lorentzian para-Sasakian manifold $\bar{M}$, respectively. First we discuss the warped products $M = N_{\perp}\times_f N_T$, here two possible cases arise:

(i) ξ is tangent to N_T ,

(ii) ξ is tangent to $N_{\perp}$,

We start with the case (i).

Theorem 3.1. *Let $\bar{M}$ be a nearly Lorentzian para-Sasakian manifold $\bar{M}$. Then there do not exist warped product submanifold $M = N_{\perp}\times_f N_T$ such that N_T is an invariant submanifold tangent to ξ and $N_{\perp}$ is an anti invariant submanifold, unless $\bar{M}$ is nearly Sasakian .*

Proof. Consider $\xi \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_{\perp})$, then by the structure equation of nearly Lorentzian para-Sasakian manifold, we have $(\bar{\nabla}_Z\phi)\xi + (\bar{\nabla}_\xi\phi)Z = -Z$. Using (2.4), we obtain $-\phi\bar{\nabla}_Z\xi + \bar{\nabla}_\xi\phi Z - \phi\bar{\nabla}_\xi Z = -Z$. Then from Lemma 2.1 (ii) and (2.5), we derive

$$\bar{\nabla}_\xi\phi Z - 2\phi h(Z,\xi) = -Z. \tag{3.1}$$

Taking the inner product with ϕZ in (3.1) and then using (2.2) and the fact that $\xi \in \Gamma(TN_T)$, we get $\|Z\|^2 = 0$ and hence we conclude that M is invariant, which proves the theorem. □

Now, we will discuss the other case, when ξ is tangent to $N_{\perp}$.

Theorem 3.2. *Let $\bar{M}$ be a nearly Lorentzian para-Sasakian manifold. Then there do not exist warped product submanifolds $M = N_{\perp}\times_f N_T$ such that $N_{\perp}$ is an anti-invariant submanifold tangent to ξ and N_T is an invariant submanifold of $\bar{M}$, unless $\bar{M}$ is nearly cosympletic.*

Proof. Consider $\xi \in \Gamma(TN_T)$ and $X \in \Gamma(TN_{\perp})$, then we have $(\bar{\nabla}_X\phi)\xi + (\bar{\nabla}_\xi\phi)X = -X$. Using (2.4), we get

$$-\phi\bar{\nabla}_X\xi + \bar{\nabla}_\xi\phi X - \phi\bar{\nabla}_\xi X = -X. \tag{3.2}$$

Taking the inner product with X in (3.2) and using (2.2), (2.5), Lemma 2.1 (ii) and the fact that ξ is tangent to $N_{\perp}$, we obtain $\|X\|^2 = 0$. Thus, we conclude that M is anti-invariant submanifold of a nearly Lorentzian para-Sasakian manifold $\bar{M}$ otherwise $\bar{M}$ is nearly cosympletic. This completes the proof. □

Now, we will discuss the warped product $M = N_T\times_f N_{\perp}$ such that the structure vector field ξ is tangent to $N_{\perp}$.

Theorem 3.3. *Let $\bar{M}$ be a nearly Lorentzian para-Sasakian manifold. Then there do not exist warped product submanifolds $M = N_T \times_f N_\perp$ such that $N_\perp$ is an anti-invariant submanifold tangent to ξ and N_T is an invariant submanifold of $\bar{M}$.*

Proof. If we consider $X \in \Gamma(TN_T)$ and the structure vector field ξ is tangent to $N_\perp$, then by (2.3), we have $(\bar{\nabla}_X \phi)\xi + (\bar{\nabla}_\xi \phi)X = -X$. Using (2.4), we obtain $\bar{\nabla}_\xi \phi X - \phi \bar{\nabla}_X \xi - \phi \bar{\nabla}_\xi X = -X$. Then by (2.5) and Lemma 2.1 (ii), we derive

$$(\phi X \ln f)\xi - 2\phi h(X, \xi) + h(\phi X, \xi) = -X. \tag{3.3}$$

Hence, the result is obtained by taking the inner product with ξ in (3.3). □

If we consider the structure vector field ξ tangent to N_T for the warped product $M = N_T \times_f N_\perp$, then we prove the following result for later use.

Lemma 3.4. *Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a nearly Lorentzian para-Sasakian $\bar{M}$ such that N_T and $N_\perp$ are invariant and antiinvariant submanifolds of $\bar{M}$, respectively. Then, we have*
(i) $\xi(\ln f) = 0$,
(ii) $g(h(X, Y), \phi Z) = 0$,
(iii) $g(h(X, \omega), \phi Z) = g(h(X, Z), \phi \omega) = -\frac{1}{3}\{\eta(X) g(Z, \omega) - (\phi X \ln f) g(Z, \omega)\}$,
(iv) $3g(h(\xi, Z), \phi \omega) = g(Z, \omega)$
for every $X, Y \in \Gamma(TN_T)$ and $Z, \omega \in \Gamma(TN_\perp)$.

Proof. If ξ is tangent to N_T, then for any $Z \in \Gamma(TN_\perp)$, we have $(\bar{\nabla}_\xi \phi)Z + (\bar{\nabla}_Z \phi)\xi = -Z$. Then from (2.4), (2.5) and Lemma 2.1 (ii), we obtain

$$2(\xi \ln f)\phi Z + 2\phi h(Z, \xi) - \bar{\nabla}_\xi \phi Z = Z. \tag{3.4}$$

Taking the inner product with ϕZ in (3.4) and using (2.2), we derive

$$2(\xi \ln f)\|Z\|^2 - g(\bar{\nabla}_\xi \phi Z, \phi Z) = 0. \tag{3.5}$$

On the other hand, by the property of Riemannian connection, we have $\xi g(\phi Z, \phi Z) = 2g(\bar{\nabla}_\xi \phi Z, \phi Z)$. By (2.2) and the property of Riemannian connection, we get

$$g(\bar{\nabla}_\xi Z, Z) = g(\bar{\nabla}_\xi \phi Z, \phi Z).$$

Using this fact in (3.5) and then from (2.5) and Lemma 2.1 (ii), we deduce that $(\xi \ln f)\|Z\|^2 = 0$ for any $Z \in \Gamma(TN_\perp)$, which gives (i). For the other parts of the lemma, we have $(\bar{\nabla}_X \phi)Z + (\bar{\nabla}_Z \phi)X = \eta(X)Z$, for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\perp)$. Using (2.4), (2.5) and (2.6), we derive

$$\eta(X)Z = -A_{\phi Z}X + \nabla^\perp_X \phi Z - 2(X \ln f)\phi Z + (\phi X \ln f)Z + h(\phi X, Z) - 2\phi h(X, Z). \tag{3.6}$$

Thus, the second part can be obtained by taking the inner product in (3.6) with Y , for any $Y \in \Gamma(TN_T)$. Again, taking the inner product in (3.6) with ω for any $\omega \in \Gamma(TN_\perp)$, we get

$$\eta(X) g(Z, \omega) = -g(h(X, \omega), \phi Z) + (\phi X \ln f) g(Z, \omega) - 2g(h(X, Z), \phi \omega). \tag{3.7}$$

By polarization identity, we get

$$\eta(X) g(Z, \omega) = -g(h(X, Z), \phi \omega) + (\phi X \ln f) g(Z, \omega) - 2g(h(X, \omega), \phi Z). \tag{3.8}$$

Then from (3.7) and (3.8), we obtain

$$g(h(X, Z), \phi \omega) = g(h(X, \omega), \phi Z) \tag{3.9}$$

which is the first equality of (iii). Using (3.9) either in (3.7) or in (3.8), we get the second equality of (iii). Now, for the last part, replacing X by ξ in the third part of this lemma. This proves the lemma completely. □

Now, we have the following characterization theorem.

Theorem 3.5. *Let M be a contact CR-submanifold of a nearly Lorentzian para-Sasakian manifold $\bar{M}$ with integrable invariant and anti-invariant distribution $D \oplus \langle \xi \rangle$ and $D^\perp$. Then M is locally a contact CR-warped product if and only if the shape operator of M satisfies*

$$A_{\phi \omega} X = \frac{1}{3}\{(\phi X \mu)\omega - \eta(X)\omega\} \quad \forall X \in \Gamma(D \oplus \langle \xi \rangle), \quad \omega \in \Gamma(D^\perp) \tag{3.10}$$

for some smooth function μ on M satisfying $V(\mu) = 0$ for every $V \in \Gamma(D^\perp)$.

Proof. Direct part follows from the Lemma 3.4 (iii). For the converse, suppose that M is contact CR-submanifold satisfying (3.10), then we have $g(h(X,Y),\phi\omega) = g(A_{\phi\omega}X,Y) = 0$ for any $X,Y \in \Gamma(D \oplus \langle\xi\rangle)$ and $\omega \in \Gamma(D^{\perp})$. Using (2.2) and (2.5), we get $g(\bar{\nabla}_X Y, \phi\omega) = -g(\phi\bar{\nabla}_X Y, \omega) = 0$. Then from (2.4), we obtain

$$g((\bar{\nabla}_X \phi)Y, \omega) = g(\bar{\nabla}_X \phi Y, \omega). \tag{3.11}$$

Similarly, we have

$$g((\bar{\nabla}_Y \phi)X, \omega) = g(\bar{\nabla}_Y \phi X, \omega). \tag{3.12}$$

Then from (3.11) and (3.12), we derive

$$g((\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X, \omega) = g(\bar{\nabla}_X \phi Y + \bar{\nabla}_Y \phi X, \omega).$$

Using (2.3) and the fact that ξ is tangent to N_T , then by orthogonality of two distributions, we obtain

$$g(\bar{\nabla}_X \phi Y + \bar{\nabla}_Y \phi X, \omega) = 0.$$

This means that $\nabla_X \phi Y + \nabla_Y \phi X \in \Gamma(D \oplus \langle\xi\rangle)$, for any $X,Y \in \Gamma(D \oplus \langle\xi\rangle)$, that is $D \oplus \langle\xi\rangle$ is integrable and its leaves are totally geodesic in M. So far as the anti-invariant distribution $D^{\perp}$ is concerned, it is integrable on M (cf. [3]). Morevoer, for any $X \in \Gamma(D \oplus \xi)$ and $Z, \omega \in \Gamma(D^{\perp})$, we have

$$g(\nabla_Z \omega, X) = g(\bar{\nabla}_Z \omega, X) = g(\phi\bar{\nabla}_Z \omega, \phi X) - \eta(\bar{\nabla}_Z \omega)\eta(X)$$

$$= g(\bar{\nabla}_Z \phi\omega, \phi X) - g((\bar{\nabla}_Z \phi)\omega, \phi X) = -g(A_{\phi\omega}Z, \phi X) - g((\bar{\nabla}_Z \phi)\omega, \phi X).$$

The second term in the right hand side of the above equation vanishes in view (2.3) and the fact that ξ tangential to N_T and the first term will be

$$-g(A_{\phi\omega}Z, \phi X) = -g(h(Z, \phi X), \phi\omega) = -g(A_{\phi\omega}\phi X, Z).$$

Making use of (2.1), (3.7) and Lemma 3.4 (i), the above equation takes the form

$$g(\nabla_Z \omega, X) = -g(A_{\phi\omega}Z, \phi X) = \frac{1}{3} X\mu g(\omega, Z). \tag{3.13}$$

Now, by Gauss formula

$$g(h^{'}(Z,\omega), X) = g(\nabla_Z \omega, X)$$

where $h^{'}$ denotes the second fundamental form of the immersion of $N_{\perp}$ into M. On using (3.13), the last equation gives

$$g(h^{'}(Z,\omega), X) = \frac{1}{3} X\mu g(\omega, Z). \tag{3.14}$$

The above relation shows that the leaves of $D^{\perp}$ are totally umbilical in M with mean curvature vector $\nabla\mu$. Moreover, the condition $V\mu = 0$, for any $V \in \Gamma(D^{\perp})$ implies that the leaves of $D^{\perp}$ are extrinsic spheres in M, that is the integral manifold $N_{\perp}$ of $D^{\perp}$ is umbilical and its mean curvature vector field is non zero and parallel along $N_{\perp}$. Hence, by a result of [11] M is locally a warped product $M = N_T \times_f N_{\perp}$, where N_T and $N_{\perp}$ denote the integral manifolds of the distributions $D \oplus \langle\xi\rangle$ and $D^{\perp}$, respectively and f is the warping function. Thus, the theorem is proved completely. □

4. Inequality for Contact CR-Warped Products

In the following section we obtain a general sharp inequality for the length of second fundamental form of warped product submanifold. We prove the following main result of this section.

Theorem 4.1. *Let $M = N_T \times_f N_{\perp}$ be a contact CR-warped product submanifold of a nearly Lorentzian para-Sasakian manifold $\bar{M}$ such that N_T is an invariant submanifold tangent to ξ and $N_{\perp}$ an anti-invariant submanifold of $\bar{M}$. Then, we have*

(i) The second fundamental form of M satisfies the inequality

$$\|h\|^2 \geq \frac{2}{9} s + 2s\|\nabla \ln f\|^2 \tag{4.1}$$

where s is the dimension of $N_{\perp}$ and $\nabla \ln f$ is the gradient of $\ln f$.

(ii) If the equality sign of (4.1) holds identically, then N_T is a totally geodesic submanifold and $N_{\perp}$ is a totally umbilical submanifold of $\bar{M}$. Moreover, M is a minimal submanifold in $\bar{M}$.

Proof. Let $\bar{M}$ be a $(2n+1)$-dimensional nearly Lorentzian para-Sasakian manifold and $M = N_T \times_f N_\perp$ be an m-dimensional contact CR-warped product submanifolds of $\bar{M}$. Let us consider $dimN_T = 2p+1$ and $dimN_\perp = s$, then $m = 2p+1+s$. Let $\{e_1 ...., e_p; \phi e_1 = e_{p+1}, ..., \phi e_p = e_{2p}, e_{2p+1} = \xi\}$ and $\{e_{(2p+1)+1}, ..., e_m\}$ be the local orthonormal frames on N_T and $N_\perp$, respectively. Then the orthonormal frames in the normal bundle $T^\perp M$ of $\phi D^\perp$ and ν are $\{\phi e_{(2p+1)+1}, ..., \phi e_m\}$ and $\{e_{m+s+1}, ..., e_{2n+1}\}$, respectively. Then the length of second fundamental form h is defined as

$$||h||^2 = \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2.$$

For the assumed frames, the above equation can be written as

$$||h||^2 = \sum_{r=m+1}^{m+s} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2 + \sum_{r=m+s+1}^{2n+1} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2. \tag{4.2}$$

The first term in the right hand side of the above equality is the $\phi D^\perp$-component and the second term is ν-component. If we equate only the $\phi D^\perp$-component, then we have

$$||h||^2 \geq \sum_{r=m+1}^{m+s} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2.$$

For the given frame of $\phi D^\perp$, the above equation will be

$$||h||^2 \geq \sum_{k=(2p+1)+1}^{m} \sum_{i,j=1}^{m} g(h(e_i, e_j), \phi e_k)^2.$$

Let us decompose the above equation in terms of the components of $h(D, D), h(D, D^\perp)$ and $h(D^\perp, D^\perp)$, then we have

$$||h||^2 \geq \sum_{k=2p+2}^{m} \sum_{i,j=1}^{2p+1} g(h(e_i, e_j), \phi e_k)^2 + 2 \sum_{k=2p+2}^{m} \sum_{i=1}^{2p+1} \sum_{j=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2 \tag{4.3}$$

$$+ \sum_{k=2p+2}^{m} \sum_{i,j=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2.$$

By Lemma 3.4 (ii), the first term of the right hand side of (4.3) is identically zero and we shall compute the next term and will left the last term

$$||h||^2 \geq 2 \sum_{k=2p+2}^{m} \sum_{i=1}^{2p+1} \sum_{j=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2.$$

As $j, k = 2p+2, ..., m$ then the above equation can be written for one summation as

$$||h||^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2.$$

Making use of Lemma 3.4 (iii), the above inequality will be

$$||h||^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\frac{1}{3}(\phi e_i \ln f - \eta(e_i)))^2 g(e_j, e_k)^2.$$

The above expression can be written as

$$||h||^2 \geq \frac{2}{9} \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\phi e_i \ln f)^2 g(e_j, e_k)^2 + \frac{2}{9} \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\eta(e_i))^2 g(e_j, e_k)^2 \tag{4.4}$$

$$-\frac{4}{9} \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\phi e_i \ln f)\eta(e_i) g(e_j, e_k)^2.$$

The last term of (4.4) is identically zero for the given frames. Thus, the above relation gives

$$\|h\|^2 \geq \frac{2}{9}\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}(\phi e_i \ln f)^2 g(e_j, e_k)^2 + \frac{2}{9}s. \tag{4.5}$$

On the other hand, from (2.7), we have

$$\|\nabla \ln f\|^2 = \sum_{i=1}^{p}(e_i \ln f)^2 + \sum_{i=1}^{p}(\phi e_i \ln f)^2 + (\xi \ln f)^2. \tag{4.6}$$

Now, the equation (4.5) can be modified as

$$\|h\|^2 \geq \frac{2}{9}\sum_{i=1}^{2p}\sum_{j,k=2p+2}^{m}(\phi e_i \ln f)^2 g(e_j, e_k)^2 + \frac{2}{9}s$$

$$+2\sum_{j,k=2p+1}^{m}(\xi \ln f)^2 g(e_j, e_k)^2 - 2\sum_{j,k=2p+2}^{m}(\xi \ln f)^2 g(e_j, e_k)^2$$

or

$$\|h\|^2 \geq \frac{2}{9}s - 2\sum_{j,k=2p+2}^{m}(\xi \ln f)^2 g(e_j, e_k)^2 + 2\sum_{j,k=2p+2}^{m}(\xi \ln f)^2 g(e_j, e_k)^2$$

$$+\frac{20}{9}\sum_{i=1}^{p}\sum_{j,k=2p+2}^{m}(\phi e_i \ln f)^2 g(e_j, e_k)^2. \qquad (since \quad \phi\xi \ln f = 0)$$

Therefore, using Lemma 3.4 (i) and (4.6), we arrive at

$$\|h\|^2 \geq \frac{2}{9}s + 2s\|\nabla \ln f\|^2$$

which is the inequality (4.1). Let h^* be the second fundamental form of $N_\perp$ in M, then from (3.14), we have

$$h^*(Z, \omega) = g(Z, \omega)\nabla \ln f \tag{4.7}$$

for any $Z, W \in \Gamma(D^\perp)$. Now, assume that the equality case of (4.1) holds identically. Then from (4.2), (4.3) and (4.4), we obtain

$$h(D, D) = 0, \quad h(D^\perp, D^\perp) = 0, \quad h(D, D^\perp) \subset \phi D^\perp. \tag{4.8}$$

Since N_T is a totally geodesic submanifold in M (by Lemma 2.1 (i)), using this fact with the first condition in (4.8) implies that N_T is totally geodesic in $\bar{M}$. On the other hand, by direct calculations same as in the proof of Theorem 3.5, we deduce that $N_\perp$ is totally umbilical in M. Therefore, the second condition of (4.8) with (4.7) implies that $N_\perp$ is totally umbilical in $\bar{M}$. Moreover, all three conditions of (4.8) imply that M is minimal submanifold of $\bar{M}$. This completes the proof of the theorem. □

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