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Pell and Pell-Lucas Numbers Associated with Brocard-Ramanujan Equation

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Abstract. In this paper, the diophantine equations of the form $A_{n_1}A_{n_2}\cdots A_{n_k} \pm 1 = B_m^2$ where $(A_n)_{n\geq 0}$ and $(B_m)_{m\geq 0}$ are either the Pell sequence or Pell-Lucas sequence are solved by applying the Primitive Divisor Theorem. This is another version of Brocard-Ramanujan equation.

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1. Introduction

The problem of finding all solutions to

$$n! + 1 = m^2$$

is known as Brocard-Ramanujan problem. Some authors [1,3,4] have been worked on this problem. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The variant of this problem, the diophantine equation

$$F_nF_{n+1}\cdots F_{n+k-1} + 1 = F_m^2$$

was investigated by Marques [5]. Also, Szalay [7] and Pongsriiam [6] worked on another version of this diophantine equation.

In this article we will give a new version of Brocard-Ramanujan equation in terms of Pell and Pell-Lucas sequence.

Let $(P_n)_{n\geq 0}$ be the Pell sequence given by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ and let $(Q_n)_{n\geq 0}$ be the Pell-Lucas sequence given by the same recurrence relation as the Pell sequence with the initial values $Q_0 = Q_1 = 2$.

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2. Preliminaries and Lemmas

Before giving the Primitive Divisor Theorem, we first give some remarks about it. Let α and β be algebraic numbers such that $\alpha+\beta$ and $\alpha\beta$ are nonzero coprime integers and $\alpha\beta^{-1}$ is not a root of unity. Let $(u_n)_{n\geq 0}$ be the sequence given by

$$u_0 = 0, u_1 = 1, \text{ and } u_n = (\alpha+\beta)u_{n-1} - (\alpha\beta)u_{n-2} \text{ for } n \geq 2.$$

Then we have Binet's formula for u_n given by

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ for } n \geq 0.$$

If $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ then u_n is the Pell sequence.

A prime p is said to be a primitive divisor of u_n if $p \mid u_n$ but p does not divide $u_1 u_2 \cdots u_{n-1}$.

Theorem 2.1 (Primitive Divisor Theorem [2])**.** *Suppose α and β are real numbers such that $\alpha+\beta$ and $\alpha\beta$ are nonzero coprime integers and $\alpha\beta^{-1}$ is not a root of unity. If $n \neq 1, 2, 6$, then u_n has a primitive divisor except when $n = 12$, $\alpha+\beta = 1$ and $\alpha\beta = -1$.*

Lemma 2.2. *For every $m \geq 1$, we have*

$$P_{m-1}P_{m+1} = \begin{cases} P_m^2 - 1, & \text{if } m \text{ is odd;} \\ P_m^2 + 1, & \text{if } m \text{ is even.} \end{cases}$$

Proof. Let m be an even integer. We know that the roots of quadratic equation of Pell numbers are $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$. So by the help of Binet's formula it can be proved as follows:

$$\begin{aligned}
P_{m-1}P_{m+1} &= \frac{\alpha^{m-1}-\beta^{m-1}}{\alpha-\beta}\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta} \\
&= \frac{\alpha^{2m}+\beta^{2m}-\alpha^{m-1}\beta^{m+1}-\alpha^{m+1}\beta^{m-1}}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2m}+\beta^{2m}-(\alpha\beta)^{m-1}(\alpha^2+\beta^2)}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2m}+\beta^{2m}-6(\alpha\beta)^{m-1}}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2m}+\beta^{2m}+6(\alpha\beta)^{m}}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2m}+\beta^{2m}-2(\alpha\beta)^{m}+8}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2m}+\beta^{2m}-2(\alpha\beta)^{m}+(\alpha-\beta)^2}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2m}+\beta^{2m}-2(\alpha\beta)^{m}}{(\alpha-\beta)^2}+1 \\
&= P_m^2+1.
\end{aligned}$$

Similarly, let m be an odd integer. It can be proved as follows:

$$
\begin{aligned}
P_{m-1}P_{m+1} &= \frac{\alpha^{m-1}-\beta^{m-1}}{\alpha-\beta}\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta} \\
&= \frac{\alpha^{2m}+\beta^{2m}-\alpha^{m-1}\beta^{m+1}-\alpha^{m+1}\beta^{m-1}}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2m}+\beta^{2m}-(\alpha\beta)^{m-1}(\alpha^2+\beta^2)}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2m}+\beta^{2m}-6(\alpha\beta)^{m-1}}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2m}+\beta^{2m}+6(\alpha\beta)^{m}}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2m}+\beta^{2m}-2(\alpha\beta)^{m}-8}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2m}+\beta^{2m}-2(\alpha\beta)^{m}-(\alpha-\beta)^2}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2m}+\beta^{2m}-2(\alpha\beta)^{m}}{(\alpha-\beta)^2}-1 \\
&= P_m^2-1.
\end{aligned}
$$

□

Lemma 2.3. *For every $m \geq 1$, we have*

(i) $Q_m^2 - 1 = \begin{cases} 8P_{m-1}P_{m+1}+3, & \text{if } m \text{ is odd;} \\ P_{3m}/P_m, & \text{if } m \text{ is even.} \end{cases}$

(ii) $Q_m^2 + 1 = \begin{cases} P_{3m}/P_m, & \text{if } m \text{ is odd;} \\ 8P_{m-1}P_{m+1}-3, & \text{if } m \text{ is even.} \end{cases}$

Proof. This can be checked easily by using Binet's formula. □

3. Main Results

Theorem 3.1. *The diophantine equation*

$$P_{n_1}P_{n_2}\cdots P_{n_k}+1=P_m^2 \tag{3.1}$$

in positive integers k, m and $3 \leq n_1 < n_2 < \cdots < n_k$ has an infinite family of solutions given by

$$P_{m-1}P_{m+1}+1=P_m^2.$$

Proof. Taking a solution of (3.1) by Lemma 2.2 we get

$$P_{n_1}P_{n_2}\cdots P_{n_k}=P_{m-1}P_{m+1}.$$

Suppose that $m \geq 14$. Then $13 \leq m-1 \leq m+1$ and therefore, by Primitive Divisor Theorem, P_{m+1} has a primitive divisor. Then $n_k = m+1$ and hence (3.1) reduces to

$$P_{n_1}P_{n_2}\cdots P_{n_{k-1}}=P_{m-1}. \tag{3.2}$$

Now $P_{m-1} > 1$ and this implies $k \geq 2$. Using the same arguments linked to primitive divisors as above (3.2) provides $n_{k-1} = m-1$. As a result, $k = 2$, i.e. there are no more terms on the left hand side of (3.2). Thus we get the infinite family of solution $P_{m-1}P_{m+1}+1=P_m^2$, $m \geq 14$. □

Theorem 3.2. *The diophantine equation*

$$Q_{n_1}Q_{n_2}\cdots Q_{n_k}+1=Q_m^2 \tag{3.3}$$

in positive integer k,*even integer* m *and in non-negative integers* $n_1<n_2<\cdots<n_k$ $(n_i\neq 1)$ *has no solution.*

Proof. Since we know that $P_nQ_n=P_{2n}$, (3.3) reduces to

$$\frac{P_{2n_1}}{P_{n_1}}\frac{P_{2n_2}}{P_{n_2}}\cdots\frac{P_{2n_k}}{P_{n_k}}=\frac{P_{3m}}{P_m}. \tag{3.4}$$

Suppose that $m\geq 14$. Then P_{3m} has a primitive divisor. Thus, $2n_k=3m$, i.e. $n_k=\frac{3m}{2}>m$. If $k=1$ (3.4) reduces to $P_m=P_{n_1}$, and we get a contradiction by $m=n_1$. Supposing $k=2$, (3.4) simplies to

$$\frac{P_{2n_1}}{P_{n_1}}P_m=P_{n_2}.$$

Since $n_2>m$, P_{n_2} contains a primitive divisor. Thus $n_2=2n_1$, and $m=n_1$ follows. This contradicts to $n_2=\frac{3m}{2}$. If $k\geq 3$ then observe that $n_{k-1}<m$ holds, otherwise we could cause a contradiction by $Q_{n_{k-1}}Q_{n_k}>Q_m^2$. Thus the equation

$$\frac{P_{2n_1}}{P_{n_1}}\cdots\frac{P_{2n_{k-2}}}{P_{n_{k-2}}}P_m=P_{n_{k-1}} \tag{3.5}$$

has no solution since $m\geq 14$, therefore P_m has a primitive divisor on the left hand side of (3.5), which can not exist on the right hand side. □

Theorem 3.3. *The diophantine equation*

$$P_{n_1}P_{n_2}\cdots P_{n_k}+1=Q_m^2 \tag{3.6}$$

in positive integer k,*even integer* m *and in non-negative integers* $n_1<n_2<\cdots<n_k$ $(n_i\neq 1)$ *has no solution.*

Proof. Suppose that $m>3$. We can write (3.6) as:

$$P_{n_1}P_{n_2}\cdots P_{n_k}P_m=P_{3m}. \tag{3.7}$$

Then P_{3m} has a primitive divisor. This implies that $n_k=3m$. Then (3.7) reduces to

$$P_{n_1}P_{n_2}\cdots P_{n_k-1}P_m=1.$$

Thus, $1=P_{n_1}P_{n_2}\cdots P_{n_k-1}P_m>P_m>P_3=5$ which is a contradiction. Therefore $m<3$,it means $m=0$ or 2. In this situation one can check that there is no solution. □

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