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Some Generalized Suborbital Graphs

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Abstract. In this paper, we consider suborbital graphs formed by the group action of the normalizer of $\Gamma_0(N)$ in $PSL_2(\mathbb{R})$ which is a finitely generated Fuchsian group. Firstly, conditions for being an edge are provided, then we give necessary and sufficient conditions for the suborbital graphs to contain a circuit. This paper is an extension of some results in [4, 6].

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1. Introduction

The modular group $\Gamma = PSL_2(\mathbb{Z})$ is the group of all linear fractional transformations

$$T : z \rightarrow \frac{az+b}{cz+d}\text{, where } a, b, c \text{ and } d \text{ are integer and } ad - bc = 1.$$

In terms of matrix representation, the elements of Γ correspond to the matrices

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1.$$

The modular group is important because it forms a subgroup of the group of isometries of the hyperbolic plane. If we consider the upper half-plane model $\mathbb{H} := \{z \in \mathbb{C} : \text{Im(z)} > 0\}$ of hyperbolic plane geometry, then the group of all orientation-preserving isometries of $\mathbb{H}$ is Γ [9].

Important subgroups of the modular group Γ, called congruence subgroups, are given by imposing congruence relations on the associated matrices. One of them is

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} modN\right\}.$$

Conway and Norton gave a complete description of the elements of the normalizer Nor(N) of $\Gamma_0(N)$ in $PSL(2,\mathbb{R})$. It consists exactly of matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix},$$

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where $e \parallel \frac{N}{h^2}$ and h is the largest divisor of 24 for which $h^2|N$ with understandings that the determinant e of the matrix is positive, and that $r \parallel s$ means that $r|s$ and $(r, s/r) = 1$ (r is called an exact divisor of s) [1].

This normalizer has acquired significance because it is related to the Monster simple group. It has also played an important role in work on Weierstrass points on Riemann surfaces associated to $\Gamma_0(N)$, on modular forms and on ternary quadratic forms [1].

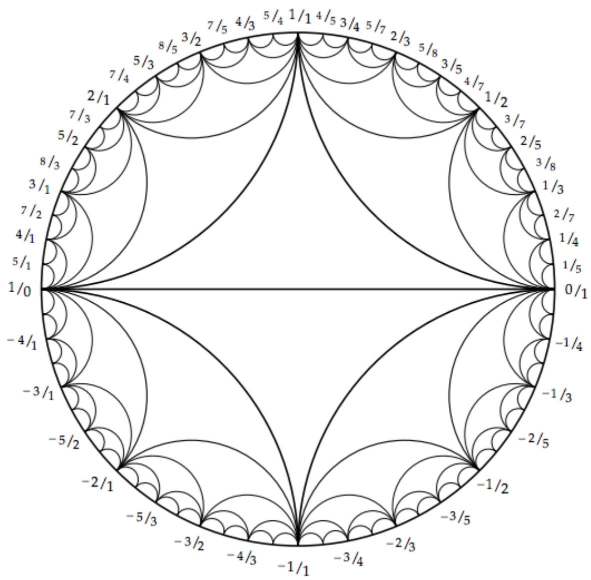

Figure 1. Farey graph

It is well-known that the graph of a group provides a method by which a group can be visualized. Such a construction is illustrated in Fig. 1, which shows the Farey graph arising from the action of Γ on $\hat{\mathbb{Q}}$. From this point of view, one can regard these combinatorial structures as pictures of the group. In this light, suborbital graphs of the normalizer have been studied under various restrictions.

- N is a square-free positive integer [7],
- N satisfy the condition of transitive action [8],
- Some non-transitive cases [3, 4, 6].

Clearly, a general statement is an open problem but seems to be not easy. We think this study can be seen as a step in the way towards the arbitrary N.

2. Group Action

Every element of the extended set of rationals $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ can be represented as a reduced fraction $\dfrac{x}{y}$, with $x, y \in \mathbb{Z}$ and $(x, y) = 1$. ∞ is represented as $\frac{1}{0} = \frac{-1}{0}$. The action of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ on $\dfrac{x}{y}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \to \frac{ax + by}{cx + dy}.$$

The following results can be found with more detailed in [3].

Lemma 2.1. *Let $\frac{k}{s}$ be an arbitrary rational number with $(k, s) = 1$. Then there exists some element $A \in \Gamma_0(N)$ such that $A(\frac{k}{s}) = \frac{k_1}{s_1}$ with $s_1|N$.*

Lemma 2.2. *Let $d|N$ and $(a, d) = (b, d) = 1$. Then $\begin{pmatrix} a \\ d \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ are conjugate under $\Gamma_0(N)$ if and only if $b \equiv a$* (mod $(d, \frac{d}{N})$).

Lemma 2.3. *Let $d|N$. Then the orbit* $\begin{pmatrix} a \\ d \end{pmatrix}$ *of* $\frac{a}{d}$ *under* $\Gamma_0(N)$ *is the set* $\frac{x}{y} \in \hat{\mathbb{Q}} : (N,y) = d, a \equiv x \cdot \frac{y}{d} \pmod{(d, \frac{d}{N})}$. *Furthermore the number of orbits* $\begin{pmatrix} a \\ d \end{pmatrix}$ *with $d|N$ under is just $\varphi(d, \frac{d}{N})$ where φ is Euler's function.*

By divisors of qp^2, direct computation shows the following result,

Theorem 2.4. *The orbits of the action of* $\Gamma_0(N)$ *on* $\hat{\mathbb{Q}}$ *are* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ q \end{pmatrix}, \begin{pmatrix} 1 \\ p^2 \end{pmatrix}, \begin{pmatrix} 1 \\ qp^2 \end{pmatrix}, \begin{pmatrix} 2 \\ p \end{pmatrix}, \dots, \begin{pmatrix} p-1 \\ p \end{pmatrix}, \begin{pmatrix} 1 \\ qp \end{pmatrix}, \begin{pmatrix} t_2 \\ qp \end{pmatrix}, \dots, \begin{pmatrix} t_{p-1} \\ qp \end{pmatrix}$ *where any two of* $1, t_2, \dots, t_{p-1}$ *are not congruent modp, and that* t_i, $i = 2, 3, \dots, p-1$, *is* i *or* $p + i$ *according as* $q \nmid i$ *or* $q|i$.

If one can just examine the actions of the elements of $\mathrm{Nor}(qp^2)$ on the orbit $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the following result is easily obtained:

Theorem 2.5. *The orbits of the action of* $\mathrm{Nor}(qp^2)$ *on* $\hat{\mathbb{Q}}$ *are as follows. Let* $l \in \{1, 2, \dots, p-1\}$. *Then*

(1) (a) *If* $q \nmid l$ *and* $l \not\equiv (\mod q)$, *then* $\begin{pmatrix} l \\ p \end{pmatrix} \cup \begin{pmatrix} p-l \\ p \end{pmatrix} \cup \begin{pmatrix} l \\ qp \end{pmatrix} \cup \begin{pmatrix} p-l \\ qp \end{pmatrix}$,

(b) *If* $q \nmid l$ *and* $l \equiv (\mod q)$, *then* $\begin{pmatrix} l \\ p \end{pmatrix} \cup \begin{pmatrix} p-l \\ p \end{pmatrix} \cup \begin{pmatrix} l \\ qp \end{pmatrix} \cup \begin{pmatrix} 2p-l \\ qp \end{pmatrix}$,

(2) *If* $q|l$, *then* $\begin{pmatrix} l \\ p \end{pmatrix} \cup \begin{pmatrix} p-l \\ p \end{pmatrix} \cup \begin{pmatrix} p+l \\ qp \end{pmatrix} \cup \begin{pmatrix} p-l \\ qp \end{pmatrix}$,

(3) $\begin{pmatrix} l \\ 1 \end{pmatrix} \cup \begin{pmatrix} l \\ p \end{pmatrix} \cup \begin{pmatrix} l \\ p^2 \end{pmatrix} \cup \begin{pmatrix} l \\ qp^2 \end{pmatrix}$.

Corollary 2.6. *The set* $\hat{\mathbb{Q}}(qp^2) = \begin{pmatrix} l \\ 1 \end{pmatrix} \cup \begin{pmatrix} l \\ q \end{pmatrix} \cup \begin{pmatrix} l \\ p^2 \end{pmatrix} \cup \begin{pmatrix} l \\ qp^2 \end{pmatrix}$ *is a maximal subset of* $\hat{\mathbb{Q}}$ *on which* $\mathrm{Nor}(qp^2)$ *acts transitively.*

A general discussion of primitivity of permutation groups was given in [5]. In this paper G is the normalizer $\mathrm{Nor}(qp^2)$, Ω is $\hat{\mathbb{Q}}$, and H is N_0 which is generated by $\Gamma_0(qp^2)$ and some element T of the form

$$T = \begin{pmatrix} qa & b \\ qp^2c & qd \end{pmatrix}.$$

That is $N_0 =< \Gamma_0(qp^2), T >$. In this case it is easily seen that we have just two blocks as
$[\infty] := \begin{pmatrix} 1 \\ p^2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ qp^2 \end{pmatrix}$ and $[0] := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cup \begin{pmatrix} 1 \\ q \end{pmatrix}$.
Therefore $\left|\mathrm{Nor}(qp^2) : N_0\right| = 2$.

3. Main Results

We now investigate the suborbital graphs for the action $\mathrm{Nor}(qp^2)$ on $\hat{\mathbb{Q}}(qp^2)$. Since the action $\mathrm{Nor}(qp^2)$ on $\hat{\mathbb{Q}}(qp^2)$ is transitive, $\mathrm{Nor}(qp^2)$ permutes the blocks transitively; so the subgraphs are all isomorphic. Hence it is sufficient to study with only one block [10]. On the other hand, it is clear that each non-trivial suborbital graph which are shown by G_{u,p^2} contains a pair $(\infty, u/p^2)$ for some $u/p^2 \in \hat{\mathbb{Q}}(qp^2)$ where $(u, p^2) = 1$. Therefore, we work on the following case: We denote by F_{u,p^2} the subgraph of G_{u,p^2} such that its vertices are in the block $[\infty]$.

Theorem 3.1. *Let r/s and x/y be in the block $[\infty]$. Then there is an edge $r/s \to x/y$ in F_{u,p^2} iff*

(i) *If* $p^2 \mid s$ *but* $qp^2 \nmid s$ *then* $x \equiv \pm qur \pmod{p^2}, y \equiv \pm qus \pmod{qp^2}, ry - sx = \pm p^2$,
(ii) *If* $qp^2 \mid s$, *then* $x \equiv \pm ur \pmod{p^2}, y \equiv \pm us \pmod{p^2}, ry - sx = \pm p^2$,

(Plus and minus sign correspond to $r/s > x/y$ and $r/s < x/y$, respectively)

Proof. Assume first that $r/s \stackrel{>}{\longrightarrow} x/y$ is an edge in F_{u,p^2}, and $p^2 \mid s$ but $qp^2 \nmid s$. It means that there exists some T in the normalizer $Nor(qp^2)$ such that T sends the pair $(\infty, u/p^2)$ to the pair $(r/s, x/y)$, that is $T(\infty) = r/s$ and $T(u/p^2) = x/y$. Since $qp^2 \nmid s$, T must be of the form $\begin{pmatrix} qa & b \\ qp^2c & qd \end{pmatrix}$. $T(\infty) = \dfrac{qa}{qp^2c} = \dfrac{r}{s}$ gives that $r = a$ and $s = p^2c$. $T(u/p^2) = \dfrac{qau + bp^2}{qp^2cu + qdp^2} = \dfrac{r}{s}$ gives that $x \equiv ur(modp^2), y \equiv us(modqp^2)$. Furthermore, we get $ry - sx = p^2$ from the equation $\begin{pmatrix} qa & b \\ qp^2c & qd \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} qr & x \\ qs & y \end{pmatrix}$. Similar calculations are done by the element $\begin{pmatrix} a & b \\ qp^2c & d \end{pmatrix}$ for (ii).

For the opposite direction, we assume that $p^2 \mid s$, $qp^2 \nmid s$ and that $x \equiv qur(modp^2), y \equiv qus(modp^2), ry - sx = p^2$. In this case, there exist $b, d \in \mathbb{Z}$ such that $x = qur + bp^2$ and $y = qus + dp^2$. If we put these equivalences in $ry - sx = p^2$, we obtain $qrd - bs = 1$. So the element $T_0 = \begin{pmatrix} qr & b \\ qs & d \end{pmatrix}$ is clearly in N_0. For minus sign and another conditions, similar calculations are done. □

Definition 3.2. [2] Let $v_1, v_2, \ldots, v_n$ be in $\hat{\mathbb{Q}}(qp^2)$. Then a configuration $v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_{n-1} \longrightarrow v_n \longrightarrow v_1$ is called a circuit of lenght n. If n is a 4, then the circuit is called a rectangle, if n is a 3 (resp. 2), then the circuit is called a triangle (resp. a self paired edge).

If we represent the edges of F_{u,p^2} as hyperbolic geodesics in the upper half plane $\mathbb{H}$, then we have:

Proposition 3.3. *The subgraph F_{u,p^2} of* Nor(qp^2) *does not cross in* $\mathbb{H}$.

Proof. Without loss generality, since the action on $\hat{\mathbb{Q}}(qp^2)$ is transitive, suppose that $\infty \longrightarrow \frac{u}{p^2}$, $\frac{x_1}{y_1p^2} \longrightarrow \frac{x_2}{y_2p^2}$ and $\frac{x_1}{y_1p^2} < \frac{u}{p^2} < \frac{x_2}{y_2p^2}$, where all letters are positive integers. Since $\frac{x_1}{y_1p^2} \longrightarrow \frac{x_2}{y_2p^2}$ and $\frac{x_1}{y_1p^2} < \frac{u}{p^2} < \frac{x_2}{y_2p^2}$ then $x_1y_2 - x_2y_1 = -1$ and $\frac{x_1}{y_1} < u < \frac{x_2}{y_2}$, respectively. Therefore $\frac{x_1}{y_1} - \frac{x_2}{y_2} < u - \frac{x_2}{y_2} < 0$. Then $\frac{x_1y_2 - x_2y_1}{y_1y_2} < \frac{uy_2 - x_2}{y_2} < 0$. So $\frac{-1}{y_1} < uy_2 - x_2 < 0$, a contradiction. □

Now we give one of our main theorems:

Theorem 3.4. *The subgraph F_{u,p^2} of* Nor(qp^2) *contains a rectangle if and only if $q = 2$ and $2u^2 \pm 2u + 1 \equiv 0 \pmod{p^2}$.*

Proof. Assume first that F_{u,p^2} has a rectangle $\frac{k_0}{l_0} \longrightarrow \frac{m_0}{n_0} \longrightarrow \frac{s_0}{t_0} \longrightarrow \frac{x_0}{y_0} \longrightarrow \frac{k_0}{l_0}$. Since N_0 permutes the vertices and edges of F_{u,p^2} transitively, the rectangle is transformed under N_0 to a rectangle $\frac{1}{0} \longrightarrow \frac{m}{p^2} \longrightarrow \frac{x}{y} \longrightarrow \frac{k}{l} \longrightarrow \frac{1}{0}$. Furthermore, without loss of generality, suppose $\frac{m}{p^2} < \frac{x}{y} < \frac{k}{l}$. Here we repeatedly use Theorem 3.1 we get $l = p^2$ from the edge $\frac{k}{l} \longrightarrow \frac{1}{0}$. From the edge $\frac{1}{0} \longrightarrow \frac{m}{p^2}$ we have $m \equiv umodp^2$. The edge $\frac{m}{p^2} \longrightarrow \frac{x}{y}$ gives that $x \equiv -qmu(\mod p^2)$, $y \equiv -qum \pmod{qp^2}$, and $my - p^2x = -p^2$. So we have that there exists some $y_1 \in \mathbb{Z}$ such that $y = qp^2y_1$, and that $x \equiv -qu^2 \pmod{p^2}$. Furthermore,

$$mqy_1 - x = -1. \tag{3.1}$$

From the edge $\frac{k}{p^2} \longrightarrow \frac{1}{0}$ we get,

$$quk \equiv -1 \pmod{p^2}. \tag{3.2}$$

Using (3.1) and the edge $\frac{x}{qp^2y_1} \longrightarrow \frac{k}{p^2}$ we conclude that

$$k \equiv -u(mqy_1 + 1) \pmod{p^2} \text{ and } (mqy_1 + 1) - qy_1k = -1. \tag{3.3}$$

Then

$$qy_1(m - k) = -2.$$

Consequently, $q = 1$ or $q = 2$. If $q = 1$ then $y_1 = 1$ or 2, $m - k = -1, -2$. If $y_1 = 1$ then $k = m + 2$. Using (3.3) we have $m + 2 \equiv -u(m + 1) \pmod{p^2}$. So we get $u^2 + 2u + 2 \equiv 0 \pmod{p^2}$. By (3.3) $u(m + 2) + 1 \equiv 0 \pmod{p^2}$. So $1 \equiv 0 \pmod{p^2}$, a contradiction. Now let $y_1 = 2$. Then $m - k = -1$, that is $k = m + 1$. From (3.2) we have $u^2 + 2u + 2 \equiv 0 \pmod{p^2}$. From (3.1) we have $x = 1 + 2m$. Since $x \equiv -qu^2 \pmod{p^2}$ then we get $2m + 1 \equiv -u^2 \pmod{p^2}$ or $u^2 + 2u + 2 \equiv 0 \pmod{p^2}$. Hence, with $u^2 + 2u + 2 \equiv 0 \pmod{p^2}$ we obtain that $u \equiv 0 \pmod{p^2}$. It gives again a contradiction. Consequently, q must be 2 and therefore $y_1 = 1$. $x \equiv -qu^2 \pmod{p^2}$ and $mqy_1 - x = -1$ gives that $2u^2+2u+1 \equiv 0(\mod p^2)$. If the inequalities $\frac{m}{p^2} > \frac{x}{y} > \frac{k}{l}$ hold then we conclude that $2u^2-2u+1 \equiv 0 \pmod{p^2}$. Conversely, suppose that $q = 2$ and $2u^2 \pm 2u + 1 \equiv 0 \pmod{p^2}$. Then, using Theorem 3.1, we see that

$$\frac{1}{0} \longrightarrow \frac{u}{p^2} \longrightarrow \frac{2u \pm 1}{2p^2} \longrightarrow \frac{u \pm 1}{p^2} \longrightarrow \frac{1}{0}$$

is a rectangle in F_{u,p^2}. □

Now we give a theorem saying that under which conditions F_{u,p^2} contains hexagons.

Theorem 3.5. *The subgraph F_{u,p^2} of* Nor(qp^2) *contains a hexagon H if and only if $q = 3$ and $3u^2 \pm 3u + 1 \equiv 0$* $\pmod{p^2}$.

Proof. Because of the transitive action, the hexagon H can be taken, as in the proof of Theorem 3.1, as:

$$\frac{1}{0} \longrightarrow \frac{u}{p^2} \longrightarrow \frac{x_1}{qy_1p^2} \longrightarrow \frac{x_2}{y_3p^2} \longrightarrow \frac{x_3}{qy_3p^2} \longrightarrow \frac{x_4}{p^2} \longrightarrow \frac{1}{0}. \tag{3.4}$$

Here all letters are positive integers and the order is in increasing order. From the second edge we get that $x_1 \equiv -qu^2($ mod $p^2)$ and that $x_1 = quy_1 + 1$. Therefore we have the congruence

$$qu^2 + quy_1 + 1 \equiv 0 \pmod{p^2}.$$

From the sixth edge it is evident that $x_4 \equiv u + y_1 \pmod{p^2}$. Since the pattern is periodic with period 1, that means that if we add 1 to each the vertex we simply get an hexagon as well, we must have $0 < y_1 \le p^2$. Because otherwise from Proposition 3.3 one of y_1, y_2, y_3 must be zero, which is impossible. From (3.4) we get that the element

$$T = \begin{pmatrix} -qu & \frac{qu^2+quy_1+1}{p^2} \\ qp^2 & qu + qy_1 \end{pmatrix}$$

is in N_0. Then it is immediate that $\frac{x_1}{qy_1p^2} = \frac{quy_1+1}{qy_1p^2} = T^2(\frac{1}{0})$. Next we show that $\frac{x_2}{y_2p^2} = T^3(\frac{1}{0})$. For this we see that $\frac{x_2}{y_2p^2} \le T^3(\frac{1}{0})$. Otherwise, suppose that $T^3(\frac{1}{0}) < \frac{x_2}{y_2p^2}$. That is, $\frac{u(qy_1^2-1)+y_1}{p^2(qy_1^2-1)} < \frac{x_2}{y_2p^2}$. So we have,

$$uy_2(qy_1^2-1) + y_1y_2 < x_2(qy_1^2-1). \tag{3.5}$$

Since $\frac{quy_1+1}{qy_1p^2} \longrightarrow \frac{x_2}{y_2p^2}$ and $\frac{u}{p^2} < \frac{x_2}{y_2p^2}$, using theorem 11, we have $quy_1^2y_2+y_1y_2 = qx_2y_1^2-y_1$ and $-uy_2 > -x_2$ respectively. From (3.5) we get that $quy_1^2y_2-uy_2+y_1y_2 < qx_2y_1^2-uy_2$. Then $-y_1+y_1y_2 < 0$, is a contradiction. Therefore $\frac{x_2}{y_2p^2} \le T^3(\frac{1}{0})$. The hexagon H must have $T^3(\frac{1}{0})$ has a vertex, otherwise we prove a contradiction by Proposition 3.3. Therefore if the equality does not hold then $T^3(\frac{1}{0})$ must equal $\frac{x_4}{p^2}(=\frac{u+y_1}{p^2})$ which is impossible. Consequently $\frac{x_2}{y_2p^2} = T^3(\frac{1}{0})$. Let us now finally obtain the equality $\frac{x_3}{qy_2p^2} = T^4(\frac{1}{0}) = \frac{quy_1(qy_1^2-2)+qy_1^2-1}{qp^2y_1(qy_1^2-2)}$. As in the above suppose that $\frac{x_3}{qy_3p^2} > T^4(\frac{1}{0})$. Then,

$$quy_3y_1(qy_1^2-2) + qy_1^2y_3 - y_3 < x_3y_1(qy_1^2-2).$$

Since $\frac{u(qy_1^2-1)+y_1}{p^2(qy_1^2-1)} \longrightarrow \frac{x_3}{qy_3p^2}$ then, by Theorem 3.1, we have,

$$y_1x_3(qy_1^2-1) = y_1 + qy_1^2y_3 + quy_1y_3(qy_1^2-1). \tag{3.6}$$

Inserting this into (3.6) we get the inequality

$$y_1(x_3 - quy_3) < y_1 + y_3.$$

Since $\frac{x_3}{qy_3p^2} \longrightarrow \frac{u+y_1}{p^2}$, we have

$$x_3 = quy_3 + qy_1y_3 - 1.$$

Then we obtain the inequality

$$qy_1^2y_3 < 2y_1 + y_3. \tag{3.7}$$

On the other hand, by Theorem 3.1, the hexagon gives the following five congruences

$$x_1 \equiv -qu^2 \pmod{p^2},$$
$$x_2 \equiv -ux_1 \pmod{p^2},$$
$$x_3 \equiv -qux_2 \pmod{p^2},$$

$$x_4 \equiv -ux_3 \pmod{p^2},$$
$$1 \equiv -qux_4 \pmod{p^2}.$$

From these congruences it is immediate that $q^3u^6 + 1 \equiv 0 \pmod{p^2}$. From this congruence and the congruence (3.2) we get that

$$y_1(qy_1^2 - 3) \equiv 0 \pmod{p^2}. \tag{3.8}$$

Now, let us turn back to (3.7). This is satisfied only if $qy_1^2 = 2, 3 or 4$. But the congruence (3.8) proves a contradiction. Therefore $\frac{x_3}{qy_3p^2} \le T^4(\frac{1}{0})$. As in above, $T^4(\frac{1}{0})$ must be a vertex in hexagon H. Hence, $T^4(\frac{1}{0}) = \frac{x_3}{qy_3p^2}$ or $\frac{x_4}{p^2}$ or $\frac{1}{0}$. If $T^4(\frac{1}{0}) = \frac{quy_1(qy_1^2-2)+qy_1^2-1}{qp^2y_1(qy_1^2-2)} = \frac{x_4}{p^2} = \frac{u+y_1}{p^2}$, then $qy_1(qy_1^2 - 2)$ is equal to 1, which is impossible. If $T^4(\frac{1}{0}) = \frac{1}{0}$ then $qy_1^2 - 2 = 0$. But this contradicts to (3.8). Therefore $T^4(\frac{1}{0}) = \frac{x_3}{qy_3p^2}$. So we have

$$\frac{quy_1(qy_1^2 - 2) + qy_1^2 - 1}{qp^2y_1(qy_1^2 - 2)} \longrightarrow \frac{x_4}{p^2} = \frac{u + y_1}{p^2}.$$

Then, $quy_1(qy_1^2 - 2) + qy_1^2 - 1 - (u + y_1)qy_1(qy_1^2 - 2) = -1$. From this we get that $qy_1^2(3 - qy_1^2) = 0$. This gives $q = 3$ and $y_1 = 1$. So, in the end, we obtain that $q = 3$ and $3u^2 + 3u + 1 \equiv 0 \pmod{p^2}$. If the decreasing order is taken then we arrive at the result $q = 3$ and $3u^2 - 3u + 1 \equiv 0 \pmod{p^2}$. To prove the converse, let $3u^2 \pm 3u + 1 \equiv 0 \pmod{p^2}$. Then, by Theorem 3.1, we have a hexagon as

$$\frac{1}{0} \longrightarrow \frac{u}{p^2} \longrightarrow \frac{3u \pm 1}{3p^2} \longrightarrow \frac{2u \pm 1}{2p^2} \longrightarrow \frac{3u \pm 2}{3p^2} \longrightarrow \frac{u \pm 1}{p^2} \longrightarrow \frac{1}{0}.$$

Therefore the proof is completed.

□

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