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Two-dimensional Chebyshev Polynomials for Solving Two-dimensional Integro-Differential Equations

Azim Rivaz[1], Samane Jahan ara[1], Farzaneh Yousefi[2]

[1 2] *Department of Applied Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran,*
[2] *Young Researchers Society, Shahid Bahonar University of Kerman, Kerman, Iran,*
e-mail: arivaz@uk.ac.ir, samanejahanara@yahoo.com, fyousefi@math.uk.ac.ir

Abstract: In this paper, we present a new approach to obtain the numerical solution of the linear two-dimensional Fredholm and Volterra integro-differential equations (2D-FIDE and 2D-VIDE). First, we introduce the two-dimensional Chebyshev polynomials and construct their operational matrices of integration. Then, both of them, two-dimensional Chebyshev polynomials and their operational matrix of integration, are used to represent the matrix form of 2D-FIDE and 2D-VIDE. The main characteristic of this approach is that it reduces 2D-FIDE and 2D-VIDE to a system of linear algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the presented technique.

Keywords: Two-dimensional Fredholm and Volterra integro-differential equations, Two-dimensional Chebyshev polynomials, Operational matrix of integration.

1. Introduction

Integral equations have been one of the principal tools in various areas of applied mathematics, physics and engineering. In this paper, we are concerned with two-dimensional integro-differential equations. Scientists have investigated the topic of integro-differential equations through their works in many scientific applications, including heat transfer, diffusion processes, neutron diffusion and biological species coexisting with increasing and decreasing rates of generation. On the other hand, two-dimensional integral equations provide an important tool for modeling numerous problems in engineering and science. These equations appear in electromagnetism, electrodynamics, molecular physics, population in addition to many other fields.

One of the main problems is how to solve integro-differential equations in one and two-dimensional space. There are several classical solution techniques to solve some of these equations; it is difficult to obtain the analytical solutions of most of these equations. Therefore, it is important to develop numerical algorithms which have sufficient accuracy. In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integro-differential equations, including the Wavelet-Galerkin method, Lagrange interpolation method, Tau method and semi-analytical numerical techniques such as Adomians decomposition method and Taylor polynomials [2, 5, 7, 11, 13].

An usual way to solve functional equations is to express the solution as a linear combination of the so-called basis functions. These basis functions can, for instance, be either orthogonal or non-orthogonal bases. Approximation by the orthogonal family of basis functions has found wide application in science and engineering. The most frequently used orthogonal functions are sine-cosine functions, block pulse functions, Legendre, Chebyshev and Laguerre polynomials. The main idea of using an orthogonal basis is that the problem under consideration reduces to a system of linear or nonlinear algebraic equations [8, 10, 11]. The main purpose of this paper is to apply the 2D orthogonal Chebyshev polynomials to solve Fredholm and Volterra integro-differential equations.

The remainder of this paper is organized as follows: in Section 2, we begin by introducing some necessary definitions. In Section 3, the two-dimensional Chebyshev polynomials and their properties are defined and their integral operational matrices are obtained. Section 4 is devoted to applying the two-dimensional Chebyshev operational matrix of integration to solve two-dimensional linear Fredholm and Volterra integro-differential equations. In Section 5, the proposed method is applied to several examples followed by conclusion in the final section.

2. Preliminaries

In this section, we give definitions and properties of Chebyshev polynomials in one-dimensional space. The well known Chebyshev polynomials of the first kind of degree n are defined by [4]:

$$T_n(x) = cos(ncos^{-1}x), \quad n \geq 0. \tag{1}$$

Also they are derived from the following recursive formula:

$$\begin{aligned} &T_0(x) = 0, \\ &T_1(x) = x, \\ &T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad n = 1, 2, 3, \cdots \end{aligned}$$

These polynomials are orthogonal on $[-1,1]$ with respect to the weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$:

$$\int_{-1}^{1} T_i(x)T_j(x)w(x)dx = \begin{cases} 0, & i \neq j, \\ \frac{\pi}{\gamma_i}, & i = j, \end{cases} \tag{2}$$

where

$$\gamma_i = \begin{cases} 1, & i = 0, \\ 2, & i \geq 1, \end{cases}$$

Chebyshev polynomials are important in approximation theory and numerical analysis [4, 6]. A function $f(x)$ over $[-1,1]$ may be represented by Chebyshev polynomials series as:

$$f(x) = \sum_{i=0}^{\infty} a_i T_i(x). \tag{3}$$

If the infinite series in (3) is truncated, then (3) can be written as:

$$f(x) \simeq \sum_{i=0}^{N} a_i T_i(x) = T(x)^t A, \tag{4}$$

where

$$\begin{aligned} T(x) &= [T_0(x), T_1(x), \cdots, T_N(x)]^t, \\ A &= [a_0, a_1, \cdots, a_N]^t, \end{aligned} \tag{5}$$

and $a_i = \dfrac{\gamma_i}{\pi} \int_{-1}^{1} f(x) T_i(x) w(x) dx$.
Chebyshev polynomials have the following useful property [6]:

$$\int_{-1}^{x} T_{N-1}(s) ds = \frac{1}{2N} T_N(x) - \frac{1}{2(N-2)} T_{N-2}(x) + \frac{(-1)^{N-1}}{1-(N-1)^2} T_0(x), \quad N \geq 3. \tag{6}$$

Moreover, for $T_0(x)$ and $T_1(x)$, we have:

$$\begin{aligned} &\textstyle\int_{-1}^{x} T_0(s) ds = T_0(x) + T_1(x), \\ &\textstyle\int_{-1}^{x} T_1(s) ds = \frac{-1}{4} T_0(x) + \frac{1}{4} T_2(x). \end{aligned} \tag{7}$$

Equations (6) and (7) allow us to write:

$$\int_{-1}^{x} T(s) ds = PT(x), \tag{8}$$

where P is the $(N+1) \times (N+1)$ operational matrix:

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \cdots & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(-1)^{N-1}}{1-(N-1)^2} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2N} \\ \frac{(-1)^N}{1-N^2} & 0 & 0 & 0 & \cdots & -\frac{1}{2N-2} & 0 \end{bmatrix}, \quad N \geq 3. \tag{9}$$

3. Two-Dimensional Chebyshev Polynomials

In this section, by considering 1D Chebyshev polynomials, we define an $(N+1)^2$ set of two-dimensional Chebyshev polynomials as:

$$T_{ij}(x,t) = T_i(x)T_j(t),\ i,j = 0,\cdots,N. \tag{10}$$

Therefore, the two-dimensional Chebyshev basis vector is as follows:

$$\begin{aligned} T(x,t) &= [T_0(x)T_0(t),\cdots,T_0(x)T_N(t),\ T_1(x)T_0(t),\cdots,T_1(x)T_N(t),\cdots,T_N(x)T_N(t)]^t \\ &= (C_N \otimes B_N)^t, \end{aligned} \tag{11}$$

in which $C_N = [T_0(x),T_1(x),\cdots,T_N(x)]$, and $B_N = [T_0(t),T_1(t),\cdots,T_N(t)]$ are one dimensional Chebyshev vectors.
The orthogonality property for these polynomials with respect to the weight function $w(x,t) = \dfrac{1}{\sqrt{1-x^2}\sqrt{1-t^2}}$ on the interval $[-1,1]\times[-1,1]$ is:

$$(T_{i,j}(x,t),T_{k,l}(x,t))_{w(x,t)} = \int_{-1}^{1}\int_{-1}^{1} T_{i,j}(x,t)T_{k,l}(x,t)w(x,t)dxdt = \begin{cases} \frac{\pi^2}{4},\ i=k\neq 0, j=l\neq 0 \\ \frac{\pi^2}{2},\ i=k=0, j=l\neq 0 \\ \frac{\pi^2}{2},\ i=k\neq 0, j=l=0 \\ \pi^2,\ i=k=0, j=l=0 \\ 0,\ else \end{cases}$$

Similarly to the one-dimensional case, a function $f(x,t)$ on $[-1,1]\times[-1,1]$ can be expanded by two-dimensional Chebyshev polynomials as the following equation:

$$f(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}T_{i,j}(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}T_i(x)T_j(t), \tag{12}$$

where

$$a_{i,j} = \frac{(f(x,t),T_{ij}(x,t))_{w(x,t)}}{(T_{i,j}(x,t),T_{ij}(x,t))_{w(x,t)}},\ i,j = 0,1,...,N. \tag{13}$$

In practice, only the finite terms of the above series are considered, so we have:

$$f(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}T_i(x)T_j(t), \tag{14}$$

For the two-dimensional Chebyshev polynomials $T_0(x)T_0(t)$, $T_0(x)T_1(t)$, $T_1(x)T_0(t)$ and $T_1(x)T_1(t)$, we have the following property:

$$\int_{-1}^{x}\int_{-1}^{t} T_0(s)T_0(r)dsdr = T_1(x)T_1(t) + T_0(x)T_1(t) + T_1(x)T_0(t) + T_0(x)T_0(t), \tag{15}$$

$$\int_{-1}^{x}\int_{-1}^{t} T_0(s)T_1(r)dsdr = \frac{-1}{4}T_0(x)T_0(t) + \frac{1}{4}T_0(x)T_2(t) + \frac{-1}{4}T_1(x)T_0(t) + \frac{1}{4}T_1(x)T_2(t), \tag{16}$$

$$\int_{-1}^{x}\int_{-1}^{t} T_1(s)T_1(r)dsdr = \frac{1}{16}T_0(t)T_0(x) + \frac{-1}{16}T_0(x)T_2(t) + \frac{-1}{16}T_2(x)T_0(t) + \frac{1}{16}T_2(x)T_2(t). \quad (17)$$

From the above equations and by considering (8), we can write:

$$\int_{-1}^{x}\int_{-1}^{t} T(s,r)dsdr = QT(x,t), \quad (18)$$

where $Q = P\otimes P$ is the operational matrix of integration of 2D Chebyshev polynomials, and P is the matrix that represented in (9).

4. Two-dimensional Fredholm and Volterra Integro-Differential Equations

In this section, by using two-dimensional Chebyshev polynomials, we solve a special kind of two-dimensional Fredholm and Volterra integro-differential equations.

4.1. Fredholm Integro-Differential Equations (FIDE):

First, we consider the two-dimensional Fredholm integro-differential equation in the form:

$$\varphi_{xt}(x,t) + \varphi(x,t) + \int_{-1}^{1}\int_{-1}^{1} k(x,t,y,z)\varphi(y,z)dydz = f(x,t), \ \ x,t \in [-1,1], \quad (19)$$

with the initial conditions:

$$\begin{aligned} &\varphi(-1,-1) = \varphi_0, \\ &\varphi(-1,t) = g(t), \\ &\varphi(x,-1) = h(x), \end{aligned} \quad (20)$$

where $f(x,t)$ and $k(x,t,y,z)$ are known functions on $[-1,1]$ and $[-1,1]\times[-1,1]$ respectively; $\varphi(x,t)$ is an unknown function and $\varphi_{xt}(x,t)$ is the derivative of $\varphi(x,t)$ with respect to t and x. $h(x)$ and $g(t)$ are given functions and φ_0 is a given number.

The process of our method is obtaining the solution of the equation as a truncated Chebyshev series defined by:

$$\varphi(x,t) \simeq \varphi_N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}T_i(x)T_j(t), \quad (21)$$

and

$$\varphi_{xt}(x,t) \simeq \psi_N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} e_{ij}T_i(x)T_j(t), \quad (22)$$

so in the matrix form, we have:

$$\begin{aligned} \varphi_N(x,t) &= T^t(x,t)A, \\ \psi_N(x,t) &= T^t(x,t)E, \end{aligned} \quad (23)$$

such that

$$A = [a_{00}, \cdots, a_{0N}, \cdots, a_{N0}, \cdots, a_{NN}]^t, \quad (24)$$

and

$$E = [e_{00}, \cdots, e_{0N}, \cdots, e_{N0}, \cdots, e_{NN}]^t. \tag{25}$$

are two unknown vectors, and $T(x,t)$ is the 2D Chebyshev vector defined in (11).
Moreover, we can expand the functions $f(x,t)$, $g(x)$, and $h(t)$ in terms of a 2D Chebyshev basis:

$$f(x,t) \simeq T^t(x,t)F, \tag{26}$$

$$\varphi_0 \simeq T^t(x,t)B, \tag{27}$$

$$g(t) \simeq T^t(x,t)G, \tag{28}$$

$$h(x) \simeq T^t(x,t)H, \tag{29}$$

where:

$$F = [f_{00}, \cdots, f_{0N}, \cdots, f_{N0}, \cdots, f_{NN}]^t,$$
$$G = [g_{00}, \cdots, g_{0N}, \cdots, g_{N0}, \cdots, g_{NN}]^t,$$
$$H = [h_{00}, \cdots, h_{0N}, \cdots, h_{N0}, \cdots, h_{NN}]^t,$$
$$B = [b_{00}, \cdots, b_{0N}, \cdots, b_{N0}, \cdots, b_{NN}]^t,$$

and the components of these vectors can be derived from (13).
Similarly, for the function $k(x,t,y,z)$, we have:

$$k(x,t,y,z) \simeq k_N(x,t,y,z) = T^t(x,t)KT(y,z), \tag{30}$$

where K is an $(N+1)\times(N+1)$ matrix; its elements are given by:

$$k_{pqlm} = \frac{(T_{pq}(x,t),\ (k(x,t,y,z),T_{lm}(y,z))_{w(y,z)}\ )_{w(x,t)}}{(T_{pq}(x,t),T_{pq}(x,t))_{w(x,t)}(T_{lm}(y,z),T_{lm}(y,z))_{w(y,z)}},\ \ p,q,l,m = 0,1,...,N.$$

Substituting (23), (26) and (30) in (19), we get:

$$\begin{aligned} &T^t(x,t)E + T^t(x,t)A + T^t(x,t)K\ (\textstyle\int_{-1}^{1}\int_{-1}^{1} T(y,z)T^t(y,z)dydz)\ A \\ &= T^t(x,t)F, \end{aligned} \tag{31}$$

Now, by letting

$$\Psi = \int_{-1}^{1}\int_{-1}^{1} T(y,z)T^t(y,z)dydz,$$

and by using the orthogonality of the two-dimensional Chebyshev functions, we have the following matrix form of (31):

$$E + A + K\Psi A = F. \tag{32}$$

On the other hand, we can write:

$$\varphi(x,t) + \varphi(-1,-1) - \varphi(-1,t) - \varphi(x,-1) = \int_{-1}^{x}\int_{-1}^{t} T^t(\tau_1,\tau_2)E\,d\tau_1 d\tau_2,$$

then, by the integral operational matrix in (18), we have the following relation between A and E:

$$A+B-G-H=Q^{t}E, \tag{33}$$

and, from 32) and (33), we can obtain the following equation:

$$A+B-G-H=Q^{t}(F-A-K\Psi A), \tag{34}$$

which is a system of linear equations, so by solving the above system, we find the unknown vector A, and the function $\varphi(x,t)$ is obtained in terms of (21).

4.2. Volterra Integro-Differential Equations (VIDE)

In this section, we consider a special kind of 2D linear VIDE as follows:

$$\varphi_{xt}(x,t)+\varphi(x,t)+\int_{-1}^{x}\int_{-1}^{t}k(x,t,y,z)\varphi(y,z)dydz=f(x,t), \tag{35}$$

with the initial conditions:

$$\begin{aligned}&\varphi(-1,-1)=\varphi_0,\\&\varphi(-1,t)=g(t),\\&\varphi(x,-1)=h(x),\end{aligned} \tag{36}$$

where $h(x)$, $g(t)$, $k(x,t,y,z)$, $f(x,t)$ and φ_0 are known functions and $\varphi(x,t)$ is an unknown function. The 2D-VIDE (35) can be solved by using the function approximation (23) and the collocation method through the the following equations:

$$R_N(x_r,t_s)=\varphi_{xt}(x_r,t_s)+\varphi_N(x_r,t_s)+\int_{-1}^{x_r}\int_{-1}^{t_s}k(x_r,t_s,y,z)\varphi_N(y,z)-f(x_r,t_s)=0, \tag{37}$$

in which the collocation points are:

$$x_r=cos(\frac{r\pi}{N}),\;\; t_s=cos(\frac{s\pi}{N})\;\; r,s=0,1,\ldots,N.$$

The kernel function $k(x,t,y,z)$, can be expressed as a truncated Chebyshev series for each x_r and t_s in the form:

$$k(x_r,t_s,y,z)\simeq k_N(x_r,t_s,y,z)=\sum_{l=0}^{N}\sum_{m=0}^{N}k_{ml}(x_r,t_s)T_l(y)T_m(z), \tag{38}$$

where $k_{ml}(x_r,t_s),\; r,s=0,1,\ldots,N$ are determined by means of the Cleanshaw-Kurtis rule [8], as follows:

$$k_{ml}(x_r,t_s)=\frac{4}{N^2}\sum_{q=0}^{N}\sum_{p=0}^{N}k(x_r,t_s,y_p,z_q)T_l(y_p)T_m(z_q),$$

where $y_p=cos(\frac{p\pi}{N}),z_q=cos(\frac{q\pi}{N}),\; p,q=0,1,\ldots,N.$
$k_N(x_r,t_s,y,z)$ can be represented in the matrix form:

$$k_N(x_r,t_s,y,z)=T^{t}(y,z)k(x_r,t_s), \tag{39}$$

where

$$k(x_r,t_s)=[k_{00}(x_r,t_s),k_{01}(x_r,t_s),\cdots,k_{0N}(x_r,t_s),\cdots,k_{N0}(x_r,t_s),\cdots,k_{NN}(x_r,t_s)]^t. \tag{40}$$

Substituting (22) and (39) in (37), we get:

$$T^t(x_r,t_s)E+T^t(x_r,t_s)A+k^t(x_r,t_s)(\int_{-1}^{x_r}\int_{-1}^{t_s}T(y,z)T^t(y,z)dydz)A=f(x_r,t_s), \tag{41}$$

for $r,s=0,1,...,N$.
By using the following relation:

$$Z(x_r,t_s)=\int_{-1}^{x_r}\int_{-1}^{t_s}T(y,z)T^t(y,z)dydz,$$

equation (41) is written in the matrix form:

$$T^t(x_r,t_s)E+T^t(x_r,t_s)A+k^t(x_r,t_s)Z(x_r,t_s)A=f(x_r,t_s),\ \ r,s=0,1,...,N. \tag{42}$$

Substituting (33) in (42), we get:

$$(T^t(x_r,t_s)Q^{-1}+T^t(x_r,t_s)+k^t(x_r,t_s)Z(x_r,t_s))A=f(x_r,t_s)-T^t(x_r,t_s)Q^{-1}(B-G-H), \tag{43}$$

for $r,s=0,1,...,N$.
Then we have:

$$(T^t_{r,s}Q^{-1}+T^t_{r,s}+\bar{K}_{r,s}\bar{Z})A=F-T^t_{r,s}Q^{-1}(B-G-H), \tag{44}$$

where

$$T_{r,s}=\begin{bmatrix}T^t(x_0,t_0)\\ \vdots\\ T^t(x_N,t_N)\end{bmatrix}_{(N+1)^2\times(N+1)^2} \qquad F=\begin{bmatrix}f(x_0,t_0)\\ \vdots\\ f(x_N,t_N)\end{bmatrix}_{(N+1)^2\times 1}$$

and

$$\bar{Z}=\begin{bmatrix}Z(x_0,t_0)\\ \vdots\\ Z(x_N,t_N)\end{bmatrix}_{(N+1)^4\times(N+1)^2} \qquad \bar{K}_{r,s}=\begin{bmatrix}k^t(x_0,t_0) & 0 & \dots & 0\\ 0 & k^t(x_0,t_1) & & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & & k^t(x_N,t_N)\end{bmatrix}_{(N+1)^2\times(N+1)^4}$$

Therefore, (44) is a system of linear equations, so by solving this system, we find vector A.

5. Numerical Results

In this section we apply the proposed method to obtain numerical solutions of two-dimensional Fredholm and Volterra integro-differential equations of type (19) and (35).

Example 5.1. First, consider the following Fredholm integro-differential equation:

$$\varphi(x,t)+\varphi_{xt}(x,t)+\int_{-1}^{1}\int_{-1}^{1}(x+t)\varphi(y,z)dydz=2t+xt^2,$$

which is subject to the initial conditions $\varphi(-1,-1)=-1$, $\varphi(x,-1)=1$ and $\varphi(-1,t)=-t^2$. Table 1 shows that the numerical solutions are in a good agreement with the exact solution $\varphi(x,t)=xt^2$ for the small values of N.

TABLE 1. Absolute error in $\varphi(x,t)$ for different values of N for Example 5.1

(x,t)	$N=4$	$N=6$	$N=8$
(-0.5,-0.5)	3.5379E–3	1.6285E–4	1.6883E–5
(-0.4,-0.4)	4.4622E–3	5.6044E–4	1.6721E–5
(-0.3,-0.3)	4.8532E–3	3.4887E–4	2.8474E–5
(-0.2,-0.2)	4.7109E–3	1.0994E–4	1.9494E–5
(-0.1,-0.1)	4.0352E–3	1.6911E–4	9.5782E–6
(0,0)	0	0	0
(0.1,0.1)	1.0839E–3	1.3984E–4	2.8326E–6
(0.2,0.2)	1.1916E–3	2.5139 E–4	2.3270E–6
(0.3,0.3)	4.0005E–3	2.4705 E–4	3.3032E–6
(0.4,0.4)	1.1916E–3	2.5139 E–4	2.3270E–6
(0.5,0.5)	4.0005E–3	2.4705 E–4	3.3032E–6

Example 5.2. We consider the following integro-differential equation:

$$\varphi(x,t)+\varphi_{xt}(x,t)+\int_{-1}^{1}\int_{-1}^{1}(y^3+yxsin(t))\varphi(y,z)dydz=x^2sin(t)+2xcos(t),$$

where

$$\begin{aligned}\varphi(-1,-1)&=-sin(1),\\ \varphi(-1,t)&=sin(t),\\ \varphi(x,-1)&=-x^2sin(1).\end{aligned}$$

Table 2 shows the numerical solutions for various N, with the exact solution $f(x)=x^2sin(t)$.

Example 5.3. Consider a Volterra integro-differential equation as follows:

$$\varphi(x,t)+\varphi_{xt}(x,t)+\int_{-1}^{x}\int_{-1}^{t}k(x,t,y,z)\varphi(y,z)dydz=f(x,t),$$

where

$$\begin{aligned}\varphi(-1,-1)&=e^{-2},\\ \varphi(-1,t)&=e^{-1+t},\\ \varphi(x,-1)&=e^{-1+x},\end{aligned}$$

such that $f(x,t)=2e^{x+t}+\frac{1}{4}(e^{2t}-e^{-2})(e^{2x}-e^{-2})$ and $k(x,t,y,z)=e^{y+z}sin(x+t)$. The exact solution is $\varphi(x,t)=e^{x+t}$.

TABLE 2. Absolute error in $\varphi(x,t)$ for different values of N for Example 5.2

(x,t)	$N=4$	$N=7$	$N=9$
(-0.5,-0.5)	4.6379E–2	1.5165E–3	2.1286E–4
(-0.4,-0.4)	4.9622E–2	4.2034E–3	3.5921E–4
(-0.3,-0.3)	3.7532E–2	4.3788E–4	3.8375E–4
(-0.2,-0.2)	5.8109E–3	2.0459E–3	2.9984E–4
(-0.1,-0.1)	6.0051E–2	3.7811E–3	5.5782E–4
(0,0)	2.8262E–3	7.1241E–6	7.0985E–6
(0.1,0.1)	8.0839E–3	1.4983E–3	6.7336E–4
(0.2,0.2)	4.3916E–2	6.4211 E–3	7.4240E–4
(0.3,0.3)	7.0205E–2	6.3805 E–3	8.3134E–4
(0.4,0.4)	5.5913E–2	7.4228 E–3	8.2250E–4
(0.5,0.5)	5.1915E–2	7.8704 E–3	6.9012E–4

For some points in $[-1,1]\times[-1,1]$, we yield the approximate and exact solution. The numerical results are given in Table 3.

TABLE 3. Numerical results for Example 5.3 ($N=16$)

(x,t)	exact solution	our method	Absolute error
(-0.5,-0.5)	0.36788	0.36790	0.21341E–4
(-0.4,-0.4)	0.44933	0.44935	0.18796E–4
(-0.3,-0.3)	0.54881	0.54886	0.46728E–4
(-0.2,-0.2)	0.67032	0.67033	0.83274E–5
(-0.1,-0.1)	0.81873	0.81879	0.59823E–4
(0,0)	1	1.0000009	0.00951E–4
(0.1,0.1)	1.22140	1.22146	0.63565E–4
(0.2,0.2)	1.49182	1.49191	0.92407E–4
(0.3,0.3)	1.822119	1.822125	0.63348E–5
(0.4,0.4)	2.22554	2.22560	0.62504E–4
(0.5,0.5)	2.71828	2.71831	0.27822E–4

6. Conclusion

Two-dimensional integro-differential equations are usually difficult to solve analytically. In many cases, it is necessary to obtain approximate solutions. For this purpose, an orthogonal basis, named 2D Chebyshev polynomials, is introduced for approximating functions in linear two-dimensional Fredholm and Volterra integro-differential equations. This technique is simple and involves less computation.

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