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Doubly Stochastic Interval Matrices

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Abstract: Interval matrices have many applications in intelligent engineering problems such as robotics in computer science. In this paper, we will first describe the concepts of interval matrices. Next, we will introduce a new class of interval matrices, namely, doubly stochastic interval matrices. Finally, we will present some properties of this new class of matrices.

Keywords: Stochastic matrix, interval matrix, doubly stochastic interval matrix.

1. Introduction

Many real-life problems originate from diverse uncertainties, for example due to data measurement errors. The elements of a matrix, occurring in practice, are usually obtained from experiments, hence they may appear with uncertainties. We represent the uncertain elements in interval forms instead of fixed real numbers. The first systematic treatment of interval vectors and matrices was given by Apostolatos and Kulisch (1968).

An interval matrix A^I is a matrix whose elements are intervals and will be written as

$$A^I = [\underline{A}, \overline{A}] = (a^I_{ij})_{(m\times n)},$$

where $a^I_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}]$. For this matrix, $\mid A^I \mid$ is a real matrix defined as

$$\mid A^I \mid = (\mid a^I_{ij} \mid)_{(m\times n)},$$

where $\mid a^I_{ij} \mid = max\{\mid \underline{a}_{ij} \mid, \mid \overline{a}_{ij} \mid\}$.
For the interval matrix $A^I = [\underline{A}, \overline{A}]$, the center matrix denoted by A_c and the radius matrix denoted by Δ are respectively defined as

$$A_c = \frac{1}{2}(\underline{A} + \overline{A}) \ , \ \Delta = \frac{1}{2}(\overline{A} - \underline{A}).$$

We assume that the reader is familiar with basic interval arithmetic, otherwise see [8].

An interval matrix A^I is said to be nonnegative if $\underline{A} \geq 0$. We will say that an interval matrix A^I is cogredient to an interval matrix B^I if there exists a permutation matrix P such that $A^I = P^T B^I P$.

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The real eigenvalue set of a square interval matrix is defined as

$$\Lambda = \{\lambda \in \mathbb{R};\ Ax = \lambda x, x \neq 0, A \in A^I\}. \tag{1}$$

A real vector x is called a real eigenvector of A^I if it is a real eigenvector of some matrix $A \in A^I$, [10].

A real nonnegative matrix A is said to be an r-doubly stochastic matrix if each of its row and column sums is r. The set of $n \times n$ r-doubly stochastic matrices is denoted by Γ_n^r, [7].

Doubly stochastic matrices are very important in engineering and robotic problems: motion planning, localization and navigation [3] to name a few. In robotic networks, the discrete time consensus algorithm requires the adjacency matrix to be doubly stochastic [2]. On the other hand, dealing with uncertainties is unavoidable in these problems [5]. In fact, interval analysis is used to solve many robotic problems such as the clearance effect, robot reliability, motion planning, localization and navigation [6].

Therefore, we motivate the concept of a doubly stochastic interval matrix by the above observation. In the next section, we will introduce doubly stochastic interval matrices and study some of the properties of these matrices. Finally we show the application of these matrices in the field of robotics.

2. Basic Definitions and Main Results

As mentioned previously, real r-doubly stochastic matrices have useful and interesting properties. In this section, we apply the existence of uncertainties in the elements of these matrices. At first, we introduce the doubly stochastic interval matrices and then we present some results for these matrices.

Definition 1. A nonnegative $n \times n$ interval matrix $A^I = [\underline{A}, \overline{A}]$ is said to be $[\alpha, \beta]$-doubly stochastic interval matrix and denoted by $A^I_{[\alpha,\beta]}$ if $\underline{A}$ and $\overline{A}$ are α-doubly stochastic and β-doubly stochastic matrices, respectively.

Clearly, the center matrix of a doubly stochastic interval matrix $A^I_{[\alpha,\beta]}$ belongs to $\Gamma_n^{\frac{\alpha+\beta}{2}}$ and its radius matrix belongs to $\Gamma_n^{\frac{\beta-\alpha}{2}}$.

Remark 2.1. Each $[\alpha, \beta]$-doubly stochastic interval matrix contains numerous r-doubly stochastic matrices, where $\alpha \leq r \leq \beta$.

The following lemma shows the existence of at least one r-doubly stochastic matrix in A^I, for each $\alpha \leq r \leq \beta$.

Lemma 1. Let $A^I_{[\alpha,\beta]}$ be an $n\times n$ doubly stochastic interval matrix. For each $\alpha \leq r \leq \beta$, there exists at least one r-doubly stochastic matrix $A \in A^I$.

Proof. Define the map $\phi : [0,1] \longrightarrow A^I$ as follows:

$$\phi(t) = \underline{A} + t(\overline{A} - \underline{A}).$$

If we choose $t = \frac{r-\alpha}{\beta-\alpha}$ for each $\alpha \leq r \leq \beta$, it is clear that $\phi(t) \in A^I$. Moreover, we have

$$\sum_{j=1}^{n} (\phi(t))_{ij} = r,\ i = 1,\cdots,n,$$

and

$$\sum_{i=1}^{n} (\phi(t))_{ij} = r,\ j = 1,\cdots,n.$$

□

In the following example, the existence of several 5.5-doubly stochastic interval matrices in A^I has been shown.

Example 2.1. Suppose

$$A^I = \begin{pmatrix} [2,3] & [0,1] & [1,4] \\ [1,5] & [2,3] & 0 \\ 0 & [1,4] & [2,4] \end{pmatrix}.$$

Then A^I is a $[3,8]$-doubly stochastic interval matrix. $A_c \in \Gamma_3^{5.5}$ and $\Delta \in \Gamma_3^{2.5}$. some other 5.5-doubly stochastic matrices in A^I are

$$A = \begin{pmatrix} 2 & 1 & 2.5 \\ 3.5 & 2 & 0 \\ 0 & 2.5 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 0.5 & 3 \\ 3.5 & 2 & 0 \\ 0 & 3 & 2.5 \end{pmatrix}, \begin{pmatrix} 2.5 & 0 & 3 \\ 3 & 2.5 & 0 \\ 0 & 3 & 2.5 \end{pmatrix}, \begin{pmatrix} 3 & 0.5 & 2 \\ 2.5 & 3 & 0 \\ 0 & 2 & 3.5 \end{pmatrix}.$$

Arndt in [1] says that a nonnegative interval matrix A^I is called (ir)reducible if $|\, A^I \,|$ is (ir)reducible. We extend the definition of reducibility of interval matrices, as the following definition. In fact, this definition is an extension of the reducibility of real matrices [7].

Definition 2. A nonnegative n-square interval matrix A^I, for $n \geq 2$, is strongly reducible if there exists a permutation matrix P such that

$$P^T A^I P = \begin{pmatrix} A_1^I & 0 \\ A_3^I & A_2^I \end{pmatrix},$$

where A_1^I and A_2^I are (n-k)-square and k-square matrices, $1 \leq k < n$, respectively. Otherwise, A^I is weakly irreducible.

In the following lemma, some equivalent statements to reducibility of interval matrices are given.

Lemma 2. For the square nonnegative interval matrix A^I, the following statements are equivalent:

(i) A^I is reducible,

(ii) $\overline{A}$ is reducible,

(iii) all $A \in A^I$ are reducible.

Proof. Since A^I is nonnegative, we have $\mid A^I \mid = \overline{A}$. This implies (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (iii): Let $\overline{A}$ be reducible. Then there exists a permutation matrix P such that

$$P^T\overline{A}P = \begin{pmatrix} \overline{A}_1 & 0 \\ \overline{A}_3 & \overline{A}_2 \end{pmatrix},$$

where $\overline{A}_1$ and $\overline{A}_2$ are square matrices.

Now suppose $A \in A^I$. Then from $0 \leq A \leq \overline{A}$, it follows that

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \leq P^TAP = \begin{pmatrix} A_1 & A_4 \\ A_3 & A_2 \end{pmatrix} \leq P^T\overline{A}P = \begin{pmatrix} \overline{A}_1 & 0 \\ \overline{A}_3 & \overline{A}_2 \end{pmatrix}.$$

This implies that $A_4 = 0$, and hence A is reducible. □

By the following example, we will show that if A^I is irreducible, then every $A \in A^I$ is not necessarily irreducible.

Example 2.2. The matrix

$$A^I = \begin{pmatrix} [2,3] & [2,3] & [1,1] \\ [3,4] & [0,1] & [2,3] \\ [0,2] & [0,3] & [2,4] \end{pmatrix}$$

is irreducible, since $\mid A^I \mid = \overline{A}$ is irreducible, but $\underline{A}$ is reducible.

The following lemma shows that the strong reducibility is higher than reducibility of an interval matrix.

Lemma 3. Every strongly reducible interval matrix is a reducible interval matrix.

Proof. If $A^I = [\underline{A}, \overline{A}]$ is strongly reducible, then $\overline{A}$ is a reducible matrix and by Lemma 2, A^I is reducible. □

The following lemma provides a criterion for an interval matrix to be doubly stochastic.

Lemma 4. A nonnegative interval matrix A^I is an $[\alpha, \beta]$-doubly stochastic interval matrix if and only if

$$A^IJ_n = J_nA^I = ([\alpha, \beta])_{n\times n},$$

where J_n is the $n \times n$ matrix whose entries are 1 and $([\alpha, \beta])_{n\times n}$ is the $n \times n$ interval matrix whose entries are $[\alpha, \beta]$.

Proof. Since each of the row and column sums is $[\alpha,\beta]$, the assertion is easily proved by direct calculation. □

Theorem 1. The product of two doubly stochastic interval matrices is a doubly stochastic interval matrix.

Proof. If $A^I_{[\alpha,\beta]}$ and $B^I_{[\alpha',\beta']}$ are $n\times n$ doubly stochastic interval matrices, then their product is a nonnegative matrix and it is clear that

$$(A^I_{[\alpha,\beta]}B^I_{[\alpha',\beta']})J_n = J_n(A^I_{[\alpha,\beta]}B^I_{[\alpha',\beta']}) = ([\alpha\alpha^{'},\beta\beta^{'}])_{n\times n}.$$

□

Theorem 2. Every doubly stochastic interval matrix $A^I_{[\alpha,\beta]}$ is cogredient to a direct sum of weakly irreducible doubly stochastic interval matrices.

Proof. If $A^I_{[\alpha,\beta]}$ is strongly irreducible, then the proof is complete.

Suppose that $A^I_{[\alpha,\beta]}$ is strongly reducible, then by definition 2, $A^I_{[\alpha,\beta]}$ is cogredient to a matrix of the form $\begin{pmatrix} A^I_1 & 0 \\ A^I_3 & A^I_2 \end{pmatrix}$, where A^I_1 is an (n-k)-square matrix and A^I_2 is a k-square matrix.

Let $sum(A^I)$ denote the sum of all entries of the matrix A^I. Clearly, $sum(A^I_3)=0$ and so $A^I_3\equiv 0$, since A^I_3 is nonnegative. If A^I_1 and A^I_2 are weakly irreducible, the result follows. If either A^I_1 or A^I_2 happens to be strongly reducible, then by repeating this argument we will end up with a direct sum of weakly irreducible doubly stochastic interval matrices. □

One of the most important problems in interval matrices is the determination of an interval that contains the set of real eigenvalues of the matrix. The following theorem and its corollary provide some ground rules regarding the real eigenvalues of this special class of matrixes.

Theorem 3. If $A^I_{[\alpha,\beta]}$ is a doubly stochastic interval matrix, then $\Lambda(A^I_{[\alpha,\beta]})\subseteq[-\beta,+\beta]$ and $e=(1,1,\cdots,1)^T$ is a real eigenvector.

Proof. For each $A\in A^I$, we have $\rho(A)\le\rho(\overline{A})$, [9]. Moreover, we know that $\rho(\overline{A})=\beta$ and $e=(1,1,\cdots,1)^T$ is an eigenvector for $\overline{A}$. Therefore, from the definitions of the eigenvalue set and the eigenvector for $A^I_{[\alpha,\beta]}$, the result will follow. □

Corollary 1. The absolute value of each eigenvalue of $A\in A^I_{[\alpha,\beta]}$ lies in the interval $[0,\beta]$.

The following theorem says that the set of all $n \times n$ doubly stochastic interval matrices forms a polyhedron with permutation matrices as its vertices.

Theorem 4. If $A^I_{[\alpha,\beta]}$ is an $n \times n$, $[\alpha,\beta]$-doubly stochastic interval matrix, then

$$A^I_{[\alpha,\beta]} = \sum_{i=1}^{s} \theta_i^I P_i, \tag{2}$$

where $P_1, P_2, \cdots, P_s$ are permutation matrices and θ_i^I are nonnegative intervals satisfying

$$\sum_{i=1}^{s} \theta_i^I = [\alpha,\beta].$$

Proof. We have

$$A^I_{[\alpha,\beta]} = A_c + [-1,1]\Delta.$$

Since $A_c \in \Gamma_n^{\frac{\alpha+\beta}{2}}$ and $\Delta \in \Gamma_n^{\frac{\beta-\alpha}{2}}$, the Birkhoff theorem, (see [7]), implies the following relations:

$$A_c = \sum_{i=1}^{s_1} \delta_i P_i, \tag{3}$$

and

$$\Delta = \sum_{i=1}^{s_2} \eta_i P_i, \tag{4}$$

where the P_i's are permutation matrices and the δ_i's and η_i's are nonnegative numbers satisfying

$$\sum_{i=1}^{s_1} \delta_i = \frac{\alpha+\beta}{2}\ , \ \sum_{i=1}^{s_2} \eta_i = \frac{\beta-\alpha}{2}.$$

Therefore, we can write

$$A^I_{[\alpha,\beta]} = \sum_{i=1}^{s} \theta_i^I P_i,$$

where

$$\sum_{i=1}^{s} \theta_i^I = \frac{\alpha+\beta}{2}[1,1] + \frac{\beta-\alpha}{2}[-1,1] = [\alpha,\beta].$$

$\square$

Example 2.3. Let

$$A^I_{[3,8]} = \begin{pmatrix} [2,3] & [0,1] & [1,4] \\ [1,5] & [2,3] & 0 \\ 0 & [1,4] & [2,4] \end{pmatrix}$$

be an $[3,8]$-doubly stochastic interval matrix. It is clear that $A^I_{[3,8]}$ can be written as follows:

$$A^I_{[3,8]} = [2,3]\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + [0,1]\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + [1,4]\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

One important question here is the total number of permutation matrices in the form (2), or in other words, the numbers of s in this form. This problem is very difficult; however, we can compute an upper bound for s.

The upper bound for the number of permutation matrices in (3) and (4) is $(n-1)^2+1$, (Theorem 3.3 and 3.5 in [7]). This allows us to compute an upper bound for s in (2), hence we have the following result.

Theorem 5. If $n \geq 4$, then the number of permutation matrices which are needed in Theorem 4 is less than or equal to $2(n-1)^2+2$.

The next example expresses an application for the doubly stochastic interval matrices in robotic problems.

Example 2.4. A classical robotics problem is to determine various types of workspace of a given robot. For example, we may have to compute the region of the workspace of the robot in which the eigenvalues of $K^I = (J^I)^T J^I$, where J^I is the Jacobian matrix, are all included in a given range $[a,b]$; see [6].

Now, suppose the Jacobian matrix related to a robot is the following form

$$J = \begin{pmatrix} 1+\sin\theta & \cos\theta \\ \cos\theta & 1+\sin\theta \end{pmatrix},$$

where $0 \leq \theta \leq \frac{\pi}{4}$.

It is clear that this matrix is an $[1+\frac{\sqrt{2}}{2}, 2+\frac{\sqrt{2}}{2}]$-doubly stochastic interval matrix and every eigenvalues of $K = J^T J$ is real. Based on Theorem 1, $K = J^T J$ is an $[\frac{3}{2}+\sqrt{2}, \frac{9}{2}+\sqrt{2}]$-doubly stochastic interval matrix. Therefore, from Theorem 3 we obtain the interval $[-\frac{9}{2}-\sqrt{2}, \frac{9}{2}+\sqrt{2}]$ for the region of the workspace of this robot.

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