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Entropy of Countable Partitions on Effect Algebras with the Riesz Decomposition Property and Weak Sequential Effect Algebras

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Abstract: The purpose of this study is twofold. For the first part, the entropy of countable partitions on an effect algebra with the Riesz decomposition property is defined. In addition, the lower and upper entropy and the conditional entropy considering a suitable state and transformation functions are introduced. Then, some basic properties of these notions are investigated. In the second part, weak sequential effect algebra is introduced followed by a definition for the entropy of countable partitions on this structure. Furthermore, with the help of appropriate state and transformation functions, the notion of entropy, conditional entropy and relative entropy are introduced. In the final step, some properties of these concepts are studied.

Keywords: Entropy; Effect algebra; Dynamical system; Sequential effect algebra.

1. Introduction

The Kolmogorov-Sinai entropy was introduced to distinguish two dynamical systems in the classical probability theory. In fact, the K-S entropy is a dynamical invariant that can be used as a tool to measure the amount of uncertainty in random events. Every pair of isomorphic dynamical systems has the same entropy. This notion was generalized in many directions ([3,16, 18, 28], etc.). If (Ω,β,ρ) is a probability space, the entropy of a measurable partition $A=\{A_1,\ldots,A_n\}$ of Ω is defined as $H(A)=-\sum_{i=1}^{n}\rho(A_i)log\rho(A_i)$. If $T:\Omega\to\Omega$ is a measure preserving transformation, and if $\bigvee_{i=0}^{n-1}T^{-i}(A)$ denotes the common refinement of the partitions $A, T^{-1}(A),, T^{-(n-1)}(A)$, then there is the limit $h(T,A):=\lim_{n\to\infty}\frac{1}{n}\bigvee_{i=0}^{n-1}T^{-i}(A)$. The K-S entropy is $h(T):=sup\{h(T,A):$ $A\ is\ a\ measurable\ partition\ of\ \Omega\}$. Probability theory was one of the first fields of mathematics using fuzzy sets. The main idea of fuzzy entropy is replacing the partitions with fuzzy partitions. The fuzzy partition of the probability space (Ω,β,ρ) is defied as a finite system of measurable functions $f_i : \Omega \to [0,1], i = 1,2,...,n$ such that $\sum_{i=1}^{n} f_i(x) = 1, \forall x \in \Omega$. There are many possibilities for operations with fuzzy sets. One of the first models was introduced by Dumitrescu [4-11]. In this model , instead of a probability measure, a function $m : F \to [0,1]$ has been considered such that $m(\sum_{i=1}^{k} g_i) = \sum_{i=1}^{k} m(g_i)$ whenever $\sum_{i=1}^{k} g_i \leq 1$. The entropy of this fuzzy partition was given by the classical formula $H(A) = -\sum_{i=1}^{n} (m(f_i))\, log\,(m(f_i))$ whenever $m(f_i) \neq 0$ and $H(A) = 0$ when $m(f_i) = 0$. Some researchers have defined fuzzy entropy considering algebraic structures such as MV-algebras and effect algebras as a probability space [3, 20, 21]. One of the important notions of entropy is the refinement and join of two or more partitions. In classical probability theory, the common refinement of $A = \{A_1,...,A_m\}$ and $B = \{B_1,...,B_n\}$ is simply $C = \{A_i \cap B_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. However, this method cannot be used in more general algebraic structures. The algebraic structures must have some special conditions. For the first time, Malickı and B. Riecan [18] suggested a suitable refinement and join of two or more partitions for defining entropy on structures with fuzzy sets. Effect algebras have been introduced by D. J. Foulis and M. K. Bennett in 1994 [1] to model unsharp measurements in a quantum mechanical system. They are a generalization of many structures which arise in quantum physics and mathematical economics [2, 19]. In fact, effect algebras are a generalization of Boolean algebras, MV-algebras, orthomodular lattices, orthomodular posets and orthoalgebras. For relations among these structures and some other related structures see, e.g., [12]. Effect algebras with the Riesz decomposition property and sequential effect algebras are very important subclasses of effect algebras [17, 22-25]. In order to define the entropy, these subclasses have necessary conditions; therefore, some writers generalize the notion of entropy for effect algebras with the Riesz decomposition property and sequential effect algebras [3, 26, 27].

In fuzzy entropy, finite partitions have always been studied until Ebrahimi [13] introduced the entropy with countable partitions. This notion was further developed and studied in [14, 15]. According to the mentioned sources, entropy on algebraic structures has been defined with finite partitions.

The notion of countable partitions and entropy on countable partitions in effect algebra with Riesz decomposition property and weak sequential effect algebra are introduced in Sections 2 and 6, respectively. In these sections, it is proved that finer partitions have bigger entropy. In addition, if ”P” is a partition that is obtained by joining two partitions ”A” and ”B”, then the entropy of ”P” is less than that of the summation entropy ”A” and entropy ”B”. Conditional entropy and relative entropy of effect algebra with RDP and weak sequential effect algebra are defined in Sections 3 and 7, respectively. The properties of these entropies, especially the relations between the conditional entropy, the relative entropy, the entropy of partitions and the entropy of join of partitions are investigated. In Section 4, the lower and the upper entropies of a dynamical system on effect algebra with RDP are explored and it is proved that two isomorphic dynamical systems have the same lower and upper entropies. In Section 5, a product operation between two effect algebras with RDP is introduced and it is proved that the product of two effect algebras with RDP is an effect algebra with RDP. Afterwards, it is proved that generally the entropy of the product of two effect algebras with RDP is bigger than the summation entropy of the two effect algebras with RDP. The entropy of a dynamical system on a weak sequential effect algebra is defined in Section 8 and some properties of this entropy are proved. Finally, in the most important theorem of this section it is proved that two isomorphic dynamical systems have the same entropy.

2. Countable Partition and Entropy of an Effect Algebra with RDP

In this section, we first define a countable partition, the refinement of a countable partition and the join of two countable partitions of an effect algebra with RDP. Then we define an entropy on a countable partition and investigate the relations between entropies of a countable partition, the refinement of a countable partition and the join of two countable partitions.

Definition 1. An effect algebra is a partial algebra $E = (E, \oplus, \theta, 1)$ with a partially defined operation $\oplus$ and two constant elements θ and 1 such that for all $a, b, c \in E$:

(i) if $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;
(ii) if $(a \oplus b) \oplus c$ is defined, then $a \oplus (b \oplus c)$ is defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
(iii) for any $a \in E$, there exists a unique element $a' \in E$ such that $a \oplus a' = 1$;
(iv) if $a \oplus 1$ is defined in E, then $a = \theta$.

Definition 2. We say that $a \leq b$ if there exists an element $c \in E$ such that $a \oplus c = b$.

Definition 3. The effect algebra E has the Riesz decomposition property (RDP) if $x \leq y_1 \oplus y_2$ implies that there exist two elements $x_1, x_2 \in E$ with $x_1 \leq y_1$ and $x_2 \leq y_2$ such that $x = x_1 \oplus x_2$. This means E has RDP iff $x_1 \oplus x_2 = y_1 \oplus y_2$ implies there exist four elements c_{11}, c_{12}, c_{21}, $c_{22} \in E$ such that $x_1 = c_{11} \oplus c_{12}$, $x_2 = c_{21} \oplus c_{22}$, $y_1 = c_{11} \oplus c_{21}$, $y_2 = c_{12} \oplus c_{22}$.

Example 2.1. Let $E = ([0,1], \oplus, 1, 0)$. Then, $a \oplus b := min\{1, a+b\}$, $\forall a, b \in [0,1]$.

Definition 4. Let E be an effect algebra. A countable sequence $A = \{a_i\}_{i=1}^{\infty}$ of elements of E is called a countable partition of E, if $\overset{\infty}{\underset{i=1}{\oplus}} a_i$ exists in E and $\overset{\infty}{\underset{i=1}{\oplus}} a_i = 1$. ($\overset{\infty}{\underset{i=1}{\oplus}} a_i$ means $a_1 \oplus a_2 \oplus a_3 \oplus \ldots$).

Definition 5. A countable partition $B = \{b_j\}_{j=1}^{\infty}$ is a refinement of a countable partition $A = \{a_i\}_{i=1}^{\infty}$ and we write $A \prec B$, if for any a_i, there is a subset $\alpha_i \subseteq N$ such that $a_i = \oplus_{j \in \alpha_i} b_j$ and $\cup_{i=1}^{\infty} \alpha_i = N$, $\alpha_i \cap \alpha_j = \emptyset \ \forall \ i \neq j$.

Definition 6. Let E be an effect algebra. A mapping $s : E \longrightarrow [0,1]$ is said to be a state if:

(i) $s(1) = 1$;
(ii) whenever $\overset{\infty}{\underset{i=1}{\oplus}} a_i$ exist and $\overset{\infty}{\underset{i=1}{\oplus}} a_i = e$ then $s(\overset{\infty}{\underset{i=1}{\oplus}} a_i) = s(e) \leq \sum_{i=1}^{\infty} s(a_i)$;
(iii) if $a \leq_E b$ then $s(a) \leq s(b)$.

Definition 7. Let $A = \{a_i\}_{i=1}^{\infty}$, $B = \{b_i\}_{i=1}^{\infty}$ be two countable partitions of effect algebra E with RDP. We say that $\{c_{ij} | i \geq 1, j \geq 1\}$ is a Riesz join refinement of A and B if $\overset{\infty}{\underset{k=1}{\oplus}} c_{ik}$ and $\overset{\infty}{\underset{k=1}{\oplus}} c_{kj}$ exist in E and, $a_i = \overset{\infty}{\underset{k=1}{\oplus}} c_{ik}$, $b_j = \overset{\infty}{\underset{k=1}{\oplus}} c_{kj}$, $sups(c_{ij}) \geq sup(s(a_i))sup(s(b_j))$(by RDP we will be able to find smaller elements).

Definition 8. Let s be a state, and $A = \{a_i\}_{i=1}^{\infty}$ be a countable partition on an effect algebra with RDP;
we define the entropy of A by $H(A) := -log \sup_{i \in \mathbb{N}} s(a_i)$.

Example 2.2. $A = \{\theta, 1\}$ is a partition and $H(A) = 0$.

Corollary 1. If $A = \{a_i\}_{i=1}^{\infty}$ is a countable partition then $H(A) \geq 0$.

Proof. $\overset{\infty}{\underset{i=1}{\oplus}} a_i = 1$ so $1 = s(1) \leq \sum_{i=1}^{\infty} s(a_i)$ and this implies there is a_i such that $s(a_i) > 0$ and so $0 < sups(a_i) \leq 1$. ∎

Theorem 1. Let C be a Riesz join refinement of $A = \{a_i\}_{i=1}^{\infty}$ and $B = \{b_i\}_{i=1}^{\infty}$. Then

$$max\{H(A), H(B)\} \leq H(C) \leq H(A) + H(B).$$

Proof. Since $a_i = \overset{\infty}{\underset{i=1}{\oplus}} c_{ij}$ so $s(c_{ij}) \leq s(a_i)\ \forall i, j$, and this implies $sup\ s(c_{ij}) \leq sup\ s(a_i)$ thus $H(C) \geq H(A)$, with the same argument we have $H(C) \geq H(B)$. On the other hand $sup\, s(c_{ij}) \geq sup\, s(a_i) sup\, s(b_j)$, which $-log\ sup\ s(c_{ij}) \leq -log\ sup\ s(a_i) - log\ sup\ s(b_j)$, that is, $H(C) \leq H(A) + H(B)$. ∎

Corollary 2. Let $A \prec B$ then $H(A) \leq H(B)$.

Proof. Since for each i, $a_i = \underset{j \in \alpha_i}{\oplus} b_j$, $\alpha_i \subseteq \mathbb{N}, \alpha_i \cap \alpha_j = \emptyset\ i \neq j$ and $\overset{\infty}{\underset{i=1}{\cup}} \alpha_i = \mathbb{N}$,
we let $c_{ij} = \begin{cases} b_j & if\ j \in \alpha_i \\ \theta & o.w \end{cases}$
$c = \{c_{ij}\}_{i,j=1}^{\infty}$ is a Riesz join refinement of A, B because

(i) $\overset{\infty}{\underset{i,j=1}{\oplus}} c_{ij} = \overset{\infty}{\underset{i=1}{\oplus}} \underset{j \in \alpha_i}{\oplus} b_j = \overset{\infty}{\underset{i=1}{\oplus}} a_i = 1$,
(ii) $a_i = \underset{j \in \alpha_i}{\oplus} b_j$, $\quad b_j = c_{ij}, j \in \alpha_i$,
(iii) $sup\ s(c_{ij}) = sup\ s(b_j) \geq sup\ s(b_j) sup\ s(a_i)$, then $H(A) \leq H(C) = H(B)$.

∎

Example 2.3. $E = ([0,1], \oplus, 1, 0)$ is an effect algebra. Consider $s : E \to [0,1]$ as $s(t) = t$ then sequence $\{(\frac{1}{2})^n\}_{n=1}^{\infty}$ is an countable partition of E. consider partition $a_1 = \frac{1}{6}, a_2 = \frac{5}{6}, 0 = a_3 = a_4 = \cdots$ and partition $b_1 = \frac{1}{4}, b_2 = \frac{3}{4}, 0 = b_3 = b_4 = \cdots$ then partition $c_{11} = \frac{1}{8}, c_{12} = \frac{1}{24}, c_{21} = \frac{1}{8}, c_{22} = \frac{17}{24}, c_{ij} = 0, \forall\ i, j > 2$, is a refinement of sequences $\{a_i\}_{i=1}^{\infty}\ and \{b_i\}_{i=1}^{\infty}$.

3. Conditional Entropy and Relative Entropy of Effect Algebras with RDP

Let we begin this section with a definition of conditional entropy.

Definition 9. Let $C = \{c_{ij} : 1 \leq i, 1 \leq j\}$ be Riesz join refinement of two countable partitions $\{a_i\}_{i=1}^{\infty}, \{b_j\}_{j=1}^{\infty}$ of effect algebra E with RDP we define :
$H_C(A|B) := -log\ sup\ (\frac{s(c_{ij})}{s(b_j)}), s(b_j) > 0$.
Remark : $b_j = \bigoplus_{i=1}^{\infty} c_{ij}$ so $s(b_j) \geq s(c_{ij})$.

Definition 10. We say Riesz join refinement $C = \{c_{ij} : 1 \leq i, 1 \leq j\}$ of countable partitions $\{a_i\}_{i=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ is independent if $sups(c_{ij}) = sup(s(a_i))sup(s(b_j))$.

Proposition 1. Let A , B and D be countable partitions of an effect algebras with RDP. Then

(i) if $A \prec B, C^1 = A \vee D$ and $C^2 = B \vee D$ are independent then $H(A \vee D) \leq H(B \vee D)$;
(ii) if $C = A \vee B$ then $H(A \vee B) \geq H_C(A|B)$.

Proof.

(i) Since $A \prec D$, there is $\alpha_i \in N$ such that $a_i = \bigoplus_{k \in \alpha_i} d_k$. This implies $sups(c^1_{jk}) = sups(b_j)\ sups(d_k) \leq sups(b_j)\ sups(a_i) = sups(c^2_{ji})$.
(ii) $s(c_{ij}) \leq \frac{s(c_{ij})}{s(b_j)}$.

■

Definition 11. Let $A_1, A_2, ..., A_n$ be countable partitions of effect algebra E with RDP. We define

$$H_*(A_1 \vee ... \vee A_n) := inf\{H(C) : C \in Ref(A_1, ..., A_n)\};$$

$$H^*(A_1 \vee ... \vee A_n) := sup\{H(C) : C \in Ref(A_1, ..., A_n)\}.$$

In view of (2.12) $max\{H(A_1), ..., H(A_n)\} \leq H_*(A_1 \vee ... \vee A_n) \leq H^*(A_1 \vee ... \vee A_n) \leq H(A_1) + ... + H(A_n)$.

Definition 12. Let A, B be two countable partitions we define $H_*(A|B) := inf\{H_C(A|B) : C \in Ref(A,B)\}$ and $H^*(A|B) := sup\{H_C(A|C) : C \in Ref(A,B)\}$.

Proposition 2. Let $A = \{a_i\}_{i=1}^{\infty}$, $B = \{b_j\}_{j=1}^{m}$ be two countable partitions of E with RDP and $C = \{c_{ij} | \ i \geq 1, j = 1, ..., m\}$ be refinement of A, B then:

(i) $H(C) > H_C(A|B) + H(B)$,

(ii) $H^*(A \vee B) \geq H^*(A|B) + H(B)$,
$H_*(A \vee B) \geq H_*(A|B) + H(B)$.

Proof.

(i) Let $sup s(b_j) = s(b_\ell)$, $j = 1, ..., m$. It holds that $\frac{s(c_{ij})}{sup s(b_j)} = \frac{s(c_{ij})}{s(b_\ell)} \leq \frac{s(c_{ij})}{s(b_j)}$ so $\frac{s(c_{ij})}{sup s(b_j)} \leq sup(\frac{s(c_{ij})}{s(b_j)})$. This implies $\frac{sup s(c_{ij})}{sup s(b_j)} \leq sup(\frac{s(c_{ij})}{s(b_j)})$and $-log \frac{sup s(c_{ij})}{sup s(b_j)} \geq -log sup \frac{s(c_{ij})}{s(b_j)}$ so $H(C) \geq H_C(A|B) + H(B)$.

(ii) By (i) the proof is trivial.

■

Definition 13. Let $A = \{a_i\}_{i=1}^{\infty}$, $B = \{b_j\}_{j=1}^{\infty}$ be countable partitions of E with RDP. The relative entropy of A with respect to B is defined as following:

$$H(A \parallel B) := \log \sup_{i,j} (\frac{s(a_i)}{s(b_j)}), \textit{ whenever } s(b_j) \neq 0.$$

Proposition 3. Let $A = \{a_i\}_{i=1}^{\infty}$, $B = \{b_j\}_{j=1}^{\infty}$ and $C = \{c_k\}_{k=1}^{\infty}$ be countable partitions of E with RDP. If $A \prec B$ then:

(i) $H(B \parallel C) \leq H(A \parallel C)$,

(ii) $H(C \parallel A) \leq H(C \parallel B)$,

(iii) $H(A \parallel B) \geq 0$.

Proof.

(i) $\sup_{j,k} (\frac{s(b_j)}{s(c_k)}) \leq \sup_{i,k} (\frac{s(a_i)}{s(c_k)})$.

(ii) $\sup_{i,k} (\frac{s(c_k)}{s(a_i)}) \leq \sup_{j,k} (\frac{s(c_k)}{s(b_j)})$.

(iii) $\sup_{i,j} (\frac{s(a_i)}{s(b_j)}) \geq 1$.

■

Corollary 3. Let A, B, C be countable partitions of E with RDP and $A \prec B$ then $H(B \vee D \parallel C) \leq H(A \vee D \parallel C)$.

Proposition 4. Let $A = \{a_i\}_{i=1}^{\infty}$, $B = \{b_j\}_{j=1}^{\infty}$ and $C = \{c_k\}_{k=1}^{\infty}$ be countable partitions of E with RDP, then:

(i) $H(A \parallel B) \geq H(A)$,

(ii) $H(A\vee B \parallel C) \leq H(A\vee B \parallel B) + H(B \parallel C)$.

Proof.

(i) $0 \leq s(b_j) \leq 1$ thus $\sup_{i,j}(\frac{s(a_i)}{s(b_j)}) \geq \sup_{i} s(a_i)$.

(ii) Let $A\vee B = \{d_{ij} : 1 \leq i, 1 \leq j\}$. $\sup_{i,j,k}(\frac{s(d_{ij})}{s(c_k)}) \leq \sup_{i,j,l}(\frac{s(d_{ij})}{s(b_l)}) \sup_{l,k}(\frac{s(b_l)}{s(c_k)})$.

■

4. Entropy of a Dynamical System on Effect Algebras with RDP

In this section we introduce dynamical system on an effect algebra with RDP.Then we will obtain some interesting properties of lower and upper entropies on this dynamical system.

Definition 14. A mapping $T : E \longrightarrow E$ is said to be a transformation of an effect algebra E if

(i) $T(\bigoplus_{i=1}^{\infty} a_i) = \bigoplus_{i=1}^{\infty} T(a_i)$ whenever $\bigoplus_{i=1}^{\infty} a_i$ and $\bigoplus_{i=1}^{\infty} T(a_i)$ exist;

(ii) $T(1) = 1$;

(iii) $s(T(a)) = s(a) \ \forall a \in E$ that s is a state of E.

A triple (E, s, T) is said to be a dynamical system.

Proposition 5. Let $A = \{a_i\}_{i=1}^{\infty}$ be a countable partition of effect algebra E with RDP then:

(i) $T(A)$ is a countable partition;

(ii) $H(A) = H(T(A))$.

Proof.

(i) $\bigoplus_{i=1}^{\infty} T(a_i) = T(\bigoplus_{i=1}^{\infty} a_i) = T(1) = 1$.

(ii) $s(a_i) = s(T(a_i))$.

■

Definition 15. Let A be a countable partition and T be a transformation of effect algebra E with RDP. We define

$$H_*^n(A,T) := H_*(A\vee T(A)\vee \ldots \vee T^{n-1}(A));$$

$$H_n^*(A,T) := H^*(A\vee T(A)\vee \ldots \vee T^{n-1}(A)).$$

Theorem 2. If $C = A\vee B$ then $T(C) = T(A)\vee T(B)$.

Proof. $C = A\vee B$ so $a_i = \bigoplus_{k=1}^{\infty} c_{ik}$ and $b_j = \bigoplus_{k=1}^{\infty} c_{kj}$. By definition of T we have $T(a_i) = \bigoplus_{k=1}^{\infty} T(c_{ik})$, $T(b_j) = \bigoplus_{k=1}^{\infty} T(c_{kj})$ and $sup\ s(T(c_{ij})) = sup\ s(c_{ij}) \geq sup\ s(a_i)\ sup\ s(b_j) = sup\ s(T(a_i))\ sup\ s(T(b_j))$.
■

Theorem 3. Let E be an effect algebra with RDP, s be a state and T be a transformation of E . For any countable partition $A = \{a_i\}_{i=1}^{\infty}$, there exist limits

$$h_*(A,T) := \lim_{n\to\infty} \frac{1}{n} H_*^n(A,T);$$

$$h^*(A,T) := \lim_{n\to\infty} \frac{1}{n} H_n^*(A,T).$$

Proof. Let C be a refinement of partitions $A, T(A), ..., T^{n-1}(A))$ and D be a refinement of partitions $A, T(A), ..., T^{m-1}(A))$. $T^n(D)$is a refinement of $T^n(A), T^{n+1}(A), ..., T^{n+m-1}(A)$. Let now ε be a join refinement of C and $T^n(D)$ so $\varepsilon = A \vee T(A) \vee ... \vee T^{n+m-1}(A)$ and $A \prec \varepsilon, ..., T^{n+m-1}(A) \prec \varepsilon$. $H_*^{n+m}(A,T) = H_*(A \vee T(A) \vee ... \vee T^{n+m-1}(A)) = inf\{H(C) : C \varepsilon Ref(A, T(A), ..., T^{n+m-1}(A))\} \leq H(\varepsilon) \leq H(c) + H(T(D)) = H(C) + H(D)$. C is arbitrary and $H_*^{n+m}(A,T) - H(D) \leq H(c)$ so $H_*^{n+m}(A,T) - H(D) \leq H_*^n(A,T)$, since D is arbitrary too this imply $H_*^{n+m}(A,T) \leq H_*^n(A,T) + H_*^m(A,T)$ and existence of $\lim\limits_{n\to\infty} \frac{1n}{H_*^n(A,T)}$. With the same argument we can conclude the existence of $\lim\limits_{n\to\infty} \frac{1n}{H_n^*(A,T)}$. ■

Definition 16. The lower and upper entropy, $h^*(T)$ and $h_*(T)$ are defined as follow:

$$h_*(T) := sup\{h_*(A,T)\} : A \text{ is a partition of } E\},$$

$$h^*(T) := sup\{h^*(A,T)\} : A \text{ is a partition of } E\}.$$

Proposition 6. Let A be a countable partition of E then $h^*(T,A) \leq H(A)$ and also $h_*(T,A) \leq H(A)$.

Proof. If C is a Riesz join refinement of $A, T(A), T^2(A), ..., T^{n-1}(A)$ then $H(C) \leq \sum_{i=0}^{n-1} H(T^i(A)) = nH(A)$ so $sup\, H(C) \leq nH(A)$ also $inf\; H(C) \leq nH(A)$. ■

Definition 17. Two dynamical systems (E_1, s_1, T_1) and (E_2, s_2, T_2) are said to be isomorphic if there exists a bijective mapping $\psi : E_1 \to E_2$ such that:

(i) $\psi(1) = 1$;
(ii) $\psi(\bigoplus_{i=1}^{\infty} a_i) = \bigoplus_{i=1}^{\infty} \psi(a_i)$ whenever $\bigoplus_{i=1}^{\infty} a_i$ and $\bigoplus_{i=1}^{\infty} \psi(a_i)$ exist;
(iii) $s_2(\psi(a)) = s_1(a)$;
(iv) $T_2(\psi(a)) = \psi(T_1(a)) \forall a \varepsilon E$.

Theorem 4. If dynamical systems (E_1, s_1, T_1) and (E_2, s_2, T_2) are isomorphic dynamical systems, where E_1, E_2 have the property RDP then $h_*(T_1) = h_*(T_2)$ and $h^*(T_1) = h^*(T_2)$.

Proof. Let $\psi : E_1 \longrightarrow E_2$ be an isomorphism.If $A = \{a_i\}_{i=1}^{\infty}$ is a countable partition of E_1 then $\{\psi(a_i\})_{i=1}^{\infty}$ is a countable partition of E_2 and vice versa also we have $H(A) = -\log \sup_{i\in N} s_1(a_i) = -\log \sup_{i\in N} s_2 \psi(a_i) = H(\psi(A))$; therefore, $H_*^n(A, T_1) = inf\{H(c) : c \in Ref(A, T_1(A), \ldots, T_1^{n-1}(A))\}$

$= inf\{H(\psi(c)) : \psi(c) \in Ref(\psi(A), T_2(\psi(A)), \ldots, T_2^{n-1}(\psi(A)))\} = H_*^n(A, T_2)$ and this proved $h_*^n(A, T_1) = h_*^n(A, T_2)$ and $h_*^n(A) = h_*^n(A)$. In similar way we can prove the second equality. ■

Definition 18. Let $A = \{a_i\}_{i=1}^{\infty}$ and $C = \{c_j\}_{j=1}^{\infty}$ be two countable partitions of the dynamical system (E, s, T). We say $A \overset{\circ}{\subseteq} C$ if for any a_i there are c_j and $b_{ij} \in E$ such that $c_j = a_i \oplus b_{ij}$ and $s(b_{ij}) = 0$.

Theorem 5. Let $A = \{a_i\}_{i=1}^{\infty}, B = \{b_j\}_{j=1}^{\infty}, C = \{c_n\}_{n=1}^{\infty}, D = \{d_m\}_{n=1}^{\infty}$ be countable partitions of effect algebra E with RDP and $A \overset{\circ}{\subseteq} C$, then

(i) $H(C) \leq H(A)$;
(ii) $T(A) \overset{\circ}{\subseteq} T(C)$;
(iii) If $P = \{p_{ij}\}_{i,j=1}^{\infty}$ is independent Riesz join refinements of A and B and $Q = \{q_{nj}\}_{n,j=1}^{\infty}$ is Riesz join refinements of C and B then $H(Q) \leq H(P)$;
(iv) If for any n and m, Riesz join refinements of $A, T(A), T^2(A), ..., T^{n-1}(A), C, T(C), T^2(C), ..., T^{m-1}(C)$ are independent then $h^*(T, C) \leq h^*(T, A)$ and also $h_*(T, C) \leq h_*(T, A)$.

Proof.

(i) $sup\, s(a_i) \leq sup\, s(c_n)$.
(ii) For any a_i there are c_j and $b_{ij} \in E$ such that $c_j = a_i \oplus b_{ij}$ and $s(b_{ij}) = \theta$. Therefore, for any $T(a_i)$ there are $T(c_j)$ and $T(b_{ij}) \in E$ such that $T(c_j) = T(a_i) \oplus T(b_{ij})$ and $s(T(b_{ij})) = s(b_{ij}) = \theta$.
(iii) $\sup_{i,j} s(p_{ij}) = \sup_i s(a_i) \ \sup_j s(b_j) \leq \sup_n s(c_n) \ \sup_j s(b_j) \leq \sup_{n,j} s(q_{nj})$.
(iv) If P is a Riesz join refinement of $A, T(A), T^2(A), ..., T^{n-1}(A)$ and Q is a Riesz join refinement of $C, T(C), T^2(C), ..., T^{n-1}(C)$ by part three we have $H(Q) = H(C \vee T(C) \vee ... \vee T^{n-1}(C)) \leq H(P) = H(A \vee T(A) \vee ... \vee T^{n-1}(A))$.

■

5. Entropy of Product Effect Algebras with RDP

We begin this section with a proposition that introduces a product on the two effect algebras with RDP.

Proposition 7. Let $(E, \oplus_E, 0_E, 1_E)$ and $(F, \oplus_F, 0_F, 1_F)$ be two effect algebras with RDP, $C = E \otimes F = \{(e, f) : e \in E, f \in F\}$, $(e_1, f_1) \otimes (e_2, f_2) = (e_1 \oplus_E e_2, f_1 \oplus_F f_2)$ and $(e_1, f_1) \leq (e_2, f_2)$ iff $e_1 \leq_E e_2$ and $f_1 \leq_F f_2$ then C is an effect algebra with RDP and we call it the product of E and F.

Proof. We just prove the Riesz decomposition property. Let $(e,f) \le (e_1,f_1) \otimes (e_2,f_2)$. Then $e \le_E e_1 \oplus_E e_2$, $f \le f_1 \oplus_F f_2$; therefore, there exists $x_1, x_2 \in E$ and $y_1, y_2 \in F$ such that $e = x_1 \oplus_E x_2$ and $f = y_1 \oplus_F y_2$ and $x_1 \le_E e_1, x_2 \le_E e_2$, $y_1 \le_F f_1, y_2 \le_F f_2$. ■

Proposition 8. $A = \{(e_i,f_i)\}_{i=1}^{\infty}$is a countable partition of $C = E \otimes F$ iff $A_E = \{e_i\}_{i=1}^{\infty}$ and $A_F = \{f_i\}_{i=1}^{\infty}$ are countable partitions of E and F respectively. We call sequences A_E and A_F related sequences to A.

Proof. Since $\overset{\infty}{\underset{i=1}{\otimes}}(e_i,f_i) = (\overset{\infty}{\underset{i=1}{\oplus_E}} e_i, \overset{\infty}{\underset{i=1}{\oplus_F}} f_i)$, the proof is obvious. ■

Proposition 9. Let $A = \{(e_i,f_i)\}_{i=1}^{\infty}$, $B = \{(e'_i,f'_i)\}_{i=1}^{\infty}$ be two countable partitions of $C = E \otimes F$. If $A \preceq B$, then $A_E \preceq B_E$ and $A_F \preceq B_F$.

Proof. $A \preceq B$; hence, for any $(e_i,f_i) \in A$, there is a subset $\alpha_i \subseteq N$ such that $(e_i,f_i) = \underset{j\in\alpha_i}{\otimes}(e'_j,f'_j) = (\underset{j\in\alpha_i}{\oplus_E} e'_i, \underset{j\in\alpha_i}{\oplus_F} f'_i)$ and this completes the proof.

■

Definition 19. Let E and F be effect algebras with RDP and S_E , S_F be states of E and F respectively. A mapping $S_p : E \otimes F \longrightarrow [0,1]$ is said to be a product state if

(i) $S_p(e,f) = S_E(e)\, S_F(f)$;

(ii) whenever $\overset{\infty}{\underset{i=1}{\otimes}}(e_i,f_i)$ and $(e_1,f_1)\otimes(e_2,f_2)$are defined and $\overset{\infty}{\underset{i=1}{\otimes}}(e_i,f_i)) = (e,f), (e_1,f_1)\otimes(e_2,f_2) = (e,f)$ then $s_p((e_1,f_1)\otimes(e_2,f_2)) = s_p(e,f), s_p((e_1,f_1)\otimes(e_2,f_2)) \le (s_E(e_1)\oplus s_E(e_2), s_F(f_1))\oplus s_F(f_2)))$, $S_p(\overset{\infty}{\underset{i=1}{\otimes}}(e_i,f_i)) = S_p(e,f)$ and $S_p(\overset{\infty}{\underset{i=1}{\otimes}}(e_i,f_i)) \le (\sum_{i=1}^{\infty} s_E(e_i), \sum_{i=1}^{\infty} s_F(f_i))$.

Proposition 10. $S_p : E \otimes F \longrightarrow [0,1]$ is a state of the effect algebra $E \otimes F$.

Proof. By definition of product state, the first condition is true. $S_p(1_E, 1_F) = S_E(1_E)\, S_F(1_F) = 1$. If $(e,f) \le (e',f')$ then $e \le_E e',\ f \le_F f'$ so $S_E(e) \le S_E(e'),\ S_F(f) \le S_F(f')$ so $S_p(e,f) \le S_p(e',f')$. ■

Definition 20. Let S_p be a product state on $E \otimes F$ and $A = \{(e_i,f_i)\}_{i=1}^{\infty}$ be a countable partition. We define the entropy of the product effect algebra $E \otimes F$ by $H_P(A) := -log \sup_{i\in\mathbb{N}} S_p(e_i,f_i)$.

Remark: Since S_p is a state of effect algebra, all of the previous propositions are true for entropy H_P.

Proposition 11. Let $A = \{(e_i,f_i)\}_{i=1}^{\infty}$, $B = \{(e'_i,f'_i)\}_{i=1}^{\infty}$ be two countable partitions of $E \otimes F$ and $C = \{(c_{ij}, c'_{ij})\}_{i,j\ge1}$ be a Riesz join refinement of A and B. If $\sup_{ij} S_E(c_{ij}) \ge \sup_i S_E(e_i)\ \sup_i S_E(e'_i)$ and $\sup_{ij} S_F(c'_{i,j}) \ge \sup_i S_F(f_i)\ \sup_i S_F(f'_i)$, then $C_E = A_E \vee B_E$ and $C_F = A_F \vee B_F$. We call C with this property a strong Riesz join refinement of A and B.

Proof. $\overset{\infty}{\underset{k=1}{\otimes}}(c_{ik},c'_{ik}) = (\overset{\infty}{\underset{k=1}{\oplus}} c_{ik}, \overset{\infty}{\underset{k=1}{\oplus}} c'_{ik}) = (e_i, f_i)$ and $\overset{\infty}{\underset{k=1}{\otimes}}(c_{kj},c'_{kj}) = (\overset{\infty}{\underset{k=1}{\oplus}} c_{kj}, \overset{\infty}{\underset{k=1}{\oplus}} c'_{kj}) = (e'_j, f'_j)$. ■

Proposition 12. Let $A = \{(e_i, f_i)\}_{i=1}^{\infty}$ be a countable partition of $E \otimes F$. Then

$$H(A) \geq H(A_E) + H(A_F)$$

Proof. $S_E(e_i)\, S_F(f_i) \leq \underset{i}{sup}\, S_E(e_i)\ \underset{i}{sup}\, S_F(f_i)$. That is,

$$\underset{i}{sup}\, S_p(e_i, f_i) = \underset{i}{sup}\, S_E(e_i)\, S_F(f_i) \leq \underset{i}{sup}\, S_E(e_i)\ \underset{i}{sup}\, S_F(f_i).$$

■

Definition 21. Let $T_E : E \to E$ and $T_F : F \to F$ be transformations of effect algebras with RDP and S be a state of $E \otimes F$. A mapping $T_P : E \otimes F \longrightarrow E \otimes F$ is said to be a product transformation of $E \otimes F$ if:

(i) $T_p(e, f) = (T_E(e), T_F(f))$;
(ii) $S(T_P(e, f)) = S(e, f)\ \ \forall (e, f) \in E \otimes F$.

Proposition 13. $T_p : E \otimes F \longrightarrow E \otimes F$ is a transformation of effect algebras.

Proof.

(i) If $\overset{\infty}{\underset{i=1}{\oplus_E}} e_i$ and $\overset{\infty}{\underset{i=1}{\oplus_F}} f_i$ are defined, then

$$T_p(\overset{\infty}{\underset{i=1}{\otimes}}(e_i, f_i)) = (T_E(\overset{\infty}{\underset{i=1}{\oplus_E}} e_i), T_F(\overset{\infty}{\underset{i=1}{\oplus_F}} f_i)) = (\overset{\infty}{\underset{i=1}{\oplus_E}} T_E(e_i), \overset{\infty}{\underset{i=1}{\oplus_F}} T_F(f_i)).$$

(ii) $T_p(1_E, 1_F) = (T_E(1_E), T_F(1_F)) = (1_E, 1_F)$.

■

Definition 22. Let $A, A_1, A_2, \ldots, A_n$ be countable partitions of $E \otimes F$. We define
$H_*^p(A_1 \vee \ldots \vee A_n) := inf\{H_p(C) : C\varepsilon\ Strong\ Ref(A_1, A_2, \ldots, A_n)\}$.
$H_*^p(A, T_p) := H_*^p(A \vee T_p(A) \vee \ldots \vee T_p^{n-1}(A))$.
$h_*(A, T_p) := lim_{n\to\infty} \frac{1}{n} H_*^p(A, T_p)$.

Proposition 14. $H_*^p(A_1 \vee \ldots \vee A_n) \geq H_*^n((A_E)_1 \vee \ldots \vee (A_E)_n) + H_*^n((A_F)_1 \vee \ldots \vee (A_F)_n)$ and $h_*(A, T_p) \geq h_*(A_E, (T_p)_E) + h_*(A_F, (T_p)_F)$.

Proof. $H_p(C) \geq H_E(C) + H_F(C)$ for all $C \in\ Strong\ Ref(A_1, A_2, \ldots, A_n)$ so
$inf\{H_p(C) : C \in\ Strong\ Ref(A_1, A_2, \ldots, A_n)\} \geq inf\{H_E(C) : C \in\ Strong\ Ref(A_1, A_2, \ldots, A_n)\} +$
$inf\{H_F(C) : C \in\ Strong\ Ref(A_1, A_2, \ldots, A_n)\}$. ■

6. Countable Partition and Entropy of Weak Sequential Effect Algebra

We start this section with a definition of weak sequential effect algebra, followed by the definition of the countable partition and the join of two countable partitions. Afterwards, some notable propositions in this section are given.

Definition 23. Let $(E,\oplus,\theta,1)$ be an effect algebra, define another binary operation $\circ$ on E, satisfying:

(i) If $b\oplus c$ is defied then $a\circ b\oplus a\circ c$ and $b\circ a\oplus c\circ a$ are defined, $a\circ(b\oplus c)=a\circ b\oplus a\circ c$ and $(b\oplus c)\circ a=b\circ a\oplus c\circ a$ for all $a\in E$;

(ii) $1\circ a=a$ for any $a\in E$;

(iii) If $a\circ b=\theta$, then $a\circ b=b\circ a$;

(iv) If $a\circ b=b\circ a$, then $a\circ b\prime=b\prime\circ a$ and for each $c\in E$, $a\circ(b\circ c)=(a\circ b)\circ c$;

(v) If $c\circ a=a\circ c$ and $c\circ b=b\circ c$, then $c\circ(a\circ b)=(a\circ b)\circ c$ and $c\circ(a\oplus b)=(a\oplus b)\circ c$ whenever $a\oplus b$ is defined;

(vi) If $\bigoplus\limits_{i=1}^{\infty} a_i$ is defined, then $\bigoplus\limits_{i=1}^{\infty} a_i\circ b$ and $\bigoplus\limits_{i=1}^{\infty} b\circ a_i$ are defined and $b\circ(\oplus_{i=1}^{\infty}a_i)=\bigoplus\limits_{i=1}^{\infty}(b\circ a_i)$, $(\bigoplus\limits_{i=1}^{\infty} a_i)\circ b=\bigoplus\limits_{i=1}^{\infty}(a_i\circ b)$.

We call $(E,\oplus,\circ,\theta,1)$ weak sequential effect algebra and its short form WSEA will be used throughout the article.

Example 6.1. Let $E=[0,1]$, $a\oplus b$ = min{ 1 , a+b } and $a\circ b=ab$. $(E=[0,1],\oplus,\circ,0,1)$ is a WSEA.

Definition 24. Let $(E,\oplus,\circ,\theta,1)$ be a WSEA. A countable sequence $A=\{a_i\}_{i=1}^{\infty}$ of elements of E is called a countable partition if $\bigoplus\limits_{i=1}^{\infty} a_i$ exists in E and $\bigoplus\limits_{i=1}^{\infty} a_i=1$. and we say countable partition $B=\{b_j\}_{j=1}^{\infty}$ is a refinement of the partition $A=\{a_i\}_{i=1}^{\infty}$, if for any a_i there is a subset $\alpha_i\subseteq\mathbb{N}$ such that $a_i=\bigoplus\limits_{j\in\alpha_i} b_j$ and $\bigcup\limits_{i=1}^{\infty}\alpha_i=\mathbb{N}$, $\alpha_i\cap\alpha_j=\emptyset\ \forall\ i\neq j$ and we write $A\prec B$.

Proposition 15. Let $(E,\oplus,\circ,\theta,1)$ be a WSEA, $A=\{a_i\}_{i=1}^{\infty}$ and $B=\{b_i\}_{i=1}^{\infty}$ be two countable partitions of E. Then $A\circ B=\{a_i\circ b_i : a_i\varepsilon A, b_i\varepsilon B, i=1,2,...\}$ is a countable partition of E, $A\prec A\circ B$ and $B\prec A\circ B$. We call $A\circ B$ the join refinement of A and B.

Proof. Since $\bigoplus\limits_{i=1}^{\infty} a_i$ and $\bigoplus\limits_{j=1}^{\infty} b_j$ are defined by property (vi) of Definition 23, $\bigoplus\limits_{j=1}^{\infty}\bigoplus\limits_{i=1}^{\infty} a_i\circ b_j$ is defined and $\oplus_{j=1}^{\infty}\bigoplus\limits_{i=1}^{\infty} a_i\circ b_j=\bigoplus\limits_{i=1}^{\infty} a_i\circ\bigoplus\limits_{j=1}^{\infty} b_j=1\circ 1=1$; therefore, $A\circ B$ is a partition. For any $a_i\varepsilon A$, $a_i=\bigoplus\limits_{j=1}^{\infty} a_i\circ b_j$ and for any b_j, $b_j=\bigoplus\limits_{i=1}^{\infty} a_i\circ b_j$, which means $A\prec A\circ B$ and $B\prec A\circ B$. ■

Definition 25. Let E be a WSEA. A mapping $s : E \longrightarrow [0,1]$ is said to be a state if

(i) s(1)=1;
(ii) whenever $\overset{\infty}{\underset{i=1}{\oplus}} a_i$, $a \oplus b$ exist and $\overset{\infty}{\underset{i=1}{\oplus}} a_i = e$ and $a \oplus b = f$ then $s(\overset{\infty}{\underset{i=1}{\oplus}} a_i) = s(e) \leq \sum_{i=1}^{\infty} s(a_i)$ and $s(a \oplus b) = s(f) \leq s(a) + s(b)$;
(iii) If $a \leq_E b$, then $s(a) \leq s(b)$;
(iv) $s(a \circ b) \geq s(a)s(b)$.

Definition 26. Let s be a state on WSEA, E, and $A = \{a_i\}_{i=1}^{\infty}$ be a countable partition of unity 1. We define the entropy of A by $H(A) := -log \sup_{i \in \mathbb{N}} s(a_i)$.

Proposition 16. Let $A \circ B$ be a join refinement of $A = \{a_i\}_{i=1}^{\infty}$ and $B = \{b_i\}_{i=1}^{\infty}$. Then

$$max\{H(A), H(B)\} \leq H(A \circ B) \leq H(A) + H(B)$$

Proof. $a_i = a_i \circ b_j \oplus a_i \circ b'_j$ so $a_i \circ b_j \leq a_i$. By definitions of state and entropy, we have $H(A) \leq H(A \circ B)$; moreover, $H(B) \leq H(A \circ B)$ with the same argument.

$s(a_i \circ b_j) \geq s(a_i)s(b_j)$; therefore, $\sup_{i,j}(s(a_i \circ b_j)) \geq s(a_i)s(b_j)$, which implies $\sup_{i,j}(s(a_i \circ b_j)) \geq \sup_i(s(a_i))s(b_j)$ and also $\sup_{i,j}(s(a_i \circ b_j))_{i,j} \geq \sup_i(s(a_i)) \sup_j(s(b_j))$.

■

Proposition 17. If a WSEA $(E, \oplus, \circ, \theta, 1)$ has the Riesz decomposition property,$A = \{a_i\}_{i=1}^{\infty}$ and $B = \{b_i\}_{i=1}^{\infty}$ are two countable partitions of unity 1, then $A \circ B$ is a Riesz join refinement of A and B.

Proof. Since $\overset{\infty}{\underset{i=1}{\oplus}} a_i$ and $\overset{\infty}{\underset{j=1}{\oplus}} b_j$ are defined by property (6) of Definition 23 $\overset{\infty}{\underset{i=1}{\oplus}} a_i \circ b_j$ and $\overset{\infty}{\underset{j=1}{\oplus}} a_i \circ b_j$ exist and $b_i = \overset{\infty}{\underset{i=1}{\oplus}} a_i \circ b_j$, $a_i = \overset{\infty}{\underset{j=1}{\oplus}} a_i \circ b_j$

■

Proposition 18. If $A \preceq B$ then $H(A) \leq H(B)$.

Proof. Since $A \preceq B$ then for any $a_i \in A$ there is $\alpha_i \subseteq N$, such that $a_i = \underset{j \in \alpha_i}{\oplus} b_j$ so $b_j \leq_E a_i$ and this implies $\sup_j s(b_j) \leq \sup_i s(a_i)$. ■

7. Conditional Entropy and Relative Entropy of Weak Sequential Effect Algebras

As we introduce the conditional entropy and the relative entropy on an effect algebra with RDP, in this section we will also define the conditional entropy and the relative entropy on weak sequential effect algebra. Furthermore, we will investigate the relation between these entropies.

Definition 27. Let $A \circ B = \{a_i \circ b_j : a_i \in A, b_j \in B\}$ be a join refinement of A and B of WSEA $(E, \oplus, \circ, \theta, 1)$. We define conditional e4ntropy as follows:

$$H(A|B) := -log\, sup\, (\frac{s(a_i \circ b_j)}{s(b_j)}), s(b_j) > 0.$$

Proposition 19. Let $A = \{a_i\}_{i=1}^{\infty}$, $B = \{b_i\}_{i=1}^{\infty}$ and $C = \{c_i\}_{i=1}^{\infty}$ be three countable partitions of unity 1 of WSEA $(E, \oplus, \circ, \theta, 1)$. Then

(1) $H(A|B) \geq 0$;
(2) If $A \preceq C$ then $H(A|B) \leq H(C|B)$;
(3) If $A \preceq B$ then $H(A|B) \leq H(A)$;
(4) $H(A \circ B|C) \geq H(A|C) + H(B|A \circ C)$;
(5) $H(A \circ C) \geq H(A) + H(C|A)$;
(6) $H(A) \geq H(A|C)$;
(7) $H(A|C) \leq H(A \circ B|C)$;
(8) If $A \preceq B$ then $H(A \circ C) \leq H(B \circ C)$;
(9) $H(A \circ B) \geq H(A|B)$;
(10) If $A \prec B$ and $C \prec D$ then $A \circ C \prec B \circ D$.

Proof. (1) $a_i \circ b_j \leq_E b_j$.

(2) Since $A \preceq C$, for all $a_i \in A$ there is α_i such that $a_i = \bigoplus_{j \in \alpha_i} c_j$, $a_i \circ b_k = \bigoplus_{j \in \alpha_i} c_j \circ b_k$ so $s(c_j \circ b_k) \leq s(a_i \circ b_k)$ and $\sup_{j,k} \frac{s(c_j \circ b_k)}{s(b_k)} \leq \sup_{i,k} \frac{s(a_i \circ b_k)}{s(b_k)}$.

(3) By definition of state, $s(a_i \circ b_j) \geq s(a_i)s(b_j)$; therefore, $sup\, \frac{s(a_i \circ b_j)}{s(b_j)} \geq sup\, s(a_i)$.

(4) $\frac{s((a_i \circ b_j) \circ c_k)}{s(c_k)} = \frac{s((a_i \circ b_j) \circ c_k)}{s(c_k)} \frac{s(a_i \circ c_k)}{s(a_i \circ c_k)}$ and this implies $sup\, \frac{s((a_i \circ b_j) \circ c_k)}{s(c_k)} \leq sup\, \frac{s((a_i \circ b_j) \circ c_k)}{s(a_i \circ c_k)}\, sup\, \frac{s(a_i \circ c_k)}{s(c_k)}$.

(5) $s(a_i \circ c_j) = \frac{s(a_i \circ c_j)}{s(a_i)} s(a_i)$ so $sup\, s(a_i \circ c_j) \leq sup\, \frac{s(a_i \circ c_j)}{s(a_i)} sup\, s(a_i)$.

(6) $\frac{s(a_i \circ c_j)}{s(c_j)} \geq s(a_i)$.

(7) $A \preceq A \circ B$.

(8) $A \preceq B$ implies for any i and k we have, $a_i = \bigoplus_{j \in \alpha_i} b_j$ and $a_i \circ c_k = (\bigoplus_{j \in \alpha_i} b_j) \circ c_k = \bigoplus_{j \in \alpha_i} (b_j \circ c_k)$; therefore, $A \circ C \preceq B \circ C$.

(9) $H(A|B) \leq H(A) \leq H(A \circ B)$.

(10) For any i and k, we have $a_i = \bigoplus_{j \in \alpha_i} b_j$ and $c_k = \bigoplus_{m \in \alpha_k} d_m$ $a_i \circ c_k = (\bigoplus_{j \in \alpha_i} b_j) \circ (\bigoplus_{m \in \alpha_k} d_m) = \bigoplus_{j \in \alpha_i, m \in \alpha_k} (b_j \circ d_m)\ \forall i, k$

■

Definition 28. Let $A = \{a_i\}_{i=1}^{\infty}$ and $B = \{b_j\}_{j=1}^{\infty}$ be countable partitions of WSEA E. The relative entropy of A with respect to B is defined as follows:

$$H(A \parallel B) := \log \sup_{i,j} (\frac{s(a_i)}{s(b_j)}),\ whenever\ s(b_j) \neq 0.$$

Proposition 20. Let $A=\{a_i\}_{i=1}^{\infty}$, $B=\{b_j\}_{j=1}^{\infty}$ and $C=\{c_k\}_{k=1}^{\infty}$ be countable partitions of WSEA E . If $A \prec B$. Then

(i) $H(B \parallel C) \leq H(A \parallel C)$,
(ii) $H(C \parallel A) \leq H(C \parallel B)$,
(iii) $H(A \parallel B) \geq 0$.

Proof. The proof is similar to the proof of proposition 3. ■

Corollary 4. Let A,B,C and D be countable partitions of WSEA E. Then

(i) if $A \prec B$, then $H(B \circ D \parallel C) \leq H(A \circ D \parallel C)$;
(ii) if $A \prec B$ and $C \prec D$, then $H(B \circ D \parallel E) \leq H(A \circ C \parallel E)$.

Proposition 21. Let $A=\{a_i\}_{i=1}^{\infty}$, $B=\{b_j\}_{j=1}^{\infty}$ and $C=\{c_k\}_{k=1}^{\infty}$ be countable partitions of WSEA E. Then

(i) $H(A \parallel B) \geq H(A)$;
(ii) $H(A \circ B \parallel C) \leq H(A \circ B \parallel B) + H(B \parallel C)$.

Proof. (i) $0 \leq s(b_j) \leq 1$.
(ii) $\sup\limits_{i,j,k}(\frac{s(a_i \circ b_j)}{s(c_k)}) \leq \sup\limits_{i,j,l}(\frac{s(a_i \circ b_j)}{s(b_l)}) \sup\limits_{l,k}(\frac{s(b_l)}{s(c_k)})$.

■

8. Entropy of Dynamical Systems on WSEA

In this section, we define the entropy of dynamical systems on weak sequential effect algebra. Then, we show that the two isomorphic dynamical systems have the same entropy.

Definition 29. A mapping $T : E \longrightarrow E$ is said to be a transformation of a weak sequential effect algebra E if:

(i) $T(\bigoplus_{i=1}^{\infty} a_i) = \bigoplus_{i=1}^{\infty} T(a_i)$ whenever $\bigoplus_{i=1}^{\infty} a_i$ and $\bigoplus_{i=1}^{\infty} T(a_i)$ exist;

(ii) $T(1) = 1$;

(iii) $s(T(a)) = s(a)\ \ \forall a \in E$ that S is a state of E;

(iv) $T(a \circ b) = T(a) \circ T(b)$.

Proposition 22. Let $A=\{a_i\}_{i=1}^{\infty}$ be a countable partition of unity 1. Then

(i) $T(A)$ is a countable partition of unity 1,

(ii) $H(A) = H(T(A))$.

Theorem 6. Let E be a WSEA, s be a state and T be a transformation of E . For any countable partition $A = \{a_i\}_{i=1}^{\infty}$ there exists the limit

$$h(T,A) := \lim_{n\to\infty} \frac{1}{n} H(A \circ T(A) \circ \ldots \circ T^{n-1}(A))$$

Proof. Let $C = A \circ T(A) \circ \ldots \circ T^{n-1}(A)$, $D = A \circ T(A) \circ \ldots \circ T^{m-1}(A)$ and $T^n(D) = T^n(A) \circ T^{n+1}(A) \circ \ldots \circ T^{m+n-1}(A)$. By part b of the previous proposition,we have $H(T^n(D)) = H(D)$ on the other hand $H(A \circ T(A) \circ \ldots \circ T^{m+n-1}(A)) = H(C + T^n(D)) \leq H(C) + H(T^n(D)) = H(C) + H(D)$. ■

The dynamical entropy $h(T)$ is defined as follows:

$$h(T) := sup\{h(A,T)\} : A \text{ is a partition of } E\}$$

Theorem 7. Let $A = \{a_i\}_{i=1}^{\infty}, B = \{b_j\}_{j=1}^{\infty}$ and $C = \{c_n\}_{n=1}^{\infty}$ be countable partitions of E. Then

(1) $h(T,A) \leq H(A)$;

(2) if $a \circ b = b \circ a$ for any $a, b \in E$ then $h(T, A \circ C) \leq h(T,A) + h(T,C)$;

(3) $h(T, T(A)) = h(T,A)$;

(4) if $a \circ b = b \circ a$ for any $a, b \in E$ then $h(T, A \circ T(A) \circ \ldots \circ T^{n-1}(A)) \leq nh(T,A), n \geq 1$;

(5) if $A \prec B$ then $T(A) \prec T(B)$;

(6) if $A \prec B$ then $h(T,A) \prec h(T,B)$;

(7) $h(T^k, A \circ T(A) \circ \ldots T^{k-1}(A)) = kh(T,A)$ for $k > 0$;

(8) $h(T^k) = kh(T)$ for $k > 0$.

Proof. (1) $h(T,A) = \lim\limits_{n\to\infty} \frac{1}{n} H(A \circ T(A) \circ \ldots \circ T^{n-1}(A)) \leq \lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=0}^{n-1} H(T^i(A)) = H(A)$.

(2) $H((A \circ C) \circ T(A \circ C) \circ \ldots \circ T^{n-1}(A \circ C)) = H(A \circ T(A) \circ \ldots . T^{n-1}(A) \circ C \circ T(C) \circ \ldots \circ T^{n-1}(C)) \leq H(A \circ T(A) \circ \ldots \circ T^{n-1}(A)) + H(C \circ T(C) \circ \ldots \circ T^n(C))$.

(3) $H(T(A) \circ T^2(A) \circ \ldots \circ T^{n-1}(A)) = H(T(A \circ T(A) \circ \ldots \circ T^{n-1}(A))) = H(A \circ T(A) \circ \ldots \circ T^{n-1}(A))$.

(4) By part b and c the proof is trivial.

(5) $A \prec B$; therefore, for any i, we have $a_i = \bigoplus\limits_{j\in\alpha_i} b_j$ and this implies $T(a_i) = \bigoplus\limits_{j\in\alpha_i} T(b_j), \forall i$.

(6) By part e, we have $T^i(A) \leq T^i(B)$ for any $i = 1, \ldots, n-1$. Part m of 19 proposition implies that $H(A \circ T(A) \circ \ldots \circ T^{n-1}(A)) \leq H(B \circ T(B) \circ \ldots \circ T^{n-1}(B))$.

(7) $h(T^k, A \circ T(A) \circ \ldots \circ T^{k-1}(A)) = \lim\limits_{n\to\infty} \frac{1}{n} H(A \circ T(A) \circ \ldots \circ T^{nk-1}(A)) = \lim\limits_{n\to\infty} \frac{k}{kn} H(A \circ T(A) \circ \ldots \circ T^{nk-1}(A)) = kh(T,A)$.

(8) $kh(T) = k\sup_A h(T,A) = \sup_A h(T^k, A \circ T(A) \circ \ldots \circ T^{k-1}(A)) \leq \sup_C h(T^k, C) = h(T^k)$. On the other hand, since $A \prec A \circ T(A) \circ \ldots \circ T^{k-1}(A)$ by part f, $h(T^k, A) \leq h(T^k, A \circ T(A) \circ \ldots \circ T^{k-1}(A)) = kh(T,A)$.

■

Definition 30. Let $(E, \oplus, \circ, \theta, 1)$ and $(E', \oplus', \circ', \theta', 1')$ be two WSEA. The two dynamical systems (E, s, T) and (E', s', T') are said to be isomorphic if there exist, a bijective map $\psi : E \to E'$ such that

(i) $\psi(1) = 1'$;
(ii) $\psi(\bigoplus_{i=1}^{\infty} a_i) = \bigoplus'^{\infty}_{i=1} \psi(a_i)$ whenever $\bigoplus_{i=1}^{\infty} a_i$ and $\bigoplus'^{\infty}_{i=1} \psi(a_i)$ exist;
(iii) $s'(\psi(a)) = s(a)$;
(iv) $T'(\psi(a)) = \psi(T(a)) \forall a \in E$;
(v) $\psi(a \circ b) = \psi(a) \circ \psi'(b)$.

Proposition 23. Let two dynamical systems (E, s, T) and (E', s', T') be isomorphic. Then

(i) $A = \{a_i\}_{i=1}^{\infty}$ is a countable partition of E iff $\psi(A) = \{\psi(a_i)\}_{i=1}^{\infty}$ is a countable partition of E';
(ii) $H(A) = H(\psi(A))$;
(iii) $h(T, A) = h(T', \psi(A))$.

Proof. (i) By property *i* and *ii* of the above definition, the proof is trivial.

(ii) $s'(\psi(a)) = s(a)$ imply $H(A) = H(\psi(A))$.

(iii) $H(A \circ T(A) \circ \ldots \circ T^{n-1}(A)) = H(\psi(A \circ T(A) \circ \ldots \circ T^{n-1}(A))) = H(\psi(A) \circ \psi(T(A)) \circ \ldots \circ \psi(T^{n-1}(A))) = H(\psi(A) \circ T'(\psi(A)) \circ \ldots \circ T'^{n-1}(\psi(A)))$ so $h(T, A) = h(T', \psi(A))$.

■

Theorem 8. If dynamical systems (E, s, T) and (E', s', T') are isomorphic dynamical systems, where E, E' are WSEA, then $h(T) = h(T')$.

Proof. By the previous proposition for any countable partitions A of E and B of E', we have $h(T, A) = h(T', \psi(A))$ and $h(T, \psi^{-1}(B)) = h(T', B)$; therefore, $\sup_A h(T, A) = \sup_B h(T', B)$. ■

Definition 31. Let E be a WSEA, $A = \{a_i\}_{i=1}^{\infty}$ and $C = \{c_j\}_{j=1}^{\infty}$ be two countable partitions . We say $A \overset{\circ}{\subseteq} C$ if for any a_i there are c_j and $b_{ij} \in E$ such that $c_j = a_i \oplus b_{ij}$ and $s(b_{ij}) = \theta$.

Theorem 9. Let E be a WSEA, $A = \{a_i\}_{i=1}^{\infty}, C = \{c_n\}_{n=1}^{\infty}$ be countable partitions of E and $A \overset{\circ}{\subseteq} C$ then

(i) $H(C) \leq H(A)$,
(ii) $T(A) \overset{\circ}{\subseteq} T(C)$.

Proof.

(i) $sup\, s(a_i) \leq sup\, s(c_n)$.
(ii) For any a_i there are c_j and $b_{ij} \in E$ such that $c_j = a_i \oplus b_{ij}$ and $s(b_{ij}) = 0$ so for any $T(a_i)$ there are $T(c_j)$ and $T(b_{ij}) \in E$ such that $T(c_j) = T(a_i) \oplus T(b_{ij})$ and $s(T(b_{ij})) = s(b_{ij}) = 0$.

■

Theorem 10. Let E be a WSEA, $A = \{a_i\}_{i=1}^{\infty}, C = \{c_n\}_{n=1}^{\infty}, D = \{d_m\}_{n=1}^{\infty}$ be countable partitions of E, $A \overset{\circ}{\subseteq} C$ and for any $a, b \in E$, $a \circ b = b \circ a$ then

(i) $H(COD) \leq H(AOD)$;
(ii) $H(C|D) \leq H(A|D)$;
(iii) $h(T,C) \leq h(T,A)$.

Proof.

(i) For any a_i there are c_j and $b_{ij} \in E$ such that $c_j = a_i \oplus b_{ij}$ so for any k , $c_j \circ d_k = (a_i \circ d_k) \oplus (b_{ij} \circ d_k)$ this implies $\sup_{i,k} s(a_i \circ d_k) \leq \sup_{j,k} s(c_j \circ d_k)$.
(ii) By part a, $s(a_i \circ d_k) \leq s(c_j \circ d_k)$ so $\sup_{i,k} \frac{s(a_i \circ d_k)}{s(d_k)} \leq \sup_{j,k} \frac{s(c_j \circ d_k)}{s(d_k)}$.
(iii) By part a, $H(C \circ T(C) \circ \ldots \circ T^{n-1}(C)) \leq H(A \circ T(A) \circ \ldots \circ T^{n-1}(A))$.

■

9. Concluding Remarks

In this paper, entropy with countable partitions on two important subclasses of effect algebras was introduced and their properties were investigated. Effect algebra is an important logic model for studying unsharp quantum events. However, due to the limitations of observational tools, physicist are not able to consider every variable in their calculations. Mathematical models can provide a better understanding of the realities of the world of micro-physics. Therefore, the entropy with countable partitions defined on the algebraic structure, especially effect algebra, may be very important. The next step in this regard could be trying to define entropy with countable partitions on other subclasses of effect algebra, such as CB-effect algebra, generalized effect algebra and some algebraic structures such as BCK-algebra and C^*-algebra.

References

[1] M.K. Bennett and D.J. Foulis, Effect algebras and unsharp quantum logics, Foundation of Physics **24**, (1994), 1331- 1352.

[2] D. Butnariu and P. Klement, Triangular Norm-Based Measures and Games with Fuzzy Coalitions, Kluwer Academic Publisher, (1993).

[3] A. Dinola, A. Dvurensku, M. Hycko and C. Manara, Entropy on effect algebras with the Riesz decomposition property I: Basic properties, Kybernetika, **2**, (2005), 143-160.

[4] D. Dumitrescu, Measure-preserving transformation and the entropy of a fuzzy partition, 13th Linz Seminar on Fuzzy Set Theory, (1991), 25-27.

[5] D. Dumitrescu, Hierarchical pattern classification, Fuzzy Sets and Systems, **28**, (1988), 145-162.

[6] D. Dumitrescu, A note on fuzzy information theory, Stud. Univ. Babes - Bolyai Math, **33**, (1988), 65-69.

[7] D. Dumitrescu, Fuzzy partitions with the connectives T infinity, S infinity, Fuzzy Sets and Systems, **47**, (1992), 193-195.

[8] D. Dumitrescu, Fuzzy measures and the entropy of fuzzy partitions, J. Math. Anal. Appl, **176**, (1993), 359-373.

[9] D. Dumitrescu, Entropy of a fuzzy process, Fuzzy Sets and Systems, **55**, (1993), 169-177.

[10] D. Dumitrescu, Fuzzy conditional logic, Fuzzy Sets and Systems, **68**, (1994), 171-179.

[11] D. Dumitrescu, Entropy of fuzzy dynamical systems, Fuzzy Sets and Systems, **70**, (1995), 45-57.

[12] A. Dvurecenskij and S. Pulmannova, New trends in quantum structures, Kluwer Acad. Publ.,Dordrecht/Boston/London and Ister Science, Bratislava, 2000.

[13] M.Ebrahimi, Generators of probability dynamical systems, Differential Geometry-Dynamical Systems, **8**, (2006), 90-97.

[14] M.Ebrahimi and N. Mohamadi, The entropy function on an algebraic structure with infinte partition and m-preserving transformation generators, Applied Sciences, **12**, (2010), 48-63.

[15] M.Ebrahimi and U. Mohamadi, m-Generators of fuzzy Dynamical Systems, Cankaya University journal of Science and Engineering, **9**, (2012), 67-182.

[16] M.Ebrahimi and B.Mosapour, The concept of entropy on D-posets, cankaya University Journal of Science and Engineering, **10**, (2013), 137-151.

[17] L. Weihua and W. Junde, A uniqueness problem of the sequence product on operator effect algebra E(H), J. Phys. A: Math. Theor, **42**, (2009), 185206-185215 .

[18] P. Malicky and B. Riecan, On the entropy of dynamical systems. In: Proc. Conference Ergodic Theory and Related Topics II, Georgenthal 1986, Teubner, Leipzig, (1987), 135-138.

[19] E. PAP, Pseudo-additive measures and their applications, In: Handbook of Measure Theory, Vol. I, II, North-Holland, Amsterdam, (2002), 1403-1468.

[20] J. Petroviciova, On the entropy of partitions in product MV algebras, Soft Computing, **4**, (2000), 41- 44.

[21] J. Petroviciova, On the entropy of dynamical systems in product MV algebras. Fuzzy Sets and Systems, **121**, (2001), 347-351.

[22] K. Ravindran, On a structure theory of effect algebras, PhD. Thesis, Kansas State University, Manhattan, (1996).

[23] Sh. Jun and W. Junde, Not each sequential effect algebra is sharply dominating. Phys. Letter A., **373**, (2009), 1708-1712.

[24] Sh. Jun and W. Junde, Remarks on the sequential effect algebras,Report. Math. Phys, **63**, (2009), 441-446.

[25] Sh. Jun and W. Junde, Sequential product on standard effect algebra E(H), J. Phys. A: Math. Theor, **44**, (2009).

[26] W. Jia-Mei, W. JunDe and Ch. Minhyung, Mutual information and relative entropy of sequential effect algebras, Theor. Phys. (Beijing, China), **54**, (2010), 215-218.

[27] J. Wang, J. Wu and M. Cho, Entropy of partitions on sequential efect algebras, Communications in Theoretical Physics, **53**, (2010), 399-402.

[28] Y. Zhao and Z. Ma, Conditional entropy of partitions on quantum logic, Communications in Theoretical Physics, **48**, (2007), 11-13.