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Further Results on (Δ_s^j, f)-Lacunary Statistical Convergence of Double Sequences of order α

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Abstract: This paper defines the space $S_{\theta_{uv}}^{\alpha}(\Delta_s^j, f)$, encompassing all sequences that are (Δ_s^j, f)-lacunary statistically convergent of order α, utilizing an unbounded modulus function f, a double lacunary sequence $\theta_{uv} = \{(k_u, l_v)\}$, a generalized difference operator Δ_s^j, and a real number $\alpha \in (0,1]$. Additionally, the space $\omega_{\theta_{uv}}^{\alpha}(\Delta_s^j, f)$ is introduced to include all sequences that are strongly (Δ_s^j, f)-lacunary summable of order α. The paper investigates properties associated with these spaces, and under specific conditions, inclusion relations between the spaces $S_{\theta_{uv}}^{\alpha}(\Delta_s^j, f)$ and $\omega_{\theta_{uv}}^{\alpha}(\Delta_s^j, f)$ are established.

Key words: Lacunary sequences, Lacunary statistical convergence, Modulus function

1. Introduction

The extension of the notion of convergence for real sequences to statistical convergence was undertaken by Fast [1]. In [2], the connection between the space $|\sigma_1|$ of strongly Cesàro summable sequences and the space N_θ of strongly lacunary summable sequences, defined by a lacunary sequence θ, was established by Freedman et al.

For single sequences, Çolak [3] proposed the concepts of statistical convergence of order α and strongly p-Cesàro summability of order α. Lacunary statistical convergence was presented by Fridy and Orhan [4]. Recently, Şengül and Et [5] studied the concepts of lacunary statistical convergence of order α and strongly p-lacunary summability of order α.

Pringsheim introduced the idea of convergence for double sequences in [6]. Mursaleen and Edely [7] went on to expand this idea to statistical convergence. Patterson and Savaş [8] studied the concept of lacunary statistical convergence using the double lacunary sequence concept.

The double sequence $\theta_2 = \{(k_r, l_s)\}$ is termed double lacunary if there exist two increasing sequences of integers such that:

$$k_0 = 0, h_r = k_r - k_{k-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s, h_{r,s} = h_r \bar{h}_s$, the intervals determined by θ_2 are denoted as,

$$I_r = \{(k): k_{r-1} < k \leq k_r\},$$

$$I_s = \{(l): l_{s-1} < l \leq l_s\},$$

$$I_{r,s} = \{(k,l): k_{r-1} < k \leq k_r \& l_{s-1} < l \leq l_s\},$$

$$q_r = \frac{k_r}{k_{r-1}}, \bar{q}_s = \frac{l_s}{l_{s-1}}, \text{ and } q_{r,s} = q_r \bar{q}_s.$$

For double sequences, the notion of statistical convergence of order α was recently introduced by Çolak and Altın [9].

The difference operator Δ for ℓ_∞, c and c_0 was originally presented by Kizmaz [10]. Subsequently, the concept of the difference operator Δ was further expanded to a fixed positive integer j by Et and Çolak [11], who defined it in the following manner:

$$Y(\Delta^j) = \{w = (w_k) \in \omega : \Delta^j w \in Y\} \text{ for } Y = \ell_\infty, c \text{ and } c_0,$$

where $\Delta^j w = (\Delta^j w_k) = (\Delta^{j-1} w_k - \Delta^{j-1} w_{k+1})$ for $j \geq 1$ and $\Delta^0 w = (w_k)$.

Additionally, the expansion of this space was undertaken by Et and Esi [12], who introduced the sequence $s = (s_k)$ comprising non-zero complex numbers. The sequence space $Y(\Delta_s^j)$ was defined by them in the following manner:

$$Y(\Delta_s^j) = \{w = (w_k) \in \omega : \Delta_s^j w \in X\} \text{ for } Y = \ell_\infty, c \text{ and } c_0,$$

where

$$\Delta_s^0 w_k = (s_k w_k)$$

and

$$\Delta_s^j w_k = \sum_{m=0}^{j} (-1)^m \binom{j}{m} s_{k+m} w_{k+m}$$

for $j \geq 1$.

Using the operator Δ^j, Tripathy and Et [13] proposed the notion of Δ^j-lacunary statistical convergent sequences and Δ^j-lacunary strongly summable sequences in 2005. The concept of the difference operator has been explored from diverse perspectives over the years by several authors [14-17].

Remember that a function $f: [0, \infty) \to [0, \infty)$ is considered a modulus function if and only if (a) $f(t) = 0$ iff $t = 0$, (b) $f(t + q) \leq f(t) + f(q)$, (c) f is increasing, and (d) f is continuous from the right at 0. This is in agreement with Maddox [18].

In 2014, the generalization of the concept of natural density was undertaken by Aizpuru et al. [19], who introduced the f-density of a subset K of positive integers utilizing an unbounded modulus function f. The expression for the f-density is given as

$$\delta^f(T) = \lim_{r\to\infty} \frac{f(|\{p \in T: p \leq r\}|)}{f(r)},$$

provided the limit exists.

The concepts of f-density and f-statistical convergence of order α were introduced by Bhardwaj and Dhawan [20]. Moreover, f-lacunary statistical convergence and strong f-lacunary summability of order α were defined by Şengül and Et [21]. In 2019, the $\Delta_s^j(f)$-statistical convergence of order α was introduced by Et and Gidemen [6]. We recommend [22-26] for more recent findings on ideal convergence via modulus function and related findings.

2. Material and Method

A sequence $w = (w_{kl})$ is defined to be $\Delta_s^j(f)$-statistical convergent of order α (sco α) to w_0 for all $\rho > 0$,

$$\lim_{m,n\to\infty} \frac{1}{f((mn)^\alpha)} f\left(\left|\left\{k \leq m, l \leq n: \left|\Delta_s^j w_{kl} - w_0\right| \geq \rho\right\}\right|\right) = 0.$$

This class of sequences, characterized by $\left(\Delta_s^j, f\right)$- sco α is demonstrated by $S^\alpha\left(\Delta_s^j, f\right)$.

Let $\theta_{uv} = \{(k_u, l_v)\}$ be a lacunary sequence and $\alpha \in (0,1]$. A sequence $w = (w_{kl})$ is considered to be $\left(\Delta_s^j, f\right)$-lacunary statistical convergent of order α (lsco α) to w_0 provided that for all $\rho > 0$,

$$\lim_{u,v\to\infty} \frac{1}{f(h_{uv}^\alpha)} f\left(\left|\left\{(k,l) \in I_{uv}: \left|\Delta_s^j w_{kl} - w_0\right| \geq \rho\right\}\right|\right) = 0.$$

In this scenario, we denote this convergence as $S_{\theta_{uv}}^\alpha\left(\Delta_s^j, f\right) - \lim w_{kl} = w_0$.

$S_{\theta_{uv}}^\alpha\left(\Delta_s^j, f\right)$ demonstrates the set of every sequences exhibiting $\left(\Delta_s^j, f\right)$- lsco α.

A sequence $w = (w_{kl})$ is considered to be strongly $\left(\Delta_s^j, f\right)$ lacunary summable sequence of order α (lsso α) if there exists w_0 such that

$$\lim_{u,v\to\infty} \frac{1}{f(h_{uv}^\alpha)} \sum_{(k,l)\in I_{uv}} f\left(\left|\Delta_s^j w_{kl} - w_0\right|\right) = 0.$$

In this context, the set encompassing every strongly $\left(\Delta_s^j, f\right)$-lsso α is indicated by $\omega_{\theta_{uv}}^\alpha\left(\Delta_s^j, f\right)$.

Furthermore, we define the space comprising all sequences that are strongly $\Delta_s^j(f)$-Cesàro summable of order α as:

$$\omega^{\alpha}\left(\Delta_s^j, f\right) = \left\{ w \in \omega : \lim_{n\to\infty} \frac{1}{f((mn)^{\alpha})} \sum_{k,l=1}^{m,n} f\left(\left|\Delta_s^j w_{kl} - w_0\right|\right) = 0, \text{ for some } w_0 \right\}.$$

In this paper, the objective is to generalize and consolidate well-established findings within the domain of lacunary statistical convergence and the f-statistical convergence of double sequences. In this study, the $\left(\Delta_s^j, f\right)$-lacunary statistical convergence of order α has been defined, and the space $S_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right)$ has been introduced with the assistance of the lacunary sequence $\theta_{uv} = \{(k_u, l_v)\}$, the generalized difference operator Δ_s^j, and the unbounded modulus function (ubmf) f. The space $\omega_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right)$, encompassing every strongly $\left(\Delta_s^j, f\right)$-lacunary summable double sequences of order α, has also been defined.

For the sake of this study, we assume that j is a constant positive integer, $\theta_{uv} = \{(k_u, l_v)\}$ is a double lacunary sequence, and $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha \leq \beta \leq 1$, (s_k) represents a fixed sequence of non-zero complex numbers, and the modulus function f is unbounded.

3. Results

Now we can start this section by giving our first result.

Theorem 3.1. Consider $w = (w_{kl})$ and $t = (t_{kl})$ as two arbitrary sequences. Then, followings hold:

1. If $S_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right) - \lim w_{kl} = w_0$ and $q \in \mathbb{C}$, then $S_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right) - \lim q w_{kl} = q w_0$.
2. If $S_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right) - \lim w_{kl} = w_0$ and $S_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right) - \lim t_{kl} = t_0$, then $S_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right) - \lim(w_{kl} + t_{kl}) = w_0 + t_0$.

Proof. Omitted.

Theorem 3.2. Subsequent inclusions are valid for any α, β with $0 < \alpha \leq \beta \leq 1$:

1. $S_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right) \subseteq S_{\theta_{uv}}^{\beta}\left(\Delta_s^j, f\right)$,
2. $\omega_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right) \subseteq \omega_{\theta_{uv}}^{\beta}\left(\Delta_s^j, f\right)$.

Proof. The demonstration for both parts is derived by observing that, due to the rising characteristic of the modulus function f. $f(h_{uv}^{\alpha}) \leq f\left(h_{uv}^{\beta}\right)$ for each $u, v \in \mathbb{N}$ if $0 < \alpha \leq \beta \leq 1$.

Theorem 3.3. Consider the lacunary sequence $\theta_{uv} = \{(k_u, l_v)\}$ such that $\liminf q_u > 1$ and $\liminf q_v > 1$. Let f be an ubmf, and there exists a positive constant c such that for each $\geq 0, z \geq 0$, $f(yz) \geq c f(y) f(z)$. Then, followings hold:

1. $S^{\alpha}\left(\Delta_s^j, f\right) \subseteq S_{\theta_{uv}}^{\beta}\left(\Delta_s^j, f\right)$,
2. $\omega^{\alpha}\left(\Delta_s^j, f\right) \subseteq \omega_{\theta_{uv}}^{\beta}\left(\Delta_s^j, f\right)$.

Proof. Presume that $\liminf q_u > 1$ and $\liminf q_v > 1$. So, there are $\kappa > 0$ and $\zeta > 0$ such that $q_u > 1 + \kappa$ and $q_v > 1 + \zeta$ for all u, v, which gives that

$$\frac{h_{uv}}{k_{uv}} \geq \frac{\kappa\zeta}{(1+\kappa)(1+\zeta)},$$

where $k_{uv} = k_u l_v$. Then, we obtain

$$h_{uv}^{\alpha} \geq \frac{(\kappa\zeta)^{\alpha}}{(1+\kappa)^{\alpha}(1+\zeta)^{\alpha}}(k_u l_v)^{\alpha} \Rightarrow f(h_{uv}^{\alpha}) \geq f\left(\frac{(\kappa\zeta)^{\alpha}}{(1+\kappa)^{\alpha}(1+\zeta)^{\alpha}}(k_u l_v)^{\alpha}\right),$$

as f is increasing.

This means

$$f(h_{uv}^{\alpha}) \geq cf\left(\frac{(\kappa\zeta)^{\alpha}}{(1+\kappa)^{\alpha}(1+\zeta)^{\alpha}}\right) f((k_u l_v)^{\alpha}),$$

due to assumption $f(yz) \geq cf(y)f(z)$.

Due to the increasing nature of the modulus function f and the fact that $\alpha \leq \beta$, it follows that $f\left(h_{uv}^{\beta}\right) \geq f(h_{uv}^{\alpha})$. This implies

$$f\left(h_{uv}^{\beta}\right) \geq cf\left(\frac{(\kappa\zeta)^{\alpha}}{(1+\kappa)^{\alpha}(1+\zeta)^{\alpha}}\right) f((k_u l_v)^{\alpha}). \quad (1)$$

1. Let $w \in S^{\alpha}\left(\Delta_s^j, f\right)$. By leveraging the monotonically increasing nature of the modulus function f and inequality (1), we can derive the following conclusion:

$$\begin{aligned}
&\frac{1}{f((k_u l_v)^{\alpha})} f\left(\left|\left\{k \leq k_u, l \leq l_v : \left|\Delta_s^j w_{kl} - w_0\right| \geq \rho\right\}\right|\right) \\
&\quad \geq \frac{1}{f((k_u l_v)^{\alpha})} f\left(\left|\left\{(k,l) \in I_{uv} : \left|\Delta_s^j w_{kl} - w_0\right| \geq \rho\right\}\right|\right) \\
&\quad \geq cf\left(\frac{(\kappa\zeta)^{\alpha}}{(1+\kappa)^{\alpha}(1+\zeta)^{\alpha}}\right) \frac{1}{f\left(h_{uv}^{\beta}\right)} f\left(\left|\left\{(k,l) \in I_{uv} : \left|\Delta_s^j w_{kl} - w_0\right| \geq \rho\right\}\right|\right).
\end{aligned}$$

We find $w \in S_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right)$ when we take the limit as $u, v \to \infty$, which results in the inclusion.

2. Let $w \in \omega^{\alpha}\left(\Delta_s^j, f\right)$. Then, according to the inequality (1), we get

$$\begin{aligned}
&\frac{1}{f((k_u l_v)^{\alpha})} \sum_{k \leq k_u, l \leq l_v} f\left(\left|\Delta_s^j w_{kl} - w_0\right|\right) \\
&\quad \geq cf\left(\frac{(\kappa\zeta)^{\alpha}}{(1+\kappa)^{\alpha}(1+\zeta)^{\alpha}}\right) \frac{1}{f\left(h_{uv}^{\beta}\right)} \sum_{(k,l) \in I_{uv}} f\left(\left|\Delta_s^j w_{kl} - w_0\right|\right).
\end{aligned}$$

The required inclusion can be obtained by taking the limit as $u, v \to \infty$ on both sides.

Theorem 3.4. Consider the lacunary sequence $\theta_{uv} = \{(k_u, l_v)\}$ such that $\limsup_u q_u^{\alpha} < \infty$, $\limsup_v q_v^{\alpha} < \infty$, and an ubmf f such that for $c > 0$, $f(r) \leq cr$ for all $r \geq 0$. Then, followings hold:

1. $S_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right) \subseteq S^{\beta}\left(\Delta_s^j, f\right)$,
2. $\omega_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right) \subseteq \omega^{\beta}\left(\Delta_s^j, f\right)$.

Proof. (i) Suppose that $\limsup_u q_u^\alpha < \infty$, $\limsup_v q_v^\alpha < \infty$. So, there exists $T > 0$ so that $q_u^\alpha < T$ and $q_v^\alpha < T$ for $\forall u, v \in \mathbb{N}$. Let $w \in S_{\theta_{uv}}^\alpha(\Delta_s^j, f)$. For any $\rho > 0$, there are $u_0, v_0 \in \mathbb{N}$ so that

$$\frac{1}{f(h_{uv}^\alpha)} f\left(\left|\left\{(k,l) \in I_{uv} : \left|\Delta_s^j w_{kl} - w_0\right| \geq \rho\right\}\right|\right) < \rho$$

for all $u > u_0, v > v_0$,

$$\Rightarrow \frac{N_{uv}}{f(h_{uv}^\alpha)} < \rho \text{ for all } u > u_0, v > v_0,$$

where

$$N_{uv} = f\left(\left|\left\{(k,l) \in I_{uv} : \left|\Delta_s^j w_{kl} - w_0\right| \geq \rho\right\}\right|\right).$$

Let

$$M := \max\left\{N_{uv} : 1 \leq u \leq u_0 \text{ and } 1 \leq v \leq v_0\right\}.$$

Let n and m be such that $k_{u-1} < m \leq k_u$ and $l_{v-1} < n \leq l_v$. Therefore, we deduce the following:

$$\begin{aligned}
\frac{1}{f((mn)^\beta)} f\left(\left|\left\{k \leq m \text{ and } l \leq n : \left|\Delta_s^j w_{kl} - w_0\right| \geq \rho\right\}\right|\right) &\leq \frac{1}{f((k_{u-1} l_{v-1})^\alpha)} \Big| \{k \leq k_u \text{ and } l \\
\leq l_v : \left|\Delta_s^j w_{kl} - w_0\right| \geq \rho\} &= \frac{1}{f((k_{u-1} l_{v-1})^\alpha)} \left\{\sum_{i,j=1,1}^{r,s} N_{i,j}\right\} \\
&\leq \frac{M u_0 v_0}{f((k_{u-1} l_{v-1})^\alpha)} + \frac{1}{f((k_{u-1} l_{v-1})^\alpha)} \left\{\sum_{i,j=u_0+1,v_0+1}^{u,v} N_{i,j}\right\} \\
&\leq \frac{M u_0 v_0}{f((k_{u-1} l_{v-1})^\alpha)} + \frac{1}{f((k_{u-1} l_{v-1})^\alpha)} \left\{\sum_{i,j=u_0+1,v_0+1}^{u,v} \frac{N_{i,j} f(h_{i,j}^\alpha)}{f(h_{i,j}^\alpha)}\right\} \\
&\leq \frac{M u_0 v_0}{f((k_{u-1} l_{v-1})^\alpha)} + \frac{1}{f((k_{u-1} l_{v-1})^\alpha)} \left(\sup_{i,j \geq u_0, v_0} \frac{N_{i,j}}{f(h_{i,j}^\alpha)}\right) c. \left\{\sum_{i,j=u_0+1,v_0+1}^{u,v} h_{i,j}^\alpha\right\} \\
&\leq \frac{M u_0 v_0}{f((k_{u-1} l_{v-1})^\alpha)} + \rho. c. \left\{\sum_{i,j=u_0+1,v_0+1}^{u,v} h_{i,j}^\alpha\right\} \leq \frac{M u_0 v_0}{f((k_{u-1} l_{v-1})^\alpha)} + \rho. c. T^2.
\end{aligned}$$

The inclusion follows by taking the limit in the above inequality.

(ii) Since the evidence is identical to part (i), it is omitted.

Theorem 3.5. Let $\theta_{uv} = \{(k_u, l_v)\}$ be a double lacunary sequence such that $\lim_{u,v\to\infty} \frac{f\left(h_{uv}^\beta\right)}{f((k_u l_v)^\alpha)} > 0$ and $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha \leq \beta \leq 1$. Then, followings hold:

i. $S^\alpha\left(\Delta_s^j, f\right) \subseteq S_{\theta_{uv}}^\beta\left(\Delta_s^j, f\right)$,

ii. $\omega^\alpha\left(\Delta_s^j, f\right) \subseteq \omega_{\theta_{uv}}^\beta\left(\Delta_s^j, f\right)$.

Proof. (i) Let $w \in S^\alpha\left(\Delta_s^j, f\right)$. Then

$$\lim_{m,n\to\infty} \frac{1}{f((mn)^{\alpha})} f\left(\left|\left\{k \le m, l \le n: \left|\Delta_s^j w_{kl} - w_0\right| \ge \rho\right\}\right|\right) = 0,$$

which implies

$$\lim_{u,v\to\infty} \frac{1}{f((k_u l_v)^{\alpha})} f\left(\left|\left\{k \le k_u, l \le l_v: \left|\Delta_s^j w_{kl} - w_0\right| \ge \rho\right\}\right|\right) = 0.$$

Take $\rho > 0$. Then, for all u, v,

$$\left\{(k,l) \in I_{uv}: \left|\Delta_s^j w_{kl} - w_0\right| \ge \rho\right\} \subseteq \left\{k \le k_u \text{ and } l \le l_v: \left|\Delta_s^j w_{kl} - w_0\right| \ge \rho\right\}.$$

Through the increasing property of the modulus function, it is obtained that

$$f\left(\left|\left\{(k,l) \in I_{uv}: \left|\Delta_s^j w_{kl} - w_0\right| \ge \rho\right\}\right|\right) \le f\left(\left|\left\{k \le k_u \text{ and } l \le l_v: \left|\Delta_s^j w_{kl} - w_0\right| \ge \rho\right\}\right|\right)$$
$$\Rightarrow \frac{f\left(h_{uv}^{\beta}\right)}{f((k_u l_v)^{\alpha})} \frac{1}{f\left(h_{uv}^{\beta}\right)} f\left(\left|\left\{(k,l) \in I_{uv}: \left|\Delta_s^j w_{kl} - w_0\right| \ge \rho\right\}\right|\right)$$
$$\le \frac{1}{f((k_u l_v)^{\alpha})} f\left(\left|\left\{k \le k_u \text{ and } l \le l_v: \left|\Delta_s^j w_{kl} - w_0\right| \ge \rho\right\}\right|\right).$$

By taking the limit as $u, v \to \infty$ on both sides and utilizing the given assumption, the inclusion is established.

(ii) The inclusion is followed in the same manner as in part (i) by being aware that

$$\sum_{(k,l)\in I_{uv}} f\left(\left|\Delta_s^j w_{kl} - w_0\right|\right) \le \sum_{k\le k_u \text{ and } l\le l_v} f\left(\left|\Delta_s^j w_{kl} - w_0\right|\right),$$

holds for any $u, v \in \mathbb{N}$.

Theorem 3.6. For real numbers α and β with $0 < \alpha \le \beta \le 1$, the inclusion

$$S_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right) \subseteq S_{\theta_{uv}}^{\beta}\left(\Delta_s^j\right)$$

holds.

Proof. Let $w \in S_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right)$. For all $r \in \mathbb{N}$, there is a $n_0 \in \mathbb{N}$ such that whenever $n \ge n_0$, we have

$$\frac{1}{f(h_{uv}^{\alpha})} f\left(\left|\left\{(k,l) \in I_{uv}: \left|\Delta_s^j w_{kl} - w_0\right| \ge \rho\right\}\right|\right) \le \frac{1}{r}$$
$$\Rightarrow f\left(\left|\left\{(k,l) \in I_{uv}: \left|\Delta_s^j w_{kl} - w_0\right| \ge \rho\right\}\right|\right) \le \frac{1}{r} f\left(\frac{r \cdot h_{uv}^{\alpha}}{r}\right).$$

Utilizing the subadditive property of the modulus function f, we obtain

$$f\left(\left|\left\{(k,l)\in I_{uv}:\left|\Delta_s^j w_{kl}-w_0\right|\geq\rho\right\}\right|\right)\leq\frac{1}{r}rf\left(\frac{h_{uv}^{\alpha}}{r}\right)$$
$$\Rightarrow\ \left|\left\{(k,l)\in I_{uv}:\left|\Delta_s^j w_{kl}-w_0\right|\geq\rho\right\}\right|$$
$$\leq\frac{h_{uv}^{\alpha}}{r},\text{ as } f \text{ is an increasing function}$$
$$\Rightarrow\ \frac{1}{h_{uv}^{\beta}}\left|\left\{(k,l)\in I_{uv}:\left|\Delta_s^j w_{kl}-w_0\right|\geq\rho\right\}\right|\leq\frac{1}{r}.$$

Taking the limit as $u,v\to\infty$, we get $w\in S_{\theta_{uv}}^{\beta}\left(\Delta_s^j\right)$.

Lemma 3.1. [18] For any modulus function f, $\lim_{q\to\infty}\frac{f(q)}{q}=\inf\left\{\frac{f(q)}{q}:q>0\right\}$.

Theorem 3.7. Let's suppose there exists a modulus function f so that $\lim_{q\to\infty}\frac{f(q)}{q}>0$. Then,

$$\omega_{\theta_{uv}}^{\alpha}\left(\Delta_s^j,f\right)\subseteq\omega_{\theta_{uv}}^{\beta}\left(\Delta_s^j\right).$$

Proof. Let $w\in\omega_{\theta_{uv}}^{\alpha}\left(\Delta_s^j,f\right)$ and $\lim_{q\to\infty}\frac{f(q)}{q}=\sigma$. Then, according to Lemma 3.1 $\sigma=\inf\left\{\frac{f(q)}{q}:q>0\right\}$. This implies $q\leq\sigma^{-1}f(q)$ for $\forall q\geq 0$. Now,

$$\frac{1}{f\left(h_{uv}^{\beta}\right)}\sum_{(k,l)\in I_{uv}}\left|\Delta_s^j w_{kl}-w_0\right|\leq\frac{1}{f\left(h_{uv}^{\beta}\right)}\sum_{(k,l)\in I_{uv}}\sigma^{-1}f\left(\left|\Delta_s^j w_{kl}-w_0\right|\right)$$
$$\leq\frac{\sigma^{-1}}{f\left(h_{uv}^{\alpha}\right)}\sum_{(k,l)\in I_{uv}}f\left(\left|\Delta_s^j w_{kl}-w_0\right|\right).$$

Taking the limit as $u,v\to\infty$, results in the inclusion.

Theorem 3.8. Given an unbounded modulus function, denoted as f, and assuming it satisfies the inequality $f(yz)\geq cf(y)f(z)$ for some positive fixed c. Then,

$$\omega_{\theta_{uv}}^{\alpha}\left(\Delta_s^j,f\right)\subseteq S_{\theta_{uv}}^{\beta}\left(\Delta_s^j,f\right).$$

Proof. Assume $w\in\omega_{\theta_{uv}}^{\alpha}\left(\Delta_s^j,f\right)$ and $\rho>0$. Using the modulus function's increasing and subadditive properties, we arrive at

$$\frac{1}{f\left(h_{uv}^{\alpha}\right)}\sum_{(k,l)\in I_{uv}}f\left(\left|\Delta_s^j w_{kl}-w_0\right|\right)\geq\frac{1}{f\left(h_{uv}^{\alpha}\right)}f\left(\sum_{(k,l)\in I_{uv}}\left|\Delta_s^j w_{kl}-w_0\right|\right)$$
$$\geq\frac{1}{f\left(h_{uv}^{\beta}\right)}f\left(\sum_{(k,l)\in I_{uv}\&\left|\Delta_s^j w_{kl}-w_0\right|\geq\rho}\left|\Delta_s^j w_{kl}-w_0\right|\right)$$
$$\geq\frac{1}{f\left(h_{uv}^{\beta}\right)}f\left(\left|\left\{(k,l)\in I_{uv}:\left|\Delta_s^j w_{kl}-w_0\right|\geq\rho\right\}\right|\rho\right)$$
$$\geq\frac{c}{f\left(h_{uv}^{\beta}\right)}f\left(\left|\left\{(k,l)\in I_{uv}:\left|\Delta_s^j w_{kl}-w_0\right|\geq\rho\right\}\right|\right)f(\rho),$$

utilizing $f(yz) \geq cf(y)f(z)$.

Taking the limit as $u,v \to \infty$, we get $w \in S_{\theta_{uv}}^{\beta}(\Delta_s^j, f)$.

Theorem 3.9. Assume that the modulus function f and the lacunary sequence $\theta_{uv} = \{(k_u, l_v)\}$ satisfy $\lim_{q\to\infty} \frac{f(q)}{q} > 0$ and $\lim_{u,v\to\infty} \frac{f(h_{uv})}{f(h_{uv}^{\alpha})} = 1$. Then, the inclusion

$$S_{\theta_{uv}}^{\alpha}(\Delta_s^j, f) \cap \ell_\infty(\Delta_s^j) \subseteq \omega_{\theta_{uv}}^{\beta}(\Delta_s^j, f) \cap \ell_\infty(\Delta_s^j),$$

holds.

Proof. Suppose $\lim_{q\to\infty} \frac{f(q)}{q} = \sigma$. According to Lemma 3.1 $\sigma = \inf\left\{\frac{f(q)}{q} : q > 0\right\}$, which implies $q \leq \sigma^{-1} f(q)$ for all $q \geq 0$. Now, let $w \in S_{\theta_{uv}}^{\alpha}(\Delta_s^j, f) \cap \ell_\infty(\Delta_s^j)$, meaning there exists $U > 0$ such that $\left|\Delta_s^j w_{kl} - w_0\right| \leq U$ for all $k,l \in \mathbb{N}$. For all $\varepsilon > 0$, consider Σ_1 and Σ_2 as the sums over $(k,l) \in I_{uv}$ $\left|\Delta_s^j w_{kl} - w_0\right| \geq \rho$ and $(k,l) \in I_{uv}$, $\left|\Delta_s^j w_{kl} - w_0\right| < \rho$, respectively. Now,

$$\begin{aligned}
&\frac{1}{f\left(h_{uv}^{\beta}\right)} \sum_{(k,l)\in I_{uv}} f\left(\left|\Delta_s^j w_{kl} - w_0\right|\right) \\
&\leq \frac{1}{f(h_{uv}^{\alpha})} \left(\sum_{1} f\left(\left|\Delta_s^j w_{kl} - w_0\right|\right) + \sum_{2} f\left(\left|\Delta_s^j w_{kl} - w_0\right|\right) \right) \\
&\leq \frac{1}{f(h_{uv}^{\alpha})} \sum_{1} f(U) + \frac{1}{f(h_{uv}^{\alpha})} \sum_{2} f(\rho) \\
&\leq \frac{1}{f(h_{uv}^{\alpha})} \left|\left\{(k,l) \in I_{uv} : \left|\Delta_s^j w_{kl} - w_0\right| \geq \rho\right\}\right| f(U) + \frac{h_{uv}}{f(h_{uv}^{\alpha})} f(\rho) \\
&\leq \frac{\sigma^{-1}}{f(h_{uv}^{\alpha})} f\left(\left|\left\{(k,l) \in I_{uv} : \left|\Delta_s^j w_{kl} - w_0\right| \geq \rho\right\}\right|\right) f(U) + \frac{\sigma^{-1} f(h_{uv})}{f(h_{uv}^{\alpha})} f(\rho).
\end{aligned}$$

By taking the limit as $u,v \to \infty$ and utilizing $\lim_{u,v\to\infty} \frac{f(h_{uv})}{f(h_{uv}^{\alpha})} = 1$, the inclusion is established.

The lacunary sequence $\theta'_{uv} = \{k'_u, l'_v\}$ is known as lacunary refinement of lacunary sequence $\theta_{uv} = \{k_u, l_v\}$ if $\{k_u, l_v\} \subseteq \{k'_u, l'_v\}$. In this case $I_{uv} \subseteq I'_{uv}$ where $I'_{uv} = \{(k', l') : k'_{u-1} < k' \leq k'_u,\ l'_{v-1} < l' \leq l'_v\}$.

Theorem 3.10. Let $\theta'_{uv} = \{k'_u, l'_v\}$ be lacunary refinement of $\theta_{uv} = \{k_u, l_v\}$ and $0 < \alpha \leq \beta \leq 1$. Additionally, suppose lacunary sequences $\theta_{uv} = \{(k_u, l_v)\}$, $\theta'_{uv} = \{k'_u, l'_v\}$ and a modulus function f satisfy $\liminf \frac{f\left(h_{uv}^{\beta}\right)}{f(l_{uv}^{\alpha})} > 0$. Then, followings hold:

1. $S_{\theta'_{uv}}^{\alpha}(\Delta_s^j, f) \subseteq S_{\theta_{uv}}^{\beta}(\Delta_s^j, f)$,
2. $\omega_{\theta'_{uv}}^{\alpha}(\Delta_s^j, f) \subseteq \omega_{\theta_{uv}}^{\beta}(\Delta_s^j, f)$.

Proof. (i) Consider $w \in S_{\theta'_{uv}}^{\alpha}(\Delta_s^j, f)$. As $I_{uv} \subseteq I'_{uv}$ for all $u,v \in \mathbb{N}$, so for any $\rho > 0$, we can conclude that

$$\{(k,l) \in I_{uv}, |\Delta_s^j w_{kl} - w_0| \geq \rho\} \subseteq \{(k,l) \in I'_{uv}: |\Delta_s^j w_{kl} - w_0| \geq \rho\},$$

$$\Rightarrow f\left(\left|\{(k,l) \in I_{uv}: |\Delta_s^j w_{kl} - w_0| \geq \rho\}\right|\right) \leq f\left(\left|\{(k,l) \in I'_{uv}: |\Delta_s^j w_{kl} - w_0| \geq \rho\}\right|\right),$$

as f is increasing

$$\Rightarrow \frac{f\left(h_{uv}^{\beta}\right)}{f(l_{uv}^{\alpha})} \frac{1}{f\left(h_{uv}^{\beta}\right)} f\left(\left|\{(k,l) \in I_{uv}, |\Delta_s^j w_{kl} - w_0| \geq \rho\}\right|\right) \leq \frac{1}{f(l_{uv}^{\alpha})} f\left(\left|\{(k,l) \in I'_{uv}: |\Delta_s^j w_{kl} - w_0| \geq \rho\}\right|\right)$$

for all $u, v \in \mathbb{N}$.

By letting $u, v \to \infty$ and utilizing $\liminf \frac{f\left(h_{uv}^{\beta}\right)}{f(l_{uv}^{\alpha})} > 0$, the desired inclusion follows.

(ii) Take $w \in \omega_{\theta'_{uv}}^{\alpha}\left(\Delta_s^j, f\right)$. Inclusion ensues through taking the limit and employing the provided assumption in the subsequent inequality

$$\frac{1}{f(l_{uv}^{\alpha})} \sum_{(k,l)\in I'_{uv}} f\left(|\Delta_s^j w_{kl} - w_0|\right) \geq \frac{f\left(h_{uv}^{\beta}\right)}{f(l_{uv}^{\alpha})} \frac{1}{f\left(h_{uv}^{\beta}\right)} \sum_{(k,l)\in I_{uv}} f\left(|\Delta_s^j w_{kl} - w_0|\right).$$

4. Conclusion

In conclusion, this paper has introduced and explored the spaces $S_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right)$ and $\omega_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right)$, which respectively encompass sequences that exhibit $\left(\Delta_s^j, f\right)$-lacunary statistical convergence of order α and strong $\left(\Delta_s^j, f\right)$-lacunary summability of order α. The definitions of these spaces involved the use of an unbounded modulus function f, a lacunary sequence $\theta_{uv} = \{(k_u, l_v)\}$, a generalized difference operator Δ_s^j, and a real number $\alpha \in (0,1]$. The properties of these spaces were thoroughly examined, including the establishment of inclusion relations under specific conditions.

The significance of this work lies in the extension and unification of concepts in the realm of statistical convergence. By introducing and studying these spaces, we provide a comprehensive framework for understanding the behavior of sequences that exhibit certain convergence properties. The spaces $S_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right)$ and $\omega_{\theta_{uv}}^{\alpha}\left(\Delta_s^j, f\right)$ offer valuable insights into the convergence patterns of sequences, facilitating a deeper understanding of their statistical and summability characteristics. This research contributes to the broader field of mathematical analysis, providing new tools and perspectives for studying the convergence behavior of sequences in various contexts.

Authorship contribution statement

Ö. Kişi: Conceptualization, Methodology; **R. Akbıyık**: Data Curation, Original Draft Writing; **M. Gürdal**: Visualization, Supervision/Observation/Advice.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Ethics Committee Approval and/or Informed Consent Information

As the authors of this study, we declare that we do not have any ethics committee approval and/or informed consent statement.

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