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Some Golden Objects in Geometry

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Abstract: In this paper, first, an induced algebra with respect to the polynomial $P(x)$ is defined and then, an induced Lie group with respect to $P(x)$ is determined. Finally, Golden Algebras, Golden Lie groups, Golden curves and Golden surfaces are introduced based on the definition of generalized Golden polynomials.

Keywords: Polynomial, algebra, Lie group, generalized golden polynomial, golden algebra, golden Lie group, golden curve, golden surface.

1. Introduction and Preliminaries

The Golden Proportion, also called the Golden Ratio, Divine Ratio, Golden Section or Golden Mean has been well known since the time of Euclid. There are many objects in the natural world that possess pentagonal symmetry, such as the inflorescence of many flowers and phyllotactic objects that have a numerical description given by the Fibonacci numbers, which are themselves based on the Golden Proportion. The Golden Proportion has also been found in the structure of musical compositions, in the ratios of harmonious sound frequencies and in dimensions of the human body [8].

Let us recall that the Golden Proportion partitions a line segment into a major subsegment and a minor subsegment in such a way that both the ratio of whole segment and the major subsegment and the ratio of major subsegment and the minor subsegment must equal the number $\varphi = \dfrac{1+\sqrt{5}}{2}$ (the Phidias number), which is the real positive root of the equation $x^2 - x - 1 = 0$. This number is often encountered when taking the ratios of distances in simple geometric figures such as the pentagram, decagon and dodecagon. In the last few years, the Golden Proportion has played a growing role in modern physics research [1, 2, 5]. The Golden Proportion also has interesting properties in Special Relativity [4]. Although the generalization of mathematics is important in its own right, the generalization of Golden numbers, which belong to the Lie group $\mathbb{R}$ and which are essential in the symmetries and beauty of natural phenomena to other sciences, especially mathematics, can produce other symmetric and beauties in them. For instance, Golden numbers appear in the representation of the finite dimensional Lie groups which have a fully physical application

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[7]. This thought, together with examples which have been mentioned in the paper, can be an essential motivation to find and study geometric Golden objects. In this paper, first, by using a polynomial, an algebra is constructed and it is shown that the inverted elements of this algebra present a Lie group. Then, the generalized Golden polynomials $x^n - F_n x - F_{n-1}$ are defined where F_n is the Fibonacci sequence. Next, using these polynomials, Golden algebras, Golden Lie groups, Golden curve and Golden surface are introduced. Finally, some of their interesting properties are proved.

2. Main results

Definition 1. Let $R[x]$ be the algebra of all polynomials, and $P(x)$ be a polynomial of degree n, in $R[x]$. We define $\mathscr{A}_P$, called the induced algebra with respect to $P(x)$, to be the set of all polynomials of degree less than n together with the addition and scalar product induced by $R[x]$. And the multiplication in $\mathscr{A}_P$ is defined in such a way that $\mathscr{A}_P$ is isomorphic to the quotient algebra $R[x]/ < P(x) >$.

Let A be a finite dimensional associative algebra over $\mathbb{R}$ with identity. Then, the group G of invertible elements in A is an open submanifold of A and with induced structure, it is a Lie group (See for instance [9]). Thus, the invertible elements of $\mathscr{A}_P$ form an Abelian Lie group which is shown by $\mathscr{G}_P$ and it is called the induced Lie group with respect to polynomial P.

For convenience, throughout the paper, we use $(a_0, \ldots, a_{n-1})$, instead of $X = a_{n-1}x^{n-1} + \ldots + a_1x^1 + a_0$.

Theorem 1. The map $\varphi_Z : \mathscr{A}_P \to \mathscr{A}_P$ with $\varphi_Z(X) = Z.X$, is a linear transformation.

Proof. The proof is trivial. ■

Definition 2. (See for instance [6]) Let G be a Lie group and V be a vector space. A linear transformation group $\psi : G \times V \to V$ or equivalently, a homomorphism $\psi : G \to GL(V)$, is also called a linear representation of G on V.

Remark 2.1. We consider the matrix representation of φ_Z for all $Z \in \mathscr{A}_P$ in the standard basis of $\mathbb{R}^n$ in Theorem 1 and it is shown by $\mathscr{A}_Z(n,\mathbb{R})$. Indeed, $\mathscr{A}_Z(n,\mathbb{R})$ is the matrix representation for $Z \in \mathscr{A}_P$. Now, we consider $\mathscr{A}_P(n,\mathbb{R}) = \{\mathscr{A}_Z(n,\mathbb{R}) | Z \in \mathscr{A}_P\}$. For convenience, we use $\mathscr{A}_P(n,\mathbb{R})$ instead of $\mathscr{A}_P$.
The set of corresponding matrices with respect to $\mathscr{G}_P$ is a subgroup of $GL(n,\mathbb{R})$.

Example 2.1. Let $P(x) = x^3 - 1$. $X = \sum_{i=0}^{2} a_i x^i$ and $Y = \sum_{i=0}^{2} b_i x^i \in \mathscr{A}_{x^3-1}$. Then

$$\varphi_X(Y) = X.Y = a_0b_0 + a_1b_2 + a_2b_1 + (a_0b_1 + a_1b_0 + a_2b_2)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2.$$

The matrix representation of φ_X in the standard basis of $\mathbb{R}^3$ is as follows:

$$\begin{pmatrix} a_0 & a_2 & a_1 \\ a_1 & a_0 & a_2 \\ a_2 & a_1 & a_0 \end{pmatrix}.$$

Example 2.2. Let $(a,b) \in \mathscr{A}_{x^2-a_0x-b_0}$. The matrix representation of (a,b) in the standard basis of $\mathbb{R}^2$ is as follows:

$$A = \begin{pmatrix} a & bb_0 \\ b & a+ba_0 \end{pmatrix}. \tag{1}$$

In this case, we have

$$detA = a^2 + aba_0 - b^2b_0.$$

If x_1 and x_2 are the roots of $x^2 - a_0x - b_0 = 0$, then

$$detA = (a+bx_1)(a+bx_2).$$

Remark 2.2. In Example 2.2, if $a_0 = 0$ and $b_0 = -1$, then

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

A corresponds to the complex number $z = a+ib$. In general, for all $a,b \in \mathbb{R}$, the matrix $\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, will always represent the complex number $a+ib$.

Definition 3. The polynomial $P_n(x) = x^n - F_nx - F_{n-1}$ (F_n is the Fibonacci sequence) is called a generalized Golden polynomial of degree n.

Remark 2.3. The generalized Golden polynomial $P_n(x) = x^n - F_nx - F_{n-1}$ is decomposed as follows:

$$P_n(x) = (x^2 - x - 1)\Big(\sum_{i=0}^{n-2} F_i x^{n-i-2}\Big),$$

where $x^2 - x - 1$ is the Golden polynomial.

Definition 4. Let P_n be a generalized Golden polynomial of degree n. The algebra $\mathscr{A}_{P_n}$ is called a Golden algebra. And also $\mathscr{G}_{P_n}$ is called a Golden Lie group with respect to P_n.

Example 2.3. Let $x^2 - x - 1$ be the Golden polynomial. By (1) the matrix representation of $(a,b) \in \mathscr{A}_{x^2-x-1}$ in the standard basis of $\mathbb{R}^2$ is as follows:

$$A = \begin{pmatrix} a & b \\ b & a+b \end{pmatrix}. \tag{2}$$

Since $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ are the roots of $x^2-x-1=0$, we have

$$detA=\Big(a+(\frac{1+\sqrt{5}}{2})b\Big)\Big(a+(\frac{1-\sqrt{5}}{2})b\Big).$$

Thus,

$$\mathscr{A}_{x^2-x-1}=\{\begin{pmatrix} a & b \\ b & a+b \end{pmatrix} \mid a,b\in\mathbb{R}\},$$

and

$$\mathscr{G}_{x^2-x-1}=\Big\{A=\begin{pmatrix} a & b \\ b & a+b \end{pmatrix} \mid detA\neq 0\Big\}.$$

We can associate to a complex 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the Möbius transformation $f(z)=\dfrac{az+b}{cz+d}$ (See for instance [10]). According to the above lines, we have the following Lemmas:

Lemma 1. The graphs of all the Möbius transformations associated with the matrices

$$\begin{pmatrix} a & bb_0 \\ b & a+ba_0 \end{pmatrix}\in\mathscr{A}_{x^2-a_0x-b_0} \quad (\forall a,b\in\mathbb{R} \text{ and } b_0\neq 0), \tag{3}$$

intersect at the points $M=(-x_1,-x_1)$ and $N=(-x_2,-x_2)$ where x_1 and x_2 are the roots of $x^2-a_0x-b_0=0$.

Proof. Let $f_1(x)=\dfrac{ax+bb_0}{bx+a+ba_0}$ and $f_2(x)=\dfrac{a'x+b'b_0}{b'x+a'+b'a_0}$ be the Möbius transformations corresponding to $\begin{pmatrix} a & bb_0 \\ b & a+ba_0 \end{pmatrix}$ and $\begin{pmatrix} a' & b'b_0 \\ b' & a'+b'a_0 \end{pmatrix}$, respectively. Suppose $f_1(x)=f_2(x)$, then

$$(ab'-ba')(x^2+a_0x-b_0)=0. \tag{4}$$

Case 1: If $ab'-ba'=0$, then $(a,b)=\lambda(a',b')$. It implies that

$$\frac{ax+bb_0}{bx+a+ba_0}=\frac{a'x+b'b_0}{b'x+a'+b'a_0}.$$

Case 2: If $ab'-ba'\neq 0$ then $(x^2+a_0x-b_0)=0$. Let x_1 and x_2 be the roots of the equation $x^2-a_0x-b_0=0$. Then $x_1+x_2=-a_0$, and $x_1x_2=-b_0$. And it is trivial that $-x_1$ and $-x_2$ are the roots of the equation (4). ■

Lemma 2. The graphs of all Möbius transformations associated with the matrices

$$\begin{pmatrix} a & b \\ bb_0 & a+ba_0 \end{pmatrix} \quad (\forall a,b\in\mathbb{R} \text{ and } b_0\neq 0),$$

which are the transpose(s) of the matrices in (3), intersect at the points $P=(A,A)$ and $Q=(B,B)$ where $1/A$ and $1/B$ are the roots of $x^2-a_0x-b_0=0$.

Proof. Let $f_1(x)=\dfrac{ax+b}{bb_0x+a+ba_0}$ and $f_2(x)=\dfrac{a'x+b'}{b'b_0x+a'+b'a_0}$ be the Möbius transformations corresponding to $\begin{pmatrix} a & b \\ bb_0 & a+ba_0 \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ b'b_0 & a'+b'a_0 \end{pmatrix}$, respectively. Suppose $f_1(x)=f_2(x)$, then

$$(ab'-ba')(b_0x^2+a_0x-1)=0. \tag{5}$$

Case 1: If $ab'-ba'=0$, then $(a,b)=\lambda(a',b')$. Consequently,

$$\frac{ax+b}{bb_0x+a+ba_0}=\frac{a'x+b'}{b'b_0x+a'+b'a_0}.$$

Case 2: If $ab'-ba'\neq 0$, then $b_0x^2+a_0x-1=0$. Let x_1 and x_2 be the roots of the equation $x^2-a_0x-b_0=0$. Then $x_1+x_2=-a_0$ and $x_1x_2=-b_0$. Suppose $A=1/x_1$ and $B=1/x_2$. Since

$$A+B=a_0/b_0 \textit{ and } AB=-1/b_0,$$

A and B are the roots of the equation (5). ∎

Theorem 2. Let $(a,b)\in\mathscr{A}_{x^2-a_0x-b_0}$. Then the tangent lines to the graph of the Mobius transformation

$$\frac{ax+b}{bb_0x+a+ba_0},$$

at the points P and Q which are introduced in the Lemma 2, intersect at a point which is on the perpendicular bisector of the segment $\overline{PQ}$. Also asymptotes of the Mobius transformation intersect at a point which is on the perpendicular bisector of the segment $\overline{PQ}$.

Proof. One of the Möbius transformations which satisfies in Lemma 2 is $y=x$. The segment $\overline{PQ}$ coincides on $y=x$. The equation of the perpendicular bisector of the segment $\overline{PQ}$ is $x+y=-a_0/b_0$. The equations of the tangent lines to the graph of $\dfrac{ax+b}{bb_0x+a+ba_0}$ at the points $P=(A,A)$ and $Q=(B,B)$ are

$$y=\frac{a^2+aba_0-b^2b_0}{(bb_0A+a+ba_0)^2}(x-A)+\frac{aA+b}{bb_0A+a+ba_0},$$
$$y=\frac{a^2+aba_0-b^2b_0}{(bb_0B+a+ba_0)^2}(x-B)+\frac{aB+b}{bb_0B+a+ba_0}.$$

These tangent lines intersect at the point

$$C=\left(x_0=\frac{-aa_0-b(2b_0+a_0^2)}{(2a+ba_0)b_0},y_0=\frac{-aa_0+2bb_0}{(2a+ba_0)b_0}\right),$$

such that $x_0+y_0=-a_0/b_0$. So, (x_0,y_0) is on the perpendicular bisector of the segment $\overline{PQ}$. The asymptotes of $y=\dfrac{ax+b}{bb_0x+a+ba_0}$ are $x=-\dfrac{a+ba_0}{bb_0}$ *and* $y=\dfrac{a}{bb_0}$. These asymptotes intersect at the point

$$D=\Big(x_1=-\frac{a+ba_0}{bb_0},y_1=\frac{a}{bb_0}\Big),$$

such that $x_1+y_1=-a_0/b_0$. Thus, (x_1,y_1) is on the perpendicular bisector of the segment $\overline{PQ}$. ■

Corollary 1. The product of the areas of the triangles DPQ and CPQ in Theorem 2, is constant and equal to $(a_0^2+4b_0)^2/4(b_0)^4$.

Proof. The length of the segment $\overline{PQ}$ is as follows:

$$|\overline{PQ}|=\sqrt{2}\Big|\frac{\sqrt{a_0^2+4b_0}}{b_0}\Big|.$$

The altitudes of the triangles CPQ and DPQ are obtained as follows:

$$h_{CPQ}=\sqrt{2}/2\Big|\frac{4bb_0+ba_0^2}{(2a+ba_0)b_0}\Big|$$

and

$$h_{DPQ}=\sqrt{2}/2\Big|\frac{2a+ba_0}{bb_0}\Big|.$$

So, the areas of the triangles CPQ and DPQ are

$$A(CPQ)=1/2\frac{b(4b_0+a_0^2)\sqrt{4b_0+a_0^2}}{b_0^2(2a+ba_0)}$$

and

$$A(DPQ)=1/2\frac{2a+ba_0\sqrt{4b_0+a_0^2}}{bb_0^2}.$$

Thus, by multiplication we obtain

$$A(CPQ)A(DPQ)=(a_0^2+4b_0)^2/4(b_0)^4.$$

■

Definition 5. Let

$$C=SL(2,\mathbb{R})\bigcap\mathscr{G}_{x^2-x-1}=\Big\{A=\begin{pmatrix} a & b \\ b & a+b \end{pmatrix}\Big|detA=1\Big\}=\Big\{(a,b)\Big|\ \ a^2+ab-b^2=1\Big\}.$$

Then, the graph of C is called the Golden curve.

We have the equation $a^2+ab-b^2=1$, by considering

$$a=a_1cos\theta-b_1sin\theta, b=a_1sin\theta+b_1cos\theta,$$

we have

$$a_1^2(cos^2\theta - sin\theta cos\theta - sin^2\theta) + a_1 b_1(-4sin\theta cos\theta + cos^2\theta - sin^2\theta) + b_1^2(sin^2\theta - sin\theta cos\theta - cos^2\theta) = 1 \tag{6}$$

and

$$cos2\theta - 2sin2\theta = 0$$

thus,

$$cos2\theta = 2\sqrt{5}/5, sin2\theta = \sqrt{5}/5. \tag{7}$$

By (6), we conclude that

$$(cos2\theta - 1/2sin2\theta)(a_1^2 - b_1^2) = 1.$$

According to (7), we get

$$3\sqrt{5}/10(a_1^2 - b_1^2) = 1.$$

Therefore,

$$a_1 = \sqrt{3\sqrt{5}/10}cosht, b_1 = \sqrt{3\sqrt{5}/10}sinht,$$

consequently,

$$a = \sqrt{3\sqrt{5}/10}coshtcos\theta - \sqrt{3\sqrt{5}/10}sinhtsin\theta,$$
$$b = \sqrt{3\sqrt{5}/10}coshtsin\theta + \sqrt{3\sqrt{5}/10}sinhtcos\theta.$$

The curvature of C is calculated as follows:

$$\kappa = \frac{\dot{a}\ddot{b} - \dot{b}\ddot{a}}{(\dot{a}^2 + \dot{b}^2)^{3/2}} = \frac{2\sqrt{(3/2)\sqrt{5}}}{3(cosht)^{3/2}}.$$

The graph of C is a hyperbola and is plotted with Maple in Fig.1:

The Golden curve C is a Lie group. Let $\gamma(t) = \begin{pmatrix} a(t) & b(t) \\ b(t) & a(t)+b(t) \end{pmatrix}$ be a curve on C. The Lie algebra of C is

$$T_I = \{V | \ \gamma(0) = I, \ \gamma'(0) = V = \begin{pmatrix} v & w \\ w & v+w \end{pmatrix} v = a'(0), w = b'(0)\}.$$

Taking the derivative of $a^2 + ab - b^2 = 1$, implies that $2aa' + a'b + ab' - 2bb' = 0$. Since $2aa' + a'b + ab' - 2bb' = 0$ and $\gamma(0) = I$, we obtain $w = -2v$. Therefore,

$$\gamma'(0) = V = v\begin{pmatrix} 1 & -1 \\ -2 & -1 \end{pmatrix}.$$

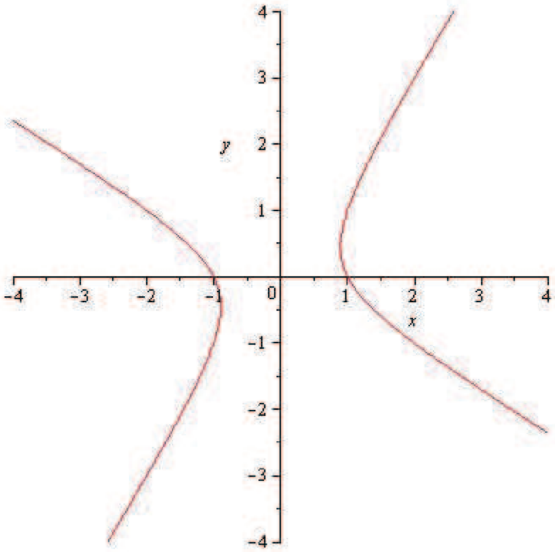

FIGURE 1. Graph for C.

Example 2.4. Let x^3-2x-1 be a generalized Golden polynomial. Suppose that $X=\sum_{i=0}^{2}a_ix^i$ and $Y=\sum_{i=0}^{2}b_ix^i\in\mathscr{A}_{x^3-2x-1}$. Using Definition 1, we obtain

$$XY=a_0b_0+a_1b_2+a_2b_1+(a_0b_1+a_1b_0+2a_1b_2+2a_2b_1+a_2b_2)x+(a_0b_2+a_1b_1+a_2b_0+2a_2b_2)x^2.$$

Hence, the matrix representation of X in the standard basis of $\mathbb{R}^3$ is as follows:

$$A=\begin{pmatrix} a_0 & a_2 & a_1\\ a_1 & a_0+2a_2 & 2a_1+a_0\\ a_2 & a_1 & a_0+2a_2\end{pmatrix}. \tag{8}$$

Therefore,

$$detA=a_0^3+a_1^3+a_2^3+4a_0^2a_2-2a_0a_1^2+4a_0a_2^2-2a_1a_2^2-3a_0a_1a_2,$$

and its decomposition is as follows:

$$detA=\left(a_0-a_1+a_2\right)\left(a_0^2-a_1^2+a_2^2+a_0a_1+3a_0a_2-a_1a_2\right)$$

$$=\left(a_0-a_1+a_2\right)\left(a_0+\phi a_1+(\phi)^2a_2\right)\left(a_0+(1-\phi)a_1+(1-\phi)^2a_2\right).$$

Interestingly $-1,\phi=\frac{1+\sqrt{5}}{2}$ *and* $1-\phi=\frac{1-\sqrt{5}}{2}$ which are the coefficients of a_1 in the above decomposition are exactly the roots of the polynomial $P(x)$.

Now,

$$\mathscr{A}_{x^3-2x-1}=\left\{\begin{pmatrix} a & c & b\\ b & a+2c & 2b+c\\ c & b & a+2c\end{pmatrix}|a,b,c\in\mathbb{R}\right\},$$

and

$$\mathscr{G}_{x^3-2x-1} = \left\{ A = \begin{pmatrix} a & c & b \\ b & a+2c & 2b+c \\ c & b & a+2c \end{pmatrix} |detA \neq 0 \right\},$$

are the Golden algebra and the Golden Lie group with respect to x^3-2x-1, respectively.

According to Examples 2.3 and 2.4, we have the following conjecture:
Let $P(x) = x^n - p_{n-1}x^{n-1} - ... - p_1x^1 - p_0$ and $\mathscr{A}_Z(n,\mathbb{R})$ be the matrix representation for $Z = (a_0,...,a_{n-1}) \in \mathscr{A}_P$. If $x_1, x_2, ..., x_n$ are the roots of the equation $x^n - p_{n-1}x^{n-1} - p_{n-2}x^{n-2} - ... - p_0 = 0$, then

$$det\mathscr{A}_Z(n,\mathbb{R}) = \left(\sum_{i=0}^{n-1} a_i x_1^i\right)\left(\sum_{i=0}^{n-1} a_i x_2^i\right)...\left(\sum_{i=0}^{n-1} a_i x_n^i\right).$$

Remark 2.4. Let

$$S = SL(3,\mathbb{R}) \bigcap \mathscr{G}_{x^3-2x-1} = \left\{ A = \begin{pmatrix} a & c & b \\ b & a+2c & 2b+c \\ c & b & a+2c \end{pmatrix} \Big| \ detA = 1 \right\},$$

namely,

$$S = \left\{ (a,b,c) \Big| \ a^3+b^3+c^3+4a^2c-2ab^2+4ac^2-2bc^2-3abc-1 = 0 \right\}.$$

By considering

$$f(a,b,c) = a^3+b^3+c^3+4a^2c-2ab^2+4ac^2-2bc^2-3abc-1,$$

we have

$$\nabla f = (3a^2+8ac-2b^2+4c^2-3bc, 3b^2-4ab-2c^2-3ac, 3c^2+4a^2+8ac-4bc-3ab) \neq 0$$

$(\forall (a,b,c) \in S)$. Hence, according to Implicit Function Theorem [11], S is a surface in $\mathbb{R}^3$.

The surface S is a Lie group. Let

$$\gamma(t) = \begin{pmatrix} a(t) & c(t) & b(t) \\ b(t) & a(t)+2c(t) & 2b(t)+c(t) \\ c(t) & b(t) & a(t)+2c(t) \end{pmatrix},$$

be a curve on S. The Lie algebra of S is

$$T_I(S) = \left\{ V \mid \gamma(0) = I,\ \gamma'(0) = V = \begin{pmatrix} x & z & y \\ y & x+2z & 2y+z \\ z & y & x+2z \end{pmatrix} \right.$$
$$\left. a'(0) = x, b'(0) = y, c'(0) = z \right\}.$$

Since $a^3+b^3+c^3+4a^2c-2ab^2+4ac^2-2bc^2-3abc-1=0$ and $\gamma(0)=I$, we get $z=-3/4x$. Consequently,

$$\gamma'(0) = V = x\begin{pmatrix} 1 & -3/4 & 0 \\ 0 & -1/2 & -3/4 \\ -3/4 & 0 & -1/2 \end{pmatrix} + y\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}.$$

Definition 6. The graph of S in Remark 2.4 is called the Golden surface.

The graph of S, plotted using Maple is shown in Fig.2:

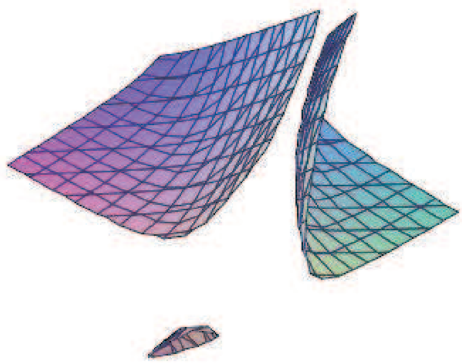

FIGURE 2. Graph for S.

The curvature of S is as follows [11]:

$$\kappa = \frac{-1}{|\nabla f|^2}\begin{pmatrix} \partial^2 f/\partial a^2 & \partial^2 f/\partial a\partial b & \partial^2 f/\partial a\partial c & \partial f/\partial a \\ \partial^2 f/\partial a\partial b & \partial^2 f/\partial b^2 & \partial^2 f/\partial b\partial c & \partial f/\partial b \\ \partial^2 f/\partial a\partial c & \partial^2 f/\partial b\partial c & \partial^2 f/\partial c^2 & \partial f/\partial c \\ \partial f/\partial a & \partial f/\partial b & \partial f/\partial c & 0 \end{pmatrix}.$$

It is calculated with Maple as follows:

$$\kappa = \frac{15}{|\nabla f|^2}.$$

Note that the curvature of S is always positive.

Theorem 3. Let $x_i,\ i = 1,2,3$, be the roots of $x^3 - 2x - 1 = 0$. Then, the maps

$$\varphi_i : \mathscr{G}_{x^3-2x-1} \to \mathbb{R} - \{0\} \ \ i = 1,2,3,$$

such that

$$\varphi_i(a + bx + cx^2) = a + bx_i + cx_i^2,$$

are group homomorphisms.
Also $\psi : \mathscr{G}_{x^3-2x-1} \to (\mathbb{R} - \{0\})^3$, such that

$$\psi(a + bx + cx^2) = (a + bx_1 + cx_1^2, a + bx_2 + cx_2^2, a + bx_3 + cx_3^2),$$

is a group isomorphism.

Proof. Suppose that $X = \sum_{i=0}^{2} a_i x^i$ and $Y = \sum_{i=0}^{2} b_i x^i \in \mathscr{G}_{x^3-2x-1}$,

$$\varphi_i(X.Y) = a_0b_0 + a_1b_2 + a_2b_1 + (a_0b_1 + a_1b_0 + 2a_1b_2 + 2a_2b_1 + a_2b_2)x_1$$
$$+(a_0b_2 + a_1b_1 + a_2b_0 + 2a_2b_2)x_1^2.$$

Since x_1 is the root of $x^3 - 2x - 1 = 0$, we have

$$\varphi_i(X)\varphi_i(Y) = a_0b_0 + a_1b_2 + a_2b_1 + (a_0b_1 + a_1b_0 + 2a_1b_2 + 2a_2b_1 + a_2b_2)x_1$$
$$+(a_0b_2 + a_1b_1 + a_2b_0 + 2a_2b_2)x_1^2.$$

Thus, φ_i is a homomorphism. Similarly, we can show that ψ is a homomorphism. Also, ψ is one to one and onto. If $\psi(\sum_{i=0}^{2} a_i x^i) = (1,1,1)$, then

$$\sum_{i=0}^{2} a_i x_1^i = 1,\ \sum_{i=0}^{2} a_i x_2^i = 1 \ \ and \ \ \sum_{i=0}^{2} a_i x_3^i = 1. \tag{9}$$

Solving the system (9), we have

$$a_0 = 1,\ a_1 = 0 \ and\ a_2 = 0.$$

Let $(r_1, r_2, r_3) \in (\mathbb{R} - \{0\})^3$. There is a polynomial $a + bx + cx^2$ in $\mathscr{G}_{x^3-2x-1}$, such that

$$a = \frac{\begin{vmatrix} r_1 & x_1 & x_1^2 \\ r_2 & x_2 & x_2^2 \\ r_3 & x_3 & x_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}}, \quad b = \frac{\begin{vmatrix} 1 & r_1 & x_1^2 \\ 1 & r_2 & x_2^2 \\ 1 & r_3 & x_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}} \ \ and \ \ c = \frac{\begin{vmatrix} 1 & x_1 & r_1 \\ 1 & x_2 & r_2 \\ 1 & x_3 & r_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}}.$$

Hence, $\psi(a + bx + cx^2) = (r_1, r_2, r_3)$. ■

Theorem 4. Let $x_i, for i = 1,2,3$ be the roots of $x^3 + ax^2 + bx + c = 0$ such that $x_i \neq x_j$, for all $i, j = 1,2,3$. Then, the maps

$$\varphi_i : \mathscr{G}_{x^3+ax^2+bx+c} \to \mathbb{R} - \{0\} \ \ i = 1,2,3,$$

such that

$$\varphi_i(a' + b'x + c'x^2) = a' + b'x_i + c'x_i^2,$$

are group homomorphisms.
Also $\psi : \mathscr{G}_{x^3+ax^2+bx+c} \to (\mathbb{R} - \{0\})^3$, such that

$$\psi(a' + b'x + c'x^2) = (a' + b'x_1 + c'x_1^2, a' + b'x_2 + c'x_2^2, a' + b'x_3 + c'x_3^2),$$

is a group isomorphism.

Proof. The proof is similar to Theorem 3. ■

One of the most important problems in linear algebra is the inverse eigenvalue problem. Let $\lambda_1, \lambda_2, ..., \lambda_n$, be given. Using the above content, we obtain the matrices whose eigenvalues are $\lambda_1, \lambda_2, ..., \lambda_n$. For example, using λ_1, λ_2 *and* λ_3, we produce the following polynomial:

$$P(x) = x^3 - (\lambda_1 + \lambda_2 + \lambda_3)x^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)x - \lambda_1\lambda_2\lambda_3.$$

According to Definition 1, we obtain an induced algebra with respect to P and consider the matrix representation of $(a_0, a_1, a_2) \in \mathscr{A}_P$. Hence, λ_1, λ_2 *and* λ_3 are the eigenvalues of the matrix representation of $(0,1,0) \in \mathscr{A}_P$. We therefore obtain a non-diagonal matrix with λ_1, λ_2 *and* λ_3 as its eigenvalues.

Example 2.5. Using the above lines, we construct the following polynomial for $\lambda_1 = -1, \lambda_2 = \phi$ *and* $\lambda_3 = 1 - \phi$:

$$x^3 - 2x - 1 = 0.$$

The matrix representation of $(a,b,c) \in \mathscr{A}_{x^3-2x-1}$ is as follows:

$$\begin{pmatrix} a & c & b \\ b & a+2c & 2b+c \\ c & b & a+2c \end{pmatrix}.$$

By setting $a=0$, $b=1$ and $c=0$, we have the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix},$$

whose the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = \phi$ and $\lambda_3 = 1-\phi$.

3. Conclusions

This paper assigns to every polynomial an Abelian Lie group. It is well known that every Lie group has a matrix representation. Hence, we can assign to every element of a Lie group a polynomial (for example: a characteristic polynomial or minimal polynomial). Therefore, we appropriate an Abelian Lie group to every element of the Lie group. Now the following question comes to mind: What is the relationship between a Lie group and an Abelian Lie group which corresponds to an element of the Lie group?

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