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Causally Simple Spacetimes and Domain Theory

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Abstract: Globally, hyperbolic spacetimes are the simplest kind of spacetimes which are studied in General Relativity. It is shown by Martin and Panangaden that it is possible to reconstruct globally hyperbolic spacetimes in a purely order theoretic manner using the causal relation J^+. Indeed these spacetimes belong to a category that is equivalent to a special category of domains known as interval domains [8]. In this paper, it is shown that this result is true for a larger superclass of spacetimes.

Keywords: Domain theory, causality, spacetime, causally simple, Alexandrov topology.

1. Introduction

Domains are spatial types of posets introduced by Dana Scott in 1970 [10]. Domain theory formalizes the ideas of approximation and convergence and is closely related to topology. It has played an important role in Computer Science.

Recently it has been proved that domain theoretic methods are useful in General Relativity. It is shown by Martin and Panangaden [7] that by using the causal relation J^+, a globally hyperbolic spacetime is a jointly bicontinuous poset with $I^+ = \ll$, whose interval topology is the manifold topology. They used globally hyperbolic spacetimes in General Relativity to introduce globally hyperbolic posets in domain theory. Indeed, an interesting observation has been made such that the causal relation relates to domain theoretical notions. In this paper, it is also proved that this is true for causally simple spacetimes (every globally hyperbolic spacetime is causally simple, however, the converse is not true [2, 5]). This result suggests that unsolved questions about spacetimes can be converted to domain theoretic form. It is worthwhile for us to use domain theory to answer these questions.

Let us recall some basic definitions in domain theory that we need in the next section. The interested reader is referred to [1, 3, 4, 6, 11] for further information. A poset is a partially ordered set, i.e, a set together with a reflexive, antisymmetric and transitive relation.

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In a poset $(P,\sqsubseteq)$, a non-empty subset $S \subseteq P$ is called directed if $(\forall x,\ y \in S)(\exists z \in S)\ x,\ y \sqsubseteq z$. The supremum of S is the least of its upper bounds provided it exists.

In addition, a non-empty subset of a poset $(P,\sqsubseteq)$ is called filtered if $(\forall x,\ y \in S)(\exists z \in S)\ z \sqsubseteq x,\ y$. The infimum of S is the greatest of its lower bounds provided it exists.

For a subset X of a poset P, set:

$$\uparrow X = \{y \in P : (\exists x \in X)\ x \sqsubseteq y\},\ \downarrow X = \{y \in P : (\exists x \in X)\ y \sqsubseteq x\}.$$

A dcpo is a poset in which every directed subset has a supremum. The least element in a poset, when it exists, is the unique element $\perp$ with $\perp \sqsubseteq x$ for all x.

Definition 1 ([1, 4])**.** For elements x, y of a poset, write $x \ll y$ if and only if for all directed sets S with a supremum,

$$y \sqsubseteq \bigsqcup S \Rightarrow (\exists s \in S)\ x \sqsubseteq s.$$

We set $\Downarrow x = \{a \in P : a \ll x\}$ and $\Uparrow x = \{a \in P : x \ll a\}$. For symbol "$\ll$", read "way below".

Definition 2 ([1, 4])**.** A poset P is continuous if $\Downarrow x$ is directed with supremum x for all $x \in P$.

Definition 3 ([1, 4])**.** For elements x, y of a poset, write $x \ll_d y$ if and only if for all filtered sets S with an infimum,

$$\bigwedge S \sqsubseteq x \Rightarrow (\exists s \in S)\ s \sqsubseteq x.$$

We set $\Downarrow_d x = \{a \in P : a \ll_d x\}$ and $\Uparrow x_d = \{a \in P : x \ll_d a\}$. For symbol "$\ll_d$", read "way above".

Definition 4 ([1, 4, 6])**.** A poset P is dual continuous if $\Uparrow_d x$ is filtered with infimum x for all $x \in P$.

A poset P is bicontinuous if it is both continuous and dual continuous. In addition, a poset is called jointly bicontinuous if it is bicontinuous and the way below relation coincides with the way above relation.

Proposition 1 ([1, 4])**.** If $x \ll y$ in a continuous poset P, then there is $z \in P$ with $x \ll z \ll y$.

Definition 5 ([1, 4, 6])**.** On a bicontinuous poset P, sets of the form

$$(a,b) := \{x \in P : a \ll x \ll_d b\}$$

form a basis for a topology called the interval topology.

2. Causal Structure of a Spacetime

We recall that a spacetime (M,g) is a real four-dimensional smooth manifold M with a Lorentz metric g.

Definition 6 ([2, 5, 9])**.** A tangent vector $v \in TM$ is classified as:

- timelike, if $g(v,v) < 0$.
- causal, if $g(v,v) \leq 0$.

A smooth curve in (M,g) is said to be timelike (resp. causal) if its tangent vector is always timelike (resp. causal).

The event p is chronologically (resp. causally) related to the event q if there is a future directed timelike (resp. future directed causal) curve connecting p with q [2, 5, 9, 12].

I^+ and J^+ are chronological future and causal future relations:

$$I^+ = \{(p,q) : \text{there is a future directed timelike curve from } p \text{ to } q\},$$
$$J^+ = \{(p,q) : \text{there is a future directed causal curve from } p \text{ to } q\}.$$

A part of the causal ladder of spacetimes according to strictly increasing requirements on its conformal structure is as follows [2, 5].

$$\text{Globally hyperbolic} \Longrightarrow \text{Causally simple} \Longrightarrow \text{Strongly causal.}$$

Definition 7 ([2, 9])**.** The spacetime M is strongly causal at p if given any neighborhood U of p there exists a neighborhood $V \subseteq U$, $p \in V$ such that any future directed (and hence also any past directed) causal curve $\gamma : I \to M$ with endpoints in V is entirely contained in U.

Theorem 1 ([2])**.** For a spacetime (M,g), the following properties are equivalent:

(a) (M,g) is strongly causal.
(b) Alexandroff topology is equal to the original topology on M.

The basis for Alexandroff topology is $\{I^+(p) \cap I^-(q) : p,q \in M\}$.

Definition 8 ([2, 9])**.** A spacetime M is globally hyperbolic if it is strongly causal and $J^+(x) \cap J^-(y)$ is compact for every x, $y \in M$.

Definition 9 ([2, 9])**.** A spacetime (M,g) is causally simple if $J^{\pm}$ are closed.

Note that every globally hyperbolic spacetime is causally simple; however, there are examples of causally simple spacetimes which are not globally hyperbolic [2].

Martin and Panangaden defined an order on the spacetime M in the following manner:

$$p \sqsubseteq q \equiv q \in J^+(p).$$

They proved the following theorem about Globally hyperbolic spacetimes:

Theorem 2 ([7, 8]). If M is a globally hyperbolic spacetime, then $(M,\sqsubseteq)$ is a jointly bicontinuous poset with $I^+ = \ll$ whose interval topology is the manifold topology.

This theorem suggests a formulation of causality independent of geometry.

3. Causally Simple Spacetimes and Bicontinuous Posets

In this section, we assume that (M,g) is a causally simple spacetime. Lemma 1, Lemma 2 and Lemma 3 are similar to those we have in [7].

Lemma 1. Let p, q and $r \in M$. Then:

i) $p \sqsubseteq q$ and $r \in I^+(q) \Rightarrow r \in I^+(p)$.

ii) $p \in I^-(q)$ and $q \sqsubseteq r \Rightarrow p \in I^-(r)$.

Lemma 2. Let y_n be a sequence in M with $y_n \sqsubseteq y$ ($y \sqsubseteq y_n$) for all n and $lim_{n\to\infty}\, y_n = y$; then $\bigsqcup y_n = y$ ($\bigwedge y_n = y$).

Proof. Let $y_n \sqsubseteq x$ for every $n \in N$. Since J^+ is closed and $y_n \in J^-(x)$, $y = lim_{n\to\infty}\, y_n \in J^-(x)$. Thus $y \sqsubseteq x$ and consequently $y = \bigsqcup y_n$. The proof for the dual case is similar to this one.

Lemma 3 ([7]). For any $x \in M$, $I^-(x)$ ($I^+(x)$) contains an increasing (decreasing) sequence with supremum (infimum) x.

The following lemma plays a key role in the proof of Theorem 3.

Lemma 4. Let S be a directed set in (M,g) with supremum $\bigsqcup S$. Then there is an increasing sequence $\{s_n\}$ in S such that $lim_{n\to\infty} s_n = \bigsqcup S$.

Proof. Let $A = \{\{s_n\} : s_n \in S, s_n \sqsubseteq s_{n+1}\ \forall n \in N\}$. We define an equivalence relation on A in the following manner:

$$\{s_n\} \sim \{s'_n\} \Leftrightarrow \exists\, m \in N\ : s_n = s'_n\ \forall\, n > m.$$

Now we define a partial order on $A/\sim$.

$$[\{s_n\}] \sqsubseteq_1 [\{s'_n\}] \Leftrightarrow \exists m \in N :\ s_n \sqsubseteq s'_n, \forall\, n \geq m.$$

Suppose that $\{a_m\}_{m\in N} = \{[\{s_{m,n}\}_{n\in N}] : m \in N\}$ is a chain in $A/\sim$. We show that it has an upper bound. We define the sequence $\{b_m\}$ in the following manner:

$$b_1 = s_{1,n_1}\ :\ s_{1,n} \sqsubseteq s_{2,n}\ \forall n > n_1,$$

$$b_i = s_{i,n_i} : s_{i,n} \sqsubseteq s_{i+1,n} \forall n > m\ and\ n_i = max\{m, n_1, ..., n_{i-1}\}.$$

It is easy to show that $[\{b_m\}]$ is an upper bound of $\{a_m\}$. Hence by Zorn's lemma, $A/\sim$ has a maximum element $c = [\{c_m\}]$. Assume by contradiction that there is a neighborhood U of $\bigsqcup S$ with compact closure such that $S \cap U = \emptyset$. Let $\{c_m\}$ be a representation of $[\{c_m\}]$. Since $c_m \sqsubseteq \bigsqcup S$, there is a causal curve γ_m from c_m to $\bigsqcup S$ which intersects ∂U in a point such as d_m. $\{d_m\}$ has an accumulation point such as d since ∂U is compact. There is $m \in N$ such that $c_n \sqsubseteq c_{n+1}, \forall n > m$ and J^+ is closed. Hence $c_i \sqsubseteq d_j, \forall i,\ j > m$ and consequently $c_i \sqsubseteq d, \forall i > m$. However, $[\{c_m\}]$ is a maximal element of $A/\sim$ which implies that d is an upper bound of S which is a contradiction of the fact that $d \sqsubseteq \bigsqcup S$ and $d \neq \bigsqcup S$.

Theorem 3. Let M be a causally simple spacetime. Then

$$x \ll y \Leftrightarrow y \in I^+(x) \Leftrightarrow x \ll_d y.$$

Proof. Let $y \in I^+(x)$. If for the directed set S, $y \sqsubseteq \bigsqcup S$, then by assumption and Lemma 1, $\bigsqcup S \in I^+(x)$. Since by Lemma 4 $\bigsqcup S$ is the limit point of a sequence in S and $I^+(x)$ is open, there exists $s \in S$ such that $s \in I^+(x)$. Consequently, $x \ll y$.

If $x \ll y$, by Lemma 3 there exists an increasing sequence y_n in $I^-(y)$ such that $\bigsqcup y_n = y$. Thus $x \sqsubseteq y_n$, for some n. By Lemma 1, $x \in I^-(y)$. The proof of the other part is similar to this.

Now we are ready to prove the main result of this paper.

Theorem 4. If M is a causally simple spacetime, then $(M, \sqsubseteq)$ is a jointly bicontinuous poset with $\ll = I^+(.)$ whose interval topology is equal to the manifold topology.

Proof. By Theorem 3, $\Downarrow x = I^-(x)$. In addition, by Lemma 3, for every $x \in M$ there is an increasing sequence $x_n \subseteq I^-(x) = \Downarrow x$ with $\bigsqcup x_n = x$. Hence M is continuous. In a similar way, we can prove that it is dually continuous. In addition, Theorem 3 and Theorem 1 imply that the interval topology is equal to the manifold topology.

4. Conclusion

Globally hyperbolic spacetimes were reconstructed by using of the relation J^+ in [7]. In this paper, we have proved that these results are true for a wider class of spacetimes and causally simple spacetimes. It is proved that causally simple spacetimes are jointly bicontinuous posets whose interval topology is equal to the manifold topology. This has the benefit that one can study these spacetimes using domain theory instead of geometry. It is now natural to ask about domain theoretic analogue results for other ladders in the causal hierarchy of spacetime.

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