Çankaya University Journal of Science and Engineering
Volume 11, No. 2 (2014) 7–17

Some Results on Stabilizers in Residuated Lattices

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Abstract: Borumand and Mohtashamnia in [1] introduced the notion of the (right and left) stabilizer in residuated lattices and proved some theorems which determine the relationship between this notion and some types of filters in residuated lattices. In this paper, we show that a part of Theorem 3.10 [1] is not correct. Borumand and Mohtashamnia proved Theorem 4.2 [1] with some conditions. We prove this theorem without any condition. Also, we prove Theorem 3.8 and part (4) of Proposition 3.3 in [1] more generally and finally obtain some new and useful theorems on stabilizers in residuated lattices.

Keywords: Residuated lattices, stabilizer, implicative filter, positive implicative filter, fantastic filter, obstinate filter.

1. Introduction

Residuated lattices are the algebraic counterparts of substructural logics, including most fuzzy logics [2]. Filters are important tools in analyzing fuzzy logics. Borumand and Mohtashamnia in [1] introduced the notion of (right and left) stabilizer in residuated lattices, stated and proved some theorems which determine the relationship between this notion and some types of filters in residuated lattices. In this paper, we correct some theorems in [1] with improvement of their conditions. For instance, Borumand and Mohtashamnia proved $(G,F)^*_R$ is a filter in any residuated lattice A, where F and G are filters of A. We show $(X,F)^*_R$ is a filter in any residuated lattice A, where X is a subset of A and obtain a quotient of residuated lattices via this filter and study its properties.

2. Preliminaries

A residuated lattice ([2],[5]) is an algebra $A = (A,\wedge,\vee,*,\rightarrow 0,1)$ with four binary operations $\wedge$, $\vee$, $*$, $\rightarrow$ and two constants 0,1 such that:

1. $(A,\vee,\wedge,0,1)$ is a bounded lattice,
2. $(A,*,1)$ is a commutative monoid,

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3. $*$ and $\rightarrow$ form an adjoint pair, i.e, $x * z \leq y$ if and only if $x \leq z \rightarrow y$, for all $x, y, z \in A$,

Lemma 1 ([3], [6])**.** In any residuated lattice A, the following relations hold for all $x, y, z \in A$:

1. $1 \rightarrow x = x$, $x \rightarrow x = 1$,
2. $x \leq y$ if and only if $x \rightarrow y = 1$,
3. $x * y \leq x \wedge y$,
4. $x \leq y \rightarrow x$,
5. $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
6. If $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$,
7. $x \vee y \leq ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$,
8. $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
9. $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$,
10. If $x \leq y$ then $x * z \leq y * z$,
11. $y \leq (y \rightarrow x) \rightarrow x$,
12. $x \leq (y \rightarrow x) \rightarrow x$,
13. $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$.

Definition 1 ([4])**.** A nonempty subset F of residuated lattice A is called a filter of A if:

1. $x * y \in F$, for all $x, y \in F$,
2. $x \leq y$ and $x \in F$ imply $y \in F$.

An alternative definition for a filter F of a residuated lattice A is the following:

1. $1 \in F$,
2. If $x \in F$ and $x \rightarrow y \in F$, then $y \in F$.

Definition 2 ([1],[7])**.** A nonempty subset F of residuated lattice A is called

1. an implicative filter if: $1 \in F$ and $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ imply $x \rightarrow z \in F$,
2. a positive implicative filter if: $1 \in F$ and $x \rightarrow ((y \rightarrow z) \rightarrow y) \in F$ and $x \in F$ imply $y \in F$,
3. a fantastic filter if: $1 \in F$ and $z \rightarrow (y \rightarrow x) \in F$ and $z \in F$ imply $((x \rightarrow y) \rightarrow y) \rightarrow x \in F$,
4. an obstinate filter if: F is a proper filter and $x, y \notin F$ imply $x \rightarrow y \in F$ and $y \rightarrow x \in F$, for all $x, y, z \in A$.

Theorem 1 ([1],[7])**.** (Extension property) Let F and G be filters of residuated lattice A such that $F \subseteq G$. If F is an (positive) implicative, fantastic or obstinate filter, then G is an (positive) implicative, fantastic or obstinate filter.

Theorem 2 ([4])**.** Let F be a filter of a residuated lattice A. Define:

$$x \equiv_F y \text{ if and only if } x \to y \in F \text{ and } y \to x \in F.$$

Then $\equiv_F$ is a congruence relation on A. The set of all congruence classes is denoted by A/F, i.e, $A/F = \{[x] \mid x \in A\}$, where $[x] = \{y \in A \mid x \equiv_F y\}$. If we define $\wedge, \vee, *, \to$ on A/F as follows:

$$[x] * [y] = [x * y], [x] \to [y] = [x \to y], [x] \wedge [y] = [x \wedge y], [x] \vee [y] = [x \vee y],$$

then $A/F = (A/F, \wedge, \vee, *, \to, [0], [1])$ is a residuated lattice which is called the quotient residuated lattice with respect to F.

3. On the Stabilizers in Residuated Lattices

Let X and Y be non-empty subsets of residuated lattice A. Borumand and Mohtashamnia in [1], defined

$$X_R^* = \{a \in A \mid a \to x = x, \forall x \in X\}$$
$$X_L^* = \{a \in A \mid x \to a = a, \forall x \in X\}$$

and denoted the stabilizer of X by $X^* = X_R^* \cap X_L^*$. They defined the stabilizer of X with respect to Y or $(X,Y)^* = (X,Y)_R^* \cap (X,Y)_L^*$, where

$$(X,Y)_R^* = \{a \in A \mid (a \to x) \to x \in Y, \forall x \in X\}$$
$$(X,Y)_L^* = \{a \in A \mid (x \to a) \to a \in Y, \forall x \in X\}.$$

Moreover, they proved $(G,F)_R^*$ is a filter in any residuated lattice A, where F and G are filters of A. In the following theorem, we prove only the condition "F be a filter" is necessary.

Theorem 3. If F is a filter of residuated lattice A and $X \subseteq A$, then $(X,F)_R^*$ is a filter of A.

Proof. Let F be a filter of residuated lattice A and $X \subseteq A$. Since $(1 \to x) \to x = 1 \in F$, for all $x \in X$ we have $1 \in (X,F)_R^*$. Let $a, b \in (X,F)_R^*$. Then $(a \to x) \to x \in F$ and $(b \to x) \to x \in F$, for all $x \in X$. Using Lemma 1,

$$((a \to x) \to x) \to (((b \to x) \to x) \to ((ab \to x) \to x)) =$$
$$(ab \to x) \to (((a \to x) \to x) \to (b \to x)) =$$
$$(ab \to x) \to (b \to (a \to x)) = 1 \in F.$$

Since F is a filter, $(ab \to x) \to x \in F$, for all $x \in X$ and so $ab \in (X,F)^*_R$. Now let $a \leq b$ and $a \in (X,F)^*_R$. Then $a \leq b$ implies $(a \to x) \to x \leq (b \to x) \to x$, for all $x \in X$ and so we get $b \in (X,F)^*_R$. Therefore $(X,F)^*_R$ is a filter for all $X \subseteq A$.

It is easy to see that

Proposition 1. In any residuated lattice A , $F \subseteq (X,F)^*((X,F)^*_R,(X,F)^*_L)$, where F is a filter of A and $X \subseteq A$.

Using the extension property, we obtain

Corollary 1. If F is an (positive) implicative, fantastic, obstinate filter of residuated lattice A then $(X,F)^*_R$ is a (positive) implicative, fantastic, obstinate filter of residuated lattice A.

Borumand and Mohtashamnia proved if F and G are filters of residuated lattice A and F is an obstinate (Boolean) filter, then $(G,F)^*_R$ is an obstinate (Boolean) filter [Theorems 4.3 and 4.4, 1]. Using the above Corollary, these theorems become clear.

Corollary 2. If F is a filter of residuated lattice A and $X \subseteq A$ then $(X,F)^*_R/F = (X/F)^*_R$.

Proof.

$$\begin{aligned}(X,F)^*_R/F = \{[a] \mid a \in (X,F)^*_R\} &= \{[a] \mid (a \to x) \to x \in F, \forall x \in X\} \\ &= \{[a] \mid [(a \to x) \to x] = [1], \forall x \in X\} \\ &= \{[a] \mid [a] \to [x] = [x], \forall x \in X\} \\ &= (X/F)^*_R.\end{aligned}$$

Borumand and Mohtashamnia in part (10) of Theorem 3.10 [1] proved $\bigcap(X_i,Y_i)^*_R = (\cap X_i, \cap Y_i)^*_R$. In the following example, we show that the equality does not hold. We note that $\bigcap(X_i,Y_i)^*_R \subseteq (\cap X_i, \cap Y_i)^*_R$. However, in general, $(\cap X_i, \cap Y_i)^*_R \not\subseteq \bigcap(X_i,Y_i)^*_R$.

Example 1. Let $A = \{0,a,b,c,1\}$. Define $*, \to$ as follows:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

then $(A,*,\to,\wedge,\vee,0,1)$ is a residuated lattice [1]. It is easy to check that $(\{1\},F)^*_R = A$ and $(\{0,1\},F)^*_R = F$, where $F = \{c,1\}$. Also $(\{1\},F)^*_R \cap (\{0,1\},F)^*_R = A \cap F = F$ and $(\{1\} \cap \{0,1\},F)^*_R = (\{1\},F)^*_R = A$. Hence $(\{1\},F)^*_R \cap (\{0,1\},F)^*_R \neq (\{1\} \cap \{0,1\},F \cap F)^*_R$.

Borumand and Mohtashamnia in Theorem 4.2 [1] proved if $X \subseteq A$ and X^*_R is a fantastic filter of A such that $(A,X^*_R)^*_R \neq \emptyset$, then $(A,X^*_R)^*_R \subseteq (A,X^*_R)^*_L$. In the following, we see that the condition "X^*_R is a fantastic filter of A such that $(A,X^*_R)^*_R \neq \emptyset$" is not necessary. Beyond that, $(A,X^*_R)^*_R = (A,X^*_R)^*_L = X^*_R$. At first we prove two Lemmas:

Lemma 2. If F is a filter of residuated lattice A, $X \subseteq A$ and $0 \in X$ then

1. $(X,F)^* = (<X>,F)^* = F$,
2. $(X,F)^*_L = (<X>,F)^*_L = F$.

Proof. 1. Let F be a filter of A, $X \subseteq A$, $0 \in X$. By Proposition 1, $F \subseteq (X,F)^*((<X>,F)^*)$. Let $a \in (X,F)^*$ then $(a \to x) \to x \in F$ and $(x \to a) \to a \in F$, for all $x \in X$. For $x = 0$, we have $a = (0 \to a) \to a \in F$. This means $(X,F)^* \subseteq F$. Therefore, $F = (X,F)^*$. Using Theorem 3.10, part 5 [1] we get $F \subseteq (<X>,F)^* \subseteq (X,F)^* = F$. Hence $F = (<X>,F)^*$.

We can similarly prove part 2.

In the following example, we show that in general, $(X,F)^*_R \neq F$, where $0 \in X$:

Example 2. Let $A = \{0,a,b,c,1\}$. Define $*,\to$ as follows:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $(A,*,\to,\wedge,\vee,0,1)$ is a residuated lattice [3]. It is easy to see that $F=\{c,1\}$ is a filter and $F\neq\{a,b,c,1\}=(\{0\},F)^*_R$.

Lemma 3. If F is a filter of residuated lattice A, then

1. $(A,F)^*_R=F$.
2. $(A,F)^*_L=F$.
3. $(A,F)^*=F$.

Proof. 1. Using Proposition 1, $F\subseteq(A,F)^*_R$. Now let $a\in(A,F)^*_R$. Then $(a\to x)\to x\in F$, for all $x\in A$. For $x=a$, $a=(a\to a)\to a\in F$. Hence $(A,F)^*_R\subseteq F$.

By Lemma 2, we get parts 2. and 3.

Let A be a residuated lattice and $X\subseteq A$. Then X^*_R is a filter of A [Theorem 3.4, 1]. Hence $(A,X^*_R)^*_R=(A,X^*_R)^*_L=X^*_R$. Therefore, we prove Theorem 4.2 [1] without any condition.

Borumand and Mohtashamnia in Proposition 3.3 (part 4) [1] proved if $h:A\longrightarrow A$ is a homomorphism and $a\in A$ then $h(\{a\})^*\subseteq\{h(a)\}^*$. In the following, we prove a stronger result:

Proposition 2. If $h:A\longrightarrow B$ is a homomorphism of residuated lattices, then for all $X\subseteq A$:

1. $h(X^*)\subseteq h(X)^*$,
2. $h(X^*_R)\subseteq h(X)^*_R$,
3. $h(X^*_L)\subseteq h(X)^*_L$.

Proof. 1. Let $h:A\longrightarrow B$ be a homomorphism of residuated lattices and $X\subseteq A$. Consider $y\in h(X^*)$ then there exists $a\in X^*$ such that $y=h(a)$. Let $z\in h(X)$ then there exists $x_0\in X$ such that $z=h(x_0)$. Since $a\in X^*$ and h is a homomorphism we get

$$y\to z=h(a)\to h(x_0)=h(a\to x_0)=h(x_0)=z,$$

$$z\to y=h(x_0)\to h(a)=h(x_0\to a)=h(a)=y.$$

Therefore, $y\in h(X)^*$.

The proof of parts 2. and 3. is similar to the proof of part 1.

Lemma 4. If X is a subset of residuated lattice A then

1. $X \cap X_R^* = \emptyset$ or $X \cap X_R^* = \{1\}$,
2. $X \cap X_L^* = \emptyset$ or $X \cap X_L^* = \{1\}$,
3. $X \cap X^* = \emptyset$ or $X \cap X^* = \{1\}$.

Proof. 1. Let $X \subseteq A$ such that $X \cap X_R^* \neq \emptyset$. Consider $a \in X \cap X_R^*$ then $a \in X$ and $a \to x = x$, for all $x \in X$. For $x = a$, we get $1 = a \to a = a$.

The proof of parts 2. and 3. is similar to the proof of part 1.

Theorem 4. If F and G are filters of residuated lattice A, then

1. $F \cap G = \{1\}$ if and only if $F \subseteq G_R^*$.
2. $F \cap G = \{1\}$ if and only if $F \subseteq G_L^*$.
3. $F \cap G = \{1\}$ if and only if $F \subseteq G^*$.

Proof. 1. Let F and G be filters of residuated lattice A. Consider $F \cap G = \{1\}$ and $a \in F$. We have $a, x \leq (a \to x) \to x$, for all $x \in G$. Hence $(a \to x) \to x \in F \cap G = \{1\}$. Therefore, $a \to x = x$, for all $x \in G$. This means $a \in G_R^*$.
Conversely, if $F \subseteq G_R^*$. Then by the above Lemma, (we note that $1 \in G \cap G_R^*$ and so $G \cap G_R^* \neq \emptyset$.) we have $F \cap G \subseteq G \cap G_R^* = \{1\}$.

The proof of part 2. is similar to the proof of part 1. Part 3. it is obtained from parts 1. and 2. and $G^* = G_R^* \cap G_L^*$.

Using the extension property and the above theorem we obtain:

Corollary 3. Let F and G be filters of residuated lattice A such that $F \cap G = \{1\}$. If F is an (positive) implicative, fantastic, obstinate filter of A then G_R^* is a (positive) implicative, fantastic, obstinate filter of A.

Proposition 3. Let A be a residuated lattice, F a filter of A and X a subset of A such that $F \subseteq X$. Then

1. $(X, F)_R^* \cap X = F$,
2. $(X, F)_L^* \cap X = F$,
3. (3)$(X, F)^* \cap X = F$.

Proof. 1. Let F be a filter of A and X a subset of A such that $F \subseteq X$. By Proposition 1 and assumption, $F \subseteq (X, F)_R^* \cap X$. Consider $a \in (X, F)_R^* \cap X$ then $a \in X$ and $(a \to x) \to x \in F$, for all $x \in X$. For $x = a$, we get $a = (a \to a) \to a \in F$. Hence $(X, F)_R^* \cap X \subseteq F$.

2. It is similar to the proof of part 1.

3. Consider

$$\begin{aligned}(X,F)^* \cap X &= (X,F)^*_L \cap (X,F)^*_R \cap X \\ &= (X,F)^*_L \cap F \text{ (by part 1.)} \\ &= F \text{ (by Proposition 1).}\end{aligned}$$

Proposition 4. If F is a filter of residuated lattice A and $X \subseteq A$, then

1. $((X,F)^*_R \cap X,F)^*_R = A$,
2. $((X,F)^*_L \cap X,F)^*_L = A$,
3. $((X,F)^* \cap X,F)^* = A$.

Proof. 1. Let $a \in A$. Consider $x \in (X,F)^*_R \cap X$. Then $(x \to y) \to y \in F$, for all $y \in X$ and $x \in X$. For $y = x$ we get $x = (x \to x) \to x \in F$. Using Lemma 1, $x \leq (a \to x) \to x$. Hence $(a \to x) \to x \in F$, for all $x \in (X,F)^*_R \cap X$. This means $a \in ((X,F)^*_R \cap X,F)^*_R$.

The proof of parts 2. and 3. is similar to the proof of part 1.

Theorem 5. If $\{F_i\}_{i\in I}$ is a family of filters of residuated lattice A and $X \subseteq A$ then $\bigcap(X,F_i)^*_R = (X,\cap F_i)^*_R$, $\bigcap(X,F_i)^*_L = (X,\cap F_i)^*_L$ and $\bigcap(X,F_i)^* = (X,\cap F_i)^*$.

Proof.

$$\begin{aligned}a \in \bigcap(X,F_i)^*_R &\Leftrightarrow a \in (X,F_i)^*_R, \forall i \in I \\ &\Leftrightarrow (a \to x) \to x \in F_i, \forall x \in X, \forall i \in I \\ &\Leftrightarrow (a \to x) \to x \in \bigcap F_i, \forall x \in X \\ &\Leftrightarrow a \in (X,\cap F_i)^*_R.\end{aligned}$$

We can similarly prove $\bigcap(X,F_i)^*_L = (X,\cap F_i)^*_L$. Hence $\bigcap(X,F_i)^* = (X,\cap F_i)^*$.

Theorem 6. If $\{F_i\}_{i\in I}$ is a chain of filters of residuated lattice A and X a finite subset of A, then $\bigcup(X,F_i)^*_R = (X,\cup F_i)^*_R$, $\bigcup(X,F_i)^*_L = (X,\cup F_i)^*_L$ and $\bigcup(X,F_i)^* = (X,\cup F_i)^*$.

Proof. Let $\{F_i\}_{i\in I}$ be a chain of filters and X a finite subset of A. It is easy to see that $\bigcup_{i\in I} F_i$ is a filter. Let $a \in \bigcup(X,F_i)^*_R$ then there exists $i \in I$ such that $a \in (X,F_i)^*_R$. This means there exists $i \in I$ such that $(a \to x) \to x \in F_i$, for all $x \in X$. Then $(a \to x) \to x \in \bigcup F_i$, for all $x \in X$ and so $a \in (X,\cup F_i)^*_R$. Therefore, $\bigcup(X,F_i)^*_R \subseteq (X,\cup F_i)^*_R$.
Conversely, let $a \in (X,\cup F_i)^*_R$. Then $(a \to x) \to x \in \bigcup F_i$, for all $x \in X$. Since X is a finite subset and $\{F_i\}_{i\in I}$ a chain of filters we get there exists $j \in I$ such that $(a \to x) \to x \in F_j$, for all $x \in X$. Hence $a \in (X,F_j)^*_R$, for some $j \in I$ and so $a \in \bigcup(X,F_i)^*_R$. Therefore, $(X,\cup F_i)^*_R \subseteq \bigcup(X,F_i)^*_R$. . Similarly $\bigcup(X,F_i)^*_L = (X,\cup F_i)^*_L$. Therefore, $\bigcup(X,F_i)^* = (X,\cup F_i)^*$.

Corollary 4. $((X,-)^*_R,\cap)$, $((X,-)^*_L,\cap)$ and $((X,-)^*_R,\cap)$, are meet semi-lattices where, $(X,-)^*_R=\{(X,F)^*_R \mid F\in Fil(A)\}$, $(X,-)^*_L=\{(X,F)^*_L \mid F\in Fil(A)\}$, $(X,-)^*=\{(X,F)^* \mid F\in Fil(A)\}$, $Fil(A)=\{F \mid\ F\ is\ a\ filter\ of\ A\}$ and X a subset of residuated lattice A.

Proposition 5. If F is a filter of residuated lattice A and $X,Y\subseteq A$ then $(X,F)^*_R\cap(Y,F)^*_R=(X\cup Y,F)^*_R$, $(X,F)^*_L\cap(Y,F)^*_L=(X\cup Y,F)^*_R$ and $(X,F)^*\cap(Y,F)^*=(X\cup Y,F)^*$.

Proof.

$$\begin{aligned}a\in(X,F)^*_R\cap(Y,F)^*_R &\Leftrightarrow (a\to x)\to x\in F,\forall x\in X \text{ and } (a\to x)\to x\in F,\forall x\in Y\\ &\Leftrightarrow (a\to x)\to x\in F,\forall x\in X\cup Y\\ &\Leftrightarrow a\in(X\cup Y,F)^*_R.\end{aligned}$$

We can similarly prove $(X,F)^*_L\cap(Y,F)^*_L=(X\cup Y,F)^*_R$ and so $(X,F)^*\cap(Y,F)^*=(X\cup Y,F)^*$.

Corollary 5. $((-,F)^*_R,\cap)$, $((-,F)^*_L,\cap)$ and $((-,F)^*,\cap)$ are meet [¡–???? HUH?] semi-lattices, where F is a filter of residuated lattice A and $(-,F)^*_R=\{(X,F)^*_R \mid X\subseteq A\}$, $(-,F)^*_L=\{(X,F)^*_L \mid X\subseteq A\}$ and $(-,F)^*=\{(X,F)^* \mid X\subseteq A\}$.

Theorem 7. If F and G are filters of residuated lattice A and $X\subseteq A$, then

$$A/(X,F\cap G)^*_R=A/(X,F)^*_R\ \bigcap{}' \ A/(X,G)^*_R,$$

where $A/(X,F)^*_R\ \bigcap{}'\ A/(X,G)^*_R=\{[x]_{(X,F)^*_R}\cap[y]_{(X,G)^*_R} \mid [x]_{(X,F)^*_R}\cap[y]_{(X,G)^*_R}\neq\emptyset\}$.

Proof. Let $[x]_{(X,F\cap G)^*_R}\in A/(X,F\cap G)^*_R$. Then

$$\begin{aligned}[x]_{(X,F\cap G)^*_R} &=\{y\in A \mid x\to y\in(X,F\cap G)^*_R\ and\ y\to x\in(X,F\cap G)^*_R\}\\ &=\{y\in A \mid x\to y\in(X,F)^*_R\cap(X,G)^*_R\ and\ y\to x\in(X,F)^*_R\cap(X,G)^*_R\}\ \text{(by Theorem 5)}\\ &=\{y\in A \mid x\to y\ and\ y\to x\in(X,F)^*_R\}\bigcap\{y\in A \mid x\to y\ and\ y\to x\in(X,G)^*_R\}\\ &=[x]_{(X,F)^*_R}\cap[x]_{(X,G)^*_R}\end{aligned}$$

Hence $A/(X,F\cap G)^*_R\subseteq A/(X,F)^*_R\ \bigcap{}'\ A/(X,G)^*_R$. Now, if $[x]_{(X,F)^*_R}\cap[y]_{(X,G)^*_R}\in A/(X,F)^*_R\ \bigcap{}'\ A/(X,G)^*_R$ then $[x]_{(X,F)^*_R}\cap[y]_{(X,G)^*_R}\neq\emptyset$. Let $z\in[x]_{(X,F)^*_R}\cap[y]_{(X,G)^*_R}$. Hence $[z]_{(X,F)^*_R}=[x]_{(X,F)^*_R}$ and $[z]_{(X,G)^*_R}=[y]_{(X,G)^*_R}$. Therefore,

$$[x]_{(X,F)^*_R}\cap[y]_{(X,G)^*_R}=[z]_{(X,F)^*_R}\cap[z]_{(X,G)^*_R}=[z]_{(X,F\cap G)^*_R}.$$

This means $[x]_{(X,F)^*_R}\cap[y]_{(X,G)^*_R}\in A/(X,F\cap G)^*_R$. Hence $A/(X,F)^*_R\ \bigcap{}'\ A/(X,G)^*_R\subseteq A/(X,F\cap G)^*_R$.

Lemma 5. Let F and G be filters of residuated lattices A and B then

$$F\times G=\{(a,b)\in A\times B \mid a\in F\ and\ b\in G\}$$

is a filter of $A \times B$ where

- $(a,b) \wedge (c,d) = (a \wedge c, b \wedge d)$
- $(a,b) \vee (c,d) = (a \vee c, b \vee d)$
- $(a,b) \to (c,d) = (a \to c, b \to d)$
- $(a,b) * (c,d) = (a * c, b * d)$
- $(a,b) \leq (c,d) \Leftrightarrow a \leq c \textit{ and } b \leq d$

for all $a,c \in A$ and $b,d \in B$.

Lemma 6. If X,Y are subsets of residuated lattices A and B, respectively and $F \in Fil(A)$, $G \in Fil(B)$ then

1. $(X \times Y, F \times G)^*_R = (X,F)^*_R \times (Y,G)^*_R$.
2. $(X \times Y, F \times G)^*_L = (X,F)^*_L \times (Y,G)^*_L$.
3. $(X \times Y, F \times G)^* = (X,F)^* \times (Y,G)^*$.

Proof.

$$\begin{aligned}(X \times Y, F \times G)^*_R &= \{(a,b) \in A \times B \mid ((a,b) \to (x,y)) \to (x,y) \in F \times G, \forall (x,y) \in X \times Y\} \\ &= \{(a,b) \in A \times B \mid ((a \to x) \to x, (b \to y) \to y) \in F \times G, \forall (x,y) \in X \times Y\} \\ &= \{(a,b) \in A \times B \mid (a \to x) \to x \in F, (b \to y) \to y \in G, \forall x \in X \text{ and } \forall y \in Y\} \\ &= \{(a,b) \in A \times B \mid a \in (X,F)^*_R \textit{ and } b \in (Y,G)^*_R\} = (X,F)^*_R \times (Y,G)^*_R.\end{aligned}$$

We can similarly prove parts 2. and 3.

Theorem 8. If X,Y are subsets of residuated lattices A and B, respectively and $F \in Fil(A)$, $G \in Fil(B)$ then

$$A \times B/(X \times Y, F \times G)^*_R \cong (A/(X,F)^*_R) \times (B/(Y,G)^*_R).$$

Proof. We define $\Psi : A \times B \longrightarrow (A/(X,F)^*_R) \times (B/(Y,G)^*_R)$ such that $\Psi(a,b) = ([a],[b])$. It is easy to see that Ψ is a well-defined and onto homomorphism. Consider

$$\begin{aligned}\ker\Psi &= \{(a,b) \mid \Psi(a,b) = ([1],[1])\} \\ &= \{(a,b) \mid [a] = [1], [b] = [1]\} \\ &= \{(a,b) \mid a \in (X,F)^*_R, b \in (Y,G)^*_R\} \\ &= (X,F)^*_R \times (Y,G)^*_R.\end{aligned}$$

Using Homomorphism Theorem and the above Lemma,$A \times B/(X \times Y, F \times G)^*_R \cong (A/(X,F)^*_R) \times (B/(Y,G)^*_R)$.

4. Conclusion

Borumand and Mohtashamnia in [1] introduced the notion of the (right and left) stabilizer in residuated lattices. They proved some theorems which determine the relationship between this notion and some types of filters in residuated lattices. In this paper, we corrected and promoted some theorems in [1] with the improvement of their conditions in addition to obtaining some results on stabilizers in residuated lattices.

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