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Relative Entropy Functional of Relative Dynamical Systems

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Abstract: In this paper, the notion of the *relative entropy functional* for relative dynamical systems on compact metric spaces is presented using the mathematical modeling of an observer. The invariance of the entropy of a system under topological conjugacy to the relative entropy functional is generalized. A new version of Jacobs Theorem concerning the entropy of a dynamical system is given. At the end, the Kolmogorov entropy from the *relative entropy functional* for dynamical systems from the view point of observer χ_X, where X denotes the base space of the system, is extracted.

Keywords: Entropy, relative dynamical system, invariant, relative generator, relative measure, relative entropy.

1. Introduction

The term entropy was first used by the German physicist Rudolf Clausius in 1865 to denote a thermodynamic function, which increase with time in all spontaneous natural processes. He introduced this in 1854. Entropy was first introduced into the theory of dynamical systems by Kolmogorov [6] in 1958. Kolmogorov's definition was improved by Sinai in 1959 [19]. The importance of entropy arises from its invariance under conjugacy. Therefore, systems with different entropies cannot be conjugate. On the other hand, in the scientific studies concerning physical systems, molding of these systems is needed. The credibility of the model given is related to the level of its precision, which can be examined with lab data. Not all the data from the lab are precise, so the role of the "observer" in this process is important. Moreover, a method is needed to rate the complexity and/or uncertainty of a system from the point of view of various observers. In order to develop a mathematical model underlying uncertainty and fuzziness in a dynamical system, which is called fuzzy mathematical modeling, we are going to apply the notion of the observer. Therefore, we first ought to mathematically identify the observer. A modeling for an observer of a set X is a fuzzy set $\Theta : X \to [0,1]$ [9, 10, 11, 15]. In fact, these kinds of fuzzy sets are called "one-dimensional observers." After this identification, the notion of the observer is used to define the *relative entropy functional* for topological dynamical systems. The idea is based on the

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relation between ”experience” and ”information” from the view point of an observer. A weight factor of $f(x)$ is assigned to any point $x \in X$, where X denotes the base space of the system. The weight factor can be considered to be the local loss of information caused by the lack of experience of any intelligent point. The *relative entropy functional* is expected to have the fundamental properties of the entropy and also coincides with the Kolmogorov entropy for dynamical systems from the view point of the observer χ_X, when there is no weight factor in the middle.

In this article the set of all probability measures on X preserving T is denoted by $M(X,T)$. We also write $E(X,T)$ for the set of all ergodic measures of T. Finally, for $\mu \in M(X,T)$, $h_\mu(T)$ denotes the Kolomogorov entropy of T.

2. Preliminary Facts

This section is devoted to providing the prerequisites that are necessary for the next section. It is assumed that X is a compact metric space and Θ is a one-dimensional observer of X [9,10,11], that is, $\Theta : X \to [0,1]$ is a fuzzy set [26]. Moreover, there is an assumption that $T : X \to X$ is a continuous map. In this case, it is said that (X,T,Θ) is a relative dynamical system. In fact, if $E \subseteq X$, then the relative probability measure of E with respect to an observer Θ is the fuzzy set $m^T_\Theta(E) : X \to [0,1]$ defined by

$$m^T_\Theta(E)(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(T^i(x))\Theta(T^i(x)),$$

where χ_E is the characteristic function of E [10].

Theorem 1. Let (X,β,m) be a probability space, and let $\Theta : X \to [0,1]$ be the characteristic function χ_X. Moreover, let $T : X \to X$ be an ergodic map, then for each $x \in X$, $m^T_\Theta(E)(x)$ is almost everywhere equal to $m(E)$ where $E \in \beta$.

Proof. See [10].

Therefore, relative probability measure is an extension of the notion of probability measure.

In the remainder of this paper, m_x is a relative measure with respect to an observer Θ at $x \in X$, i.e. $m_x(E) = m^T_\Theta(E)(x)$ for any $E \subseteq X$.

In the following, some classical results that are needed in the sequel, are recalled.

Theorem 2. (Choquet) Suppose that Y is a compact convex metrizable subset of a locally convex space E, and $x_0 \in Y$. Then, there exists a probability measure τ on Y which represents x_0 and is supported by the extreme points of Y, that is, $\Phi(x_0) = \int_Y \Phi d\tau$ for every continuous linear functional Φ on E, and $\tau(\text{ext}(Y)) = 1$.

Proof. See [16].

Let $\mu \in M(X,T)$ and $f : X \to \mathbb{R}$ be a bounded measurable function. It is known that $E(X,T)$ equals the extreme points of $M(X,T)$, applying Choquet's Theorem for $E = M(X)$, the space of finite regular Borel measures on X, and $Y = M(X,T)$, and using the linear functional $\Phi : M(X) \to \mathbb{R}$ given by $\Phi(\mu) = \int_X f d\mu$, the following corollary is reached:

Corollary 1. Suppose that $T : X \to X$ is a continuous map on the compact metric space X. Then, for each $\mu \in M(X,T)$, there is a unique measure τ on the Borel subsets of the compact metrizable space $M(X,T)$, such that $\tau(E(X,T)) = 1$ and

$$\int_X f(x)d\mu(x) = \int_{E(X,T)} (\int_X f(x)dm(x))d\tau(m)$$

for every bounded measurable function $f : X \to \mathbb{R}$.

Under the assumptions of Corollary 1, $\mu = \int_{E(X,T)} m d\tau(m)$ is written, which is called the ergodic decomposition of μ.

Theorem 3. (Jacobs) Let $T : X \to X$ be a continuous map on a compact metrizable space. If $\mu \in M(X,T)$ and $\mu = \int_{E(X,T)} m d\tau(m)$ is the ergodic decomposition of μ, then we have:

(i) If ξ is a finite Borel partition of X, then, $h_\mu(T,\xi) = \int_{E(X,T)} h_m(T,\xi)d\tau(m)$.

(ii) $h_\mu(T) = \int_{E(X,T)} h_m(T)d\tau(m)$ (both sides could be ∞).

Proof. : See [23].

3. Relative Entropy Functional of Relative Dynamical Systems

This section presents the notion of entropy from the view point of different observers and describes a relative perspective of complexity and uncertainty in fuzzy systems.

Definition 1. Suppose that $T : X \to X$ is a continuous map on the topological space $X, x \in X$ and A a Borel subset of X. Then

$$m_x(A) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(T^i(x))\Theta(T^i(x)).$$

Now, let $x \in X$ and $\xi = \{A_1, A_2, ..., A_n\}$ and $\eta = \{B_1, B_2, ..., B_m\}$ be finite Borel partitions of X. We define

$$\Omega_\Theta(x,T,\xi) := -\sum_{i=1}^{n} m_x(A_i)\log m_x(A_i),$$

and

$$\Omega_\Theta(x,T,\xi|\eta) := -\sum_{i,j} m_x(A_i)\log \frac{m_x(A_i \cap B_j)}{m_x(B_j)}.$$

(We assume that $\log 0 = -\infty$ and $0 \times \infty = 0$).

Note that the quantity $\Omega_\Theta(x,T,\xi|\eta)$ is the conditional version of $\Omega_\Theta(x,T,\xi)$. It is clear $\Omega_\Theta(x,T,\xi) \geq 0$.

Definition 2. A partition ξ is a refinement of a partition η, if every element of η is a union of elements of ξ. If ξ is a refinement of η, we write $\eta \prec \xi$.

Definition 3. Given two partitions ξ and η their common refinement is defined by

$$\xi \vee \eta = \{A_i \cap B_j; A_i \in \xi, B_j \in \eta\}.$$

Theorem 4. Let $x \in X$ and $A_1, A_2, \ldots, A_k$ be pairwise disjoint Borel subsets of X. If $A = \cup_{j=1}^{k} A_j$ then $m_x(A) = \sum_{j=1}^{k} m_x(A_j)$.

Proof.

$$\begin{aligned}
m_x(A) &= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(T^i(x))\Theta(T^i(x)) \\
&= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{\cup_{j=1}^{k} A_j}(T^i(x))\Theta(T^i(x)) \\
&= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=1}^{k} \chi_{A_j}(T^i(x))\Theta(T^i(x)) \\
&= \sum_{j=1}^{k} \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{A_j}(T^i(x))\Theta(T^i(x)) \\
&= \sum_{j=1}^{k} \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{A_j}(T^i(x))\Theta(T^i(x)) \\
&= \sum_{j=1}^{k} m_x(A_j).
\end{aligned}$$

Theorem 5. Suppose that $T : X \to X$ is a continuous map on the topological space X and $x \in X$. If ξ, η, ζ are finite Borel partitions of X, then

$$\Omega_\Theta(x,T,\xi \vee \eta|\zeta) = \Omega_\Theta(x,T,\xi|\zeta) + \Omega_\Theta(x,T,\eta|\xi \vee \zeta).$$

Proof. Let $\xi = \{A_1, A_2, \ldots, A_n\}, \eta = \{B_1, B_2, \ldots, B_m\}, \zeta = \{C_1, C_2, \ldots, C_k\}$ be finite Borel partitions of X and assume, without loss of generality, that all sets have the property that $m_x(A) \neq 0$. By definition, we have

$$\Omega_\Theta(x,T,\xi \vee \eta|\zeta) = -\sum_{i,j,k} m_x(A_i \cap B_j \cap C_k) \log \frac{m_x(A_i \cap B_j \cap C_k)}{m_x(C_k)}.$$

However, we may write

$$\frac{m_x(A_i \cap B_j \cap C_k)}{m_x(C_k)} = \frac{m_x(A_i \cap B_j \cap C_k)}{m_x(A_i \cap C_k)} . \frac{m_x(A_i \cap C_k)}{m_x(C_k)},$$

unless $m_x(A_i \cap C_k) = 0$, in the latter case the left hand side is zero and we need not consider it; therefore,

$$\begin{aligned}\Omega_\Theta(x,T,\xi \vee \eta|\zeta) &= -\sum_{i,j,k} m_x(A_i \cap B_j \cap C_k) \log \frac{m_x(A_i \cap C_k)}{m_x(C_k)} - \\ &\quad - \sum_{i,j,k} m_x(A_i \cap B_j \cap C_k) \log \frac{m_x(A_i \cap B_j \cap C_k)}{m_x(A_i \cap C_k)} \\ &= -\sum_{i,j,k} m_x(A_i \cap B_j \cap C_k) \log \frac{m_x(A_i \cap C_k)}{m_x(C_k)} + \Omega_\Theta(x,T,\eta|\xi \vee \zeta) \qquad (1)\end{aligned}$$

However, by Theorem 4, we have

$$\sum_j m_x(A_i \cap B_j \cap C_k) = m_x(A_i \cap C_k)$$

Now multiplying both sides by $-\log \frac{m_x(A_i \cap C_k)}{m_x(C_k)}$ and summing over i and k, we will obtain

$$-\sum_{i,j,k} m_x(A_i \cap B_j \cap C_k) \log \frac{m_x(A_i \cap C_k)}{m_x(C_k)} = \Omega_\Theta(x,T,\xi|\zeta). \qquad (2)$$

Combining 1 and (2), we will have

$$\Omega_\Theta(x,T,\xi \vee \eta|\zeta) = \Omega_\Theta(x,T,\xi|\zeta) + \Omega_\Theta(x,T,\eta|\xi \vee \zeta).$$

Theorem 6. Suppose that $T : X \to X$ is a continuous map on the topological space X and $x \in X$. If ξ and η are finite Borel partitions of X then

$$\Omega_\Theta(x,T,\xi \vee \eta) = \Omega_\Theta(x,T,\xi) + \Omega_\Theta(x,T,\eta|\xi).$$

Proof. Let $\xi = \{A_1, A_2, ..., A_n\}, \eta = \{B_1, B_2, ..., B_m\}$ be finite Borel partitions of X. We can write

$$m_x(A_i \cap B_j) = \frac{m_x(A_i \cap B_j)}{m_x(A_i)} . m_x(A_i).$$

Therefore, we have

$$\begin{aligned}\Omega_\Theta(x,T,\xi \vee \eta) &= -\sum_{i,j} m_x(A_i \cap B_j) \log \frac{m_x(A_i \cap B_j)}{m_x(A_i)} - \sum_{i,j} m_x(A_i \cap B_j) \log m_x(A_i) \\ &= -\sum_{i,j} m_x(A_i \cap B_j) \log m_x(A_i) + \Omega_\Theta(x,T,\eta|\xi) \\ &= \Omega_\Theta(x,T,\xi) + \Omega_\Theta(x,T,\eta|\xi).\end{aligned}$$

Theorem 7. Suppose that $T : X \to X$ is a continuous map on the topological space X and $x \in X$. If ξ, η, ζ are finite Borel partitions of X and $\xi \prec \eta$, then

$$\Omega_\Theta(x,T,\xi|\zeta) \le \Omega_\Theta(x,T,\eta|\zeta).$$

Proof. From Theorem 5, we have

$$\begin{aligned}\Omega_\Theta(x,T,\eta|\zeta) &= \Omega_\Theta(x,T,\xi \vee \eta|\zeta)\\ &= \Omega_\Theta(x,T,\xi|\zeta) + \Omega_\Theta(x,T,\eta|\xi \vee \zeta)\\ &\ge \Omega_\Theta(x,T,\xi|\zeta).\end{aligned}$$

Theorem 8. Suppose that $T : X \to X$ is a continuous map on the topological space X and $x \in X$. Let $\xi$and η be finite Borel partitions and $\xi \prec \eta$, then

$$\Omega_\Theta(x,T,\xi) \le \Omega_\Theta(x,T,\eta).$$

Proof. Since $\xi \prec \eta$, we have

$$\begin{aligned}\Omega_\Theta(x,T,\eta) &= \Omega_\Theta(x,T,\eta \vee \xi)\\ &\ge \Omega_\Theta(x,T,\xi).\end{aligned}$$

Definition 4. Suppose that $T : X \to X$ is a continuous map on the topological space $X, x \in X$ and ξ is a finite Borel partition of X. The map $h_\Theta(.,T,\xi) : X \to [0,\infty]$ is defined as

$$h_\Theta(x,T,\xi) = \limsup_{l\to\infty} \frac{1}{l}\Omega_\Theta(x,T,\vee_{i=0}^{l-1}T^{-i}\xi).$$

Theorem 9. Let ξ and η be finite partitions of X and $\xi \prec \eta$. Then, $h_\Theta(x,T,\xi) \le h_\Theta(x,T,\eta)$.

Proof. If $\xi \prec \eta$ then $\vee_{j=0}^{n-1}T^{-j}\xi \prec \vee_{j=0}^{n-1}T^{-j}\eta$ for all $n \ge 1$. This easily leads to the result.

Theorem 10. Suppose that ξ be a finite partition of X. Then for every $k \in \mathbb{N}$,

$$h_\Theta(x,T,\xi) = h_\Theta(x,T,\vee_{j=0}^{k}T^{-j}\xi).$$

Proof. We immediately obtain

$$\begin{aligned}h_\Theta(x,T,\vee_{j=0}^{k}T^{-j}\xi) &= \limsup_{n\to\infty}\frac{1}{n}\Omega_\Theta(x,T,\vee_{i=0}^{n-1}T^{-i}(\vee_{j=0}^{k}T^{-j}\xi))\\ &= \limsup_{n\to\infty}\frac{1}{n}\Omega_\Theta(x,T,\vee_{t=0}^{n+k-1}T^{-t}\xi)\\ &= \limsup_{p\to\infty}\frac{p}{p-k}.\frac{1}{p}\Omega_\Theta(x,T,\vee_{t=0}^{p-1}T^{-t}\xi)\\ &= h_\Theta(x,T,\xi).\end{aligned}$$

Definition 5. Let $T : X \to X$ be a continuous map on the topological space X. Then, a partition ξ of X is called a relative generator of T if there exists an integer $k > 0$ such that

$$\eta \prec \vee_{i=0}^{k} T^{-i}\xi$$

for every partition η of X.

Theorem 11. Let ξ be a relative generator of T, then $h_\Theta(x,T,\eta) \leq h_\Theta(x,T,\xi)$ for every partition η of X.

Proof. Since ξ is a relative generator of T, then for partition η, there exists an integer $k > 0$ such that

$$\eta \prec \vee_{i=0}^{k} T^{-i}\xi.$$

Hence

$$h_\Theta(x,T,\eta) \leq h_\Theta(x,T,\vee_{i=0}^{k} T^{-i}\xi) = h_\Theta(x,T,\xi).$$

Definition 6. Suppose that $T : X \to X$ is a continuous map on the topological space $X, x \in X$ and ξ is a finite Borel partition of X. We define the relative entropy of T at x by

$$h_\Theta(T,m_x) = \sup_{\xi} h_\Theta(x,T,\xi).$$

Theorem 12. Let ξ be a relative generator of T. Then $h_\Theta(x,T,\xi) = h_\Theta(T,m_x)$.

Proof. Obvious.

Definition 7. Suppose that $T : X \to X$ is a continuous map on the compact metric space X, and ξ is a relative generator for the relative dynamical system (X,Θ,T). Let $\mu \in M(X,T)$ be such that $h_\mu(T) < \infty$. The *relative entropy functional* of T (with respect to μ), $L_\Theta^T(.,\mu,\xi) : C(X) \to \mathbb{R}$, is defined as

$$L_\Theta^T(f,\mu,\xi) = \int_X f(x) h_\Theta(x,T,\xi) d\mu(x)$$

for all $f \in C(X)$ (again $0 \times \infty := 0$).

In the following, the independence of *relative entropy functional* from the selection of the relative generator is proved.

Theorem 13. Definition 7 is independent of the choice of relative generator, i.e., if ξ and η are two relative generators of T, then

$$L_\Theta^T(f,\mu,\xi) = L_\Theta^T(f,\mu,\eta).$$

for all $f \in C(X)$.

Proof. Let ξ, η be relative generators of T. Then by Theorem 12, we have

$$h_\Theta(x,T,\xi) = h_\Theta(T,m_x) = h_\Theta(x,T,\eta).$$

So, if $f \in C(X)$, then,

$$f(x)h_\Theta(x,T,\xi) = f(x)h_\Theta(x,T,\eta)$$

for all $x \in X$. Therefore, $L^T_\Theta(f,\mu,\xi) = L^T_\Theta(f,\mu,\eta)$.

Remark 1. By Theorem 13, we conclude that the definition of *relative entropy functional* is independent of the selection of generators. Therefore, given any invariant measure μ and any relative generator ξ, we have the unique *relative entropy functional*. Thus, we can write $L^T_\Theta(f,\mu)$ for $L^T_\Theta(f,\mu,\xi)$ without confusion.

Example 1. Let $X = \frac{\mathbb{R}}{\mathbb{Z}}$, β denote the Borel sigma-algebra, $\Theta = \chi_X$ and $f(x) = 1$. We let $T : X \to X$ be the doubling map $T(x) = 2x(mod 1)$. We know that T preserves Lebesgue measure m and is ergodic. Hence by Theorem 1, for each $x \in X$ and $A \subset X$, we have $m_x(A) = m(A)$. Let $\xi = \{[0,\frac{1}{2}),[\frac{1}{2},1)\}$; then observe that

$$\xi \vee T^{-1}\xi = \{[0,\frac{1}{4}),[\frac{1}{4},\frac{1}{2}),[\frac{1}{2},\frac{3}{4}),[\frac{3}{4},1)\}.$$

and more generally,

$$\vee_{i=1}^{l-1} T^{-i}\xi = \{[\frac{i}{2^n},\frac{i+1}{2^n}) : i = 0,1,...,2^n - 1\}.$$

Thus, ξ is a relative generator and for each $x \in X$, we can now calculate

$$\begin{aligned}
\Omega_\Theta(x,T,\vee_{i=1}^{l-1} T^{-i}\xi) &= -\sum_{i=0}^{2^l-1} m_x([\frac{i}{2^l},\frac{i+1}{2^l}))\log m_x([\frac{i}{2^l},\frac{i+1}{2^l})) \\
&= -\sum_{i=0}^{2^l-1} (\frac{1}{2^l})\log(\frac{1}{2^l}) \\
&= -2^l(\frac{1}{2^l})\log(\frac{1}{2^l}) \\
&= l\log 2.
\end{aligned}$$

Thus, we see that $\frac{1}{l}\Omega_\Theta(x,T,\vee_{i=1}^{l-1} T^{-i}\xi) = \log 2$ and thus letting $l \to \infty$ gives $h_\Theta(x,T,\xi) = \log 2$. Therefore, for each $\mu \in M(X,T)$, we have $L^T_\Theta(f,\mu,\xi) = \log 2$.

Theorem 14. Suppose that $T : X \to X$ is a continuous map on the compact metric space X. Then,

1. Given any $\mu \in M(X,T)$, the *relative entropy functional* $f \mapsto L^T_\Theta(f,\mu)$ is linear.
2. Given any $f \in C(X)$, the map $\mu \mapsto L^T_\Theta(f,\mu)$ is affine.

Proof. 1. and 2. are trivial.

Definition 8. We say that two relative dynamical systems (X,T_1,Θ_1) and (Y,T_2,Θ_2) are conjugate if there exists a homeomorphism $\varphi : X \to Y$ such that $\varphi o T_1 = T_2 o \varphi$ and $\Theta_2(T_2 o \varphi(x)) = \Theta_1(T_1(x))$ for all $x \in X$.

Theorem 15. Suppose that $T : X \to X$ is a continuous map on the compact metric space X. If two relative dynamical systems (X,T_1,Θ_1) and (Y,T_2,Θ_2) are conjugate, and $\mu \in M(X,T)$, then,

$$L^{T_1}_{\Theta_1}(f,\mu) = L^{T_2}_{\Theta_2}(f\varphi^{-1},\mu\varphi^{-1})$$

for all $f \in C(X)$.

Proof. For $x \in X$ and the Borel set $A \subset X$, we have $m^{T_1}_{\Theta}(A)(x) = m^{T_2}_{\Theta}(\varphi(A))(\varphi(x))$. Therefore, $\Omega_\Theta(x,T_1,\xi) = \Omega_\Theta(\varphi(x),T_2,\varphi(\xi))$ for any finite Borel partition ξ. By definition of $h_\Theta(.,T,\xi)$ we have $h_{\Theta_1}(.,T_1,\xi) = h_{\Theta_2}(.,T_2,\varphi(\xi))o\varphi$. Note that $\varphi(\xi) = \{\varphi(A); A \in \xi\}$. Let $\mu \in M(X,T_1)$, and $f \in C(X)$. Then,

$$\begin{aligned} L^{T_1}_{\Theta_1}(f,\mu) &= \int_X f(x)h_{\Theta_1}(x,T_1,\xi)d\mu(x) \\ &= \int_X f(x)h_{\Theta_2}(\varphi(x),T_2,\varphi(\xi))d\mu(x) \\ &= \int_Y f(\varphi^{-1}(x))h_{\Theta_2}(x,T_2,\varphi(\xi))d(\mu\varphi^{-1})(x) \\ &= L^{T_2}_{\Theta_2}(f\varphi^{-1},\mu\varphi^{-1}). \end{aligned}$$

Now we can deduce the following version of Jacobs Theorem.

Theorem 16. Suppose that $T : X \to X$ is a continuous map on the compact metric space X. If $\mu \in M(X,T)$ and $\mu = \int_{E(X,T)} m d\tau(m)$ is the ergodic decomposition of μ, then

$$L^T_\Theta(f,\mu) = \int_{E(X,T)} L^T_\Theta(f,m)d\tau(m)$$

for all $f \in C(X)$.

Proof. Let ξ be a relative generator of relative dynamical system (X,Θ,T). First, let $f \in C^+(X)$. Applying Corollary 1, we have

$$\begin{aligned} L^T_\Theta(f,\mu,\xi) &= \int_X f(x)h_\Theta(x,T,\xi)d\mu(x) \\ &= \int_{E(X,T)} (\int_X f(x)h_\Theta(x,T,\xi)dm(x))d\tau(m) \\ &= \int_{E(X,T)} (\int_X L^T_\Theta(f,m,\xi)d\tau(m). \end{aligned}$$

For $f \in C(X)$, write $f = f^+ - f^-$ where $f^+, f^- \in C^+(X)$.

Theorem 17. Suppose that $T:X\to X$ is a continuous map on the compact metric space X. Moreover, let $x\in X$ and $\mu\in M(X,T)$. Then

$$L_{\Theta}^{T}(1,\mu)=h_{\Theta}(T,m_x).$$

Proof. Let ξ be a relative generator. Let $\mu\in M(X,T)$. By Theorem 12, we have

$$h_{\Theta}(x,T,\xi)=h_{\Theta}(T,m_x)$$

for arbitrary $x\in X$. Therefore,

$$L_{\Theta}^{T}(1,\mu)=\int_X h_{\Theta}(T,m_x)d\mu(x)=h_{\Theta}(T,m_x).$$

Theorem 18. Suppose that $T:X\to X$ is a continuous map on the compact metric space X. Moreover, let $x\in X$ and $\mu\in M(X,T)$. Then the *relative entropy functional* $f\mapsto L_{\Theta}^{T}(f,\mu)$ is a continuous linear function on $C(X)$, and $\|L_{\Theta}^{T}(.,\mu)\|=h_{\Theta}(T,m_x)$.

Proof. Let ξ be a relative generator. Let $f\in C(X)$, then

$$\begin{aligned}|L_{\Theta}^{T}(f,\mu)| &= |\int_X f(x)h_{\Theta}(x,T,\xi)d\mu(x)|\leq\int_X|f(x)|h_{\Theta}(x,T,m)d\mu(x)\\ &\leq \|f\|_\infty\int_X h_{\Theta}(x,T,m)d\mu(x)=\|f\|_\infty L_{\Theta}^{T}(1,\mu)=\|f\|_\infty h_{\Theta}(T,m_x)\end{aligned}$$

Therefore, the *relative entropy functional* is a continuous function and $\|L_{\Theta}^{T}(.,m)\|\leq h_{\Theta}^{S}(T,m_x)$. The equality holds by Theorem 15.

In the following theorem, Kolmogorov entropy from the *relative entropy functional* as a special case is extracted.

Theorem 19. Suppose that $T:X\to X$ is a continuous map on the compact metric space X. If $\Theta:X\to[0,1]$ is the characteristic function χ_X, then $L_{\Theta}^{T}(1,\mu)=h_{\mu}(T)$.

Proof. Let ξ be a relative generator. By Theorem 12, we have

$$h_{\Theta}(x,T,\xi)=h_{\Theta}(T,m_x).$$

First, let $m\in E(X,T)$. For each Borel set A and $x\in X$, applying Theorem 1 we have $m_x(A)=m(A)$. Therefore, by replacing m_x with m, we have

$$h_{\Theta}(x,T,\xi)=h_m(T).$$

Therefore,

$$L_{\Theta}(1,m)=\int_X h_{\Theta}(x,T,\xi)dm(x)=h_m(T).$$

Now, let $\mu \in M(X,T)$, and $\mu = \int_{E(X,T)} md\tau(m)$ be the ergodic decomposition of μ. Applying Theorem 3 and Theorem 15 we have

$$\begin{aligned} L_\Theta^T(1,\mu) &= \int_{E(X,T)} L_\Theta^T(1,m)d\tau(m) \\ &= \int_{E(X,T)} h_m(T)d\tau(m) \\ &= h_\mu(T). \end{aligned}$$

4. Conclusions

In this paper, the new notion of *relative entropy functional* for relative dynamical systems on compact metric spaces from the view point of observer Θ is introduced. This notion is an invariant object under the conjugate relation. It can be used in order to classify relative dynamical systems $[9,11,15]$. Moreover, by using it, a new method is obtained to make comparisons between the perspectives of observers. Moreover, it can be used to measure the complexity and/or uncertainty of the system through the viewpoint of observers. This notion is a continuous linear functional on $C(X)$ such that its norm equals the relative entropy of T at each $x \in X$. Finally, it is proved that if $\Theta : X \to [0,1]$ is the characteristic function χ_X, then $L_\Theta^T(1,\mu)$ is the Kolmogorov entropy of T. With regards to further research, more work can be carried out in the field of computation of relative entropy for relative dynamical systems on non-compact metric spaces.

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