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Some Properties of the Generalized Leonardo Numbers

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Abstract — In this study, various properties of the generalized Leonardo numbers, which are one of the generalizations of Leonardo numbers, have been investigated. Additionally, some identities among the generalized Leonardo numbers have been obtained. Furthermore, some identities between Fibonacci numbers and generalized Leonardo numbers have been provided. In the last part of the study, binomial sums of generalized Leonardo numbers have been derived. The results obtained for generalized Leonardo numbers are reduced to Leonardo numbers.

Keywords *Binet's formula, Fibonacci numbers, Leonardo numbers*

Mathematics Subject Classification (2020) 11B37, 11B39

1. Introduction

Number sequences are one of the fundamental areas of study within mathematics. Amongst number sequences, the Fibonacci sequence holds a place of importance. This sequence has comprehensive applications in various fields, including mathematics, biology, art, and finance. Many authors have studied different mathematical properties of Fibonacci numbers in [1–6].

The Lucas sequence is another significant number sequence. The Lucas sequence has similar properties with the Fibonacci sequence in [5, 6]. The studies of these sequences involve investigating their properties, relationships, and applications. Mathematicians continue to investigate new properties of number sequences.

In recent years, researchers have been studying Leonardo numbers, which are similar to the recurrence relation of Fibonacci numbers. Catarino and Borges defined the Leonardo sequence in [7]. Moreover, some identities of Leonardo numbers were obtained in [8]. Recent studies on Leonardo numbers have investigated various generalizations of Leonardo numbers in [9–19].

This study investigates the k-Leonardo numbers as defined by Kuhapatanakul and Chobsorn in [13]. Some identities, including binomial sums for k-Leonardo numbers, are obtained. Additionally, some relationships between Fibonacci and k-Leonardo numbers are provided. All the results obtained in this study are reduced to Leonardo numbers for $k = 1$.

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2. Preliminaries

In this section, some definitions and identities of Fibonacci, Lucas and Leonardo numbers are provided.

Definition 2.1. [1] The Fibonacci numbers are characterized, for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}$$

with $F_0 = 0$ and $F_1 = 1$.

Fibonacci numbers correspond A000045 in OEIS [20].

Proposition 2.2. [1] The Binet's formula for Fibonacci sequence is provided as follows:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2.1}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Definition 2.3. [1] The Lucas numbers are provided the following reccurence relation, for $n \geq 2$,

$$L_n = L_{n-1} + L_{n-2}$$

with $L_0 = 2, L_1 = 1$.

Lucas numbers correspond A000032 in OEIS, [20].

Proposition 2.4. [1] The Binet's formula for Lucas sequence is provided as follows:

$$L_n = \alpha^n + \beta^n \tag{2.2}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Some identities [5, 6] relating to Fibonacci and Lucas numbers are as follows:

$$F_{m-1} + F_{m+1} = L_m \tag{2.3}$$

$$L_{m-1} + L_{m+1} = 5F_m \tag{2.4}$$

$$F_{s+t} + (-1)^t F_{s-t} = L_t F_s \tag{2.5}$$

$$F_{s+t} - (-1)^t F_{s-t} = F_t L_s \tag{2.6}$$

$$L_{s+t} + (-1)^t L_{s-t} = L_t L_s \tag{2.7}$$

$$L_{s+t} - (-1)^t L_{s-t} = 5F_s F_t \tag{2.8}$$

$$F_m F_n - F_{m+k} F_{n-k} = (-1)^{n-k} F_{m+k-n} F_k \tag{2.9}$$

$$L_{2h} - 2(-1)^h = 5F_h^2 \tag{2.10}$$

$$F_{s+t} = F_{s+1} F_{t+1} - F_{s-1} F_{t-1} \tag{2.11}$$

$$L_{2m} L_{2n} = 5(F_{m+n}^2 + F_{m-n}^2) + 4(-1)^{m+n} \tag{2.12}$$

$$\sum_{i=0}^{2n} \binom{2n}{i} F_{2i} = 5^n F_{2n} \tag{2.13}$$

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{2i} = 5^n L_{2n+1} \tag{2.14}$$

Definition 2.5. [7] The Leonardo sequence has the following recurrence relation, for $n \geq 2$,

$$Le_n = Le_{n-1} + Le_{n-2} + 1$$

and the initial conditions of this recurrence relation are $Le_0 = Le_1 = 1$.

These numbers correspond A001595 in OEIS [20].

Proposition 2.6. [7] The Binet's formula of Leonardo sequence is

$$Le_n = \frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Definition 2.7. [13] The generalized Leonardo numbers has the following recurrence:

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k$$

for $k \in \mathbb{N}$ and $n \geq 2$. In addition, the initial conditions are $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$.

Proposition 2.8. [13] The relation between Fibonacci numbers and generalized Leonardo numbers is provided as follows:

$$\mathcal{L}_{k,n} = (k+1)F_{n+1} - k \tag{2.15}$$

Proposition 2.9. [14] The Binet's formula of the generalized Leonardo sequence is

$$\mathcal{L}_{k,n} = (k+1)\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - k \tag{2.16}$$

Where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Table 1. Several terms of the Fibonacci, Leonardo, Lucas, and generalized Leonardo numbers

n	0	1	2	3	4	5	6	7
F_n	0	1	1	2	3	5	8	13
Le_n	1	1	3	5	9	15	25	41
L_n	2	1	3	4	7	11	18	29
$\mathcal{L}_{k,n}$	1	1	$2+k$	$3+2k$	$5+4k$	$8+7k$	$13+12k$	$21+20k$

3. Main Results

This section provides new identities of the generalized Leonardo numbers.

Proposition 3.1. For any non-negative integers r, s and $r \geq s$, the following identity is valid

$$\mathcal{L}^2_{k,r+s} - \mathcal{L}^2_{k,r-s} = (k+1)^2 F_{2r+2}F_{2s} - 2k(\mathcal{L}_{k,r+s} - \mathcal{L}_{k,r-s})$$

where F_r and $\mathcal{L}_{k,r}$ are rth Fibonacci and generalized Leonardo numbers, respectively.

Proof. Using (2.16) to left hand side (LHS),

$$LHS = \left((k+1)\left(\frac{\alpha^{r+s+1} - \beta^{r+s+1}}{\alpha - \beta}\right) - k\right)^2 - \left((k+1)\left(\frac{\alpha^{r-s+1} - \beta^{r-s+1}}{\alpha - \beta}\right) - k\right)^2$$

From (2.1) and (2.2),

$$LHS = \frac{(k+1)^2}{5}(L_{2r+2s+2} - L_{2r-2s+2}) - 2k(k+1)(F_{r+s+1} - F_{r-s+1})$$

Considering (2.8),

$$LHS = (k+1)^2 F_{2r+2}F_{2s} - 2k(k+1)(F_{r+s+1} - F_{r-s+1})$$

Using (2.15), the result is obtained. □

Taking $k = 1$ in Proposition 3.1, the identity [8] for Leonardo numbers is as follows:

$$Le_{r+s}^2 - Le_{r-s}^2 = 2(2F_{2r+2}F_{2s} - Le_{r+s} + Le_{r-s})$$

Proposition 3.2. For any non-negative integers r and s such that $r \geq s + 4$,

$$\begin{aligned}\mathcal{L}_{k,r+s}\mathcal{L}_{k,r+s-2} + \mathcal{L}_{k,r-s}\mathcal{L}_{k,r-s-2} =& \mathcal{L}_{k,r+s-1}^2 + \mathcal{L}_{k,r-s-1}^2 + 2(-1)^{r+s}(k+1)^2 \\ &- k(\mathcal{L}_{k,r+s-4} + \mathcal{L}_{k,r-s-4}) - 2k^2\end{aligned}$$

where $\mathcal{L}_{k,r}$ is rth generalized Leonardo number.

Proof. Using (2.16) to the left-hand side (LHS),

$$\begin{aligned}LHS =& \left((k+1)\left(\frac{\alpha^{r+s+1} - \beta^{r+s+1}}{\alpha - \beta}\right) - k\right)\left((k+1)\left(\frac{\alpha^{r+s-1} - \beta^{r+s-1}}{\alpha - \beta}\right) - k\right) + \\ &\left((k+1)\left(\frac{\alpha^{r-s+1} - \beta^{r-s+1}}{\alpha - \beta}\right) - k\right)\left((k+1)\left(\frac{\alpha^{r-s-1} - \beta^{r-s-1}}{\alpha - \beta}\right) - k\right)\end{aligned}$$

From (2.1) and (2.2),

$$LHS = \frac{(k+1)^2}{5}\left(L_{2r+2s} + L_{2r-2s} + 6(-1)^{r+s}\right) - k(k+1)(F_{r+s+1} + F_{r+s-1} + F_{r-s+1} + F_{r-s-1})$$

Using (2.3) and (2.7),

$$LHS = \frac{(k+1)^2}{5}(L_{2r}L_{2s} + 6(-1)^{r+s}) - k(k+1)(L_{r+s} + L_{r-s})$$

Considering (2.12),

$$LHS = (k+1)^2(F_{r+s}^2 + F_{r-s}^2) - k(k+1)(L_{r+s} + L_{r-s}) + 2(-1)^{r+s}(k+1)^2$$

In the final step, from (2.15),

$$LHS = \mathcal{L}_{k,r+s-1}^2 + \mathcal{L}_{k,r-s-1}^2 + 2(-1)^{r+s}(k+1)^2 - k(\mathcal{L}_{k,r+s-4} + \mathcal{L}_{k,r-s-4}) - 2k^2$$

□

Taking $k = 1$ in Proposition 3.2, the following identity [8] of Leonardo numbers is obtained:

$$Le_{r+s}Le_{r+s-2} + Le_{r-s}Le_{r-s-2} = Le_{r+s-1}^2 + Le_{r-s-1}^2 - Le_{r+s-4} - Le_{r-s-4} + 8(-1)^{r-s} - 2$$

Proposition 3.3. For any non-negative integers r and s, the following identity holds true:

$$\mathcal{L}_{k,r}\mathcal{L}_{k,s} = (\mathcal{L}_{k,r} + k)(\mathcal{L}_{k,s} + k) - k(\mathcal{L}_{k,r} + \mathcal{L}_{k,s}) - k^2$$

where $\mathcal{L}_{k,r}$ is rth generalized Leonardo number.

Proof. Using (2.16) to LHS,

$$\mathcal{L}_{k,r}\mathcal{L}_{k,s} = \left((k+1)\left(\frac{\alpha^{r+1} - \beta^{r+1}}{\alpha - \beta}\right) - k\right)\left((k+1)\left(\frac{\alpha^{s+1} - \beta^{s+1}}{\alpha - \beta}\right) - k\right)$$

From (2.1) and (2.2),

$$\mathcal{L}_{k,r}\mathcal{L}_{k,s} = \frac{(k+1)^2}{5}(L_{r+s+2} - (-1)^{s+1}L_{r-s}) - k(k+1)(F_{r+1} + F_{s+1}) + k^2$$

Using (2.8) and (2.15), the result is obtained. □

If we take $2r$ and $2s$ instead of r and s, respectively, and take $k = 1$, we obtain the following identity [8] of Leonardo numbers:

$$Le_{2r}Le_{2s} = (Le_{r+s} + 1)^2 + (Le_{r-s-1} + 1)^2 - Le_{2r} - Le_{2s} - 1$$

Proposition 3.4. For non-negative integers m, r, and s, the following holds:

$$\mathcal{L}_{k,m+r}\mathcal{L}_{k,m+s} - \mathcal{L}_{k,m}\mathcal{L}_{k,m+r+s} = (k+1)^2(-1)^{m+1}F_rF_s - k\mathcal{L}_{k,m+r} - k\mathcal{L}_{k,m+s} + k\mathcal{L}_{k,m} + k\mathcal{L}_{k,m+r+s}$$

where F_m and $\mathcal{L}_{k,m}$ are mth Fibonacci and generalized Leonardo numbers, respectively.

Proof. Using (2.16) to the left-hand side (LHS),

$$LHS = \left((k+1)\left(\frac{\alpha^{m+r+1} - \beta^{m+r+1}}{\alpha - \beta}\right) - k\right)\left((k+1)\left(\frac{\alpha^{m+s+1} - \beta^{m+s+1}}{\alpha - \beta}\right) - k\right) -$$

$$\left((k+1)\left(\frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta}\right) - k\right)\left((k+1)\left(\frac{\alpha^{m+r+s+1} - \beta^{m+r+s+1}}{\alpha - \beta}\right) - k\right)$$

Considering (2.1) and (2.2),

$$LHS = \frac{(k+1)^2}{5}(-1)^{m+1}(L_{r+s} - (-1)^sL_{r-s}) + k(k+1)F_{m+1} +$$

$$k(k+1)(F_{m+r+s+1} - F_{m+r+1} - F_{m+s+1})$$

From (2.8),

$$LHS = (k+1)^2(-1)^{m+1}F_rF_s + k(k+1)(F_{m+r+s+1} + F_{m+1} - F_{m+r+1} - F_{m+s+1})$$

Considering (2.15), the result is clear. □

Taking $k = 1$, the following identity [8] for Leonardo numbers holds true:

$$Le_{m+r}Le_{m+s} - Le_mLe_{m+r+s} = 4(-1)^{m+1}F_rF_s - Le_{m+r} - Le_{m+s} + Le_m + Le_{m+r+s}$$

Proposition 3.5. For any non-negative integers $r \geq 1$ and $s \geq r$, the following identities are valid:

$$\mathcal{L}_{k,s+r} + (-1)^r\mathcal{L}_{k,s-r} = L_r(\mathcal{L}_{k,s} + k) - k(1 + (-1)^r)$$

and

$$\mathcal{L}_{k,s+r} - (-1)^r\mathcal{L}_{k,s-r} = L_{s+1}(\mathcal{L}_{k,r-1} + k) - k(1 - (-1)^r)$$

where L_r and $\mathcal{L}_{k,r}$ are rth Lucas and generalized Leonardo numbers, respectively.

Proof. From (2.15),

$$\mathcal{L}_{k,s+r} + (-1)^r\mathcal{L}_{k,s-r} = (k+1)(F_{s+r+1} + (-1)^rF_{s-r+1}) - k(1 + (-1)^r)$$

Using (2.5), the first identity is obtained. Similarly, the other identity is derived by using (2.15) and (2.6). □

For $k = 1$, we obtain the following identities [8] of Leonardo numbers:

$$Le_{s+r} + (-1)^rLe_{s-r} = L_r(Le_s + 1) - (1 + (-1)^r)$$

and

$$Le_{s+r} - (-1)^mLe_{s-r} = L_{s+1}(Le_{r-1} + 1) - (1 - (-1)^r)$$

Proposition 3.6. For any non-negative integers $r \geq 1$ and $s \geq 1$,

$$\mathcal{L}_{k,r+1}\mathcal{L}_{k,s+1} - \mathcal{L}_{k,r-1}\mathcal{L}_{k,s-1} = (k+1)\mathcal{L}_{k,r+s+1} - k(\mathcal{L}_{k,r} + \mathcal{L}_{k,s}) - k^2 + k$$

where $\mathcal{L}_{k,r}$ is rth generalized Leonardo number.

Proof. Using (2.16) to the left-hand side (LHS),

$$LHS = \left((k+1)\left(\frac{\alpha^{r+2}-\beta^{r+2}}{\alpha-\beta}\right) - k\right)\left((k+1)\left(\frac{\alpha^{s+2}-\beta^{s+2}}{\alpha-\beta}\right) - k\right) - \left((k+1)\left(\frac{\alpha^{r}-\beta^{r}}{\alpha-\beta}\right) - k\right)\left((k+1)\left(\frac{\alpha^{s}-\beta^{s}}{\alpha-\beta}\right) - k\right)$$

Considering (2.1) and (2.2),

$$LHS = \frac{(k+1)^2}{5}(L_{r+s+4} - L_{r+s}) - k(k+1)(F_{r+1} + F_{s+1})$$

From (2.8) and (2.11), we obtain the result. □

Taking $k = 1$, we find the following identity [8] of Leonardo numbers:

$$Le_{r+1}Le_{s+1} - Le_{r-1}Le_{s-1} = 2Le_{r+s+1} - Le_r - Le_s$$

Proposition 3.7. Let r, t, and s be non-negative integers such that $r \geq t$ and $r \geq s$. Then, the following identity is valid:

$$\mathcal{L}_{k,r+t}\mathcal{L}_{k,r-t} - \mathcal{L}_{k,r+s}\mathcal{L}_{k,r-s} = (k+1)^2((-1)^{r-t}F_t^2 - (-1)^{r-s}F_s^2) - k(\mathcal{L}_{k,r+t} + \mathcal{L}_{k,r-t} - \mathcal{L}_{k,r+s} - \mathcal{L}_{k,r-s})$$

where F_r and $\mathcal{L}_{k,r}$ are rth Fibonacci and generalized Leonardo numbers, respectively.

Proof. By applying Binet's formula for the generalized Leonardo numbers to the left-hand side, we can derive the result. □

Taking $k = 1$, the following identity [8] can be found:

$$Le_{r+t}Le_{r-t} - Le_{r+s}Le_{r-s} = 4(-1)^r((-1)^tF_t^2 - (-1)^sF_s^2) + Le_{r+s} + Le_{r-s} - Le_{r+t} - Le_{r-t}$$

Proposition 3.8. For any non-negative integer r, the following holds:

$$\mathcal{L}_{k,r+1}F_{r+1} - \mathcal{L}_{k,r}F_r = \mathcal{L}_{k,r}F_{r+1} + kF_r$$

where F_r and $\mathcal{L}_{k,r}$ are rth Fibonacci and generalized Leonardo numbers, respectively.

Proof. Using (2.16) and (2.1) to the left-hand side (LHS),

$$LHS = \left((k+1)\left(\frac{\alpha^{r+2}-\beta^{r+2}}{\alpha-\beta}\right) - k\right)\left(\frac{\alpha^{r+1}-\beta^{r+1}}{\alpha-\beta}\right) - \left((k+1)\left(\frac{\alpha^{r+1}-\beta^{r+1}}{\alpha-\beta}\right) - k\right)\left(\frac{\alpha^{r}-\beta^{r}}{\alpha-\beta}\right)$$

From (2.1) and (2.2),

$$LHS = \frac{k+1}{5}(L_{2r+2} + 2(-1)^r) - kF_{r-1}$$

Considering (2.10), the following identity is obtained:

$$\mathcal{L}_{k,r+1}F_{r+1} - \mathcal{L}_{k,r}F_r = \mathcal{L}_{k,r}F_{r+1} + kF_r$$

□

For $k = 1$, we obtain the following identity [8] between Leonardo and Fibonacci number:

$$Le_{r+1}F_{r+1} - Le_rF_r = Le_rF_{r+1} + F_r$$

Proposition 3.9. For any non-negative integers s and r where $r \geq 1$ and $s \geq r+1$, the following identities are valid:

$$F_s\mathcal{L}_{k,r} - F_r\mathcal{L}_{k,s} = (-1)^r(\mathcal{L}_{k,s-r-1} + k) + k(F_r - F_s)$$

and

$$F_s\mathcal{L}_{k,r} + F_r\mathcal{L}_{k,s} = \mathcal{L}_{k,s+r-1} + F_s\mathcal{L}_{k,r-1} - kF_r + k$$

where F_r and $\mathcal{L}_{k,r}$ are rth Fibonacci and generalized Leonardo numbers, respectively.

Proof. Using (2.15),

$$F_s\mathcal{L}_{k,r} - F_r\mathcal{L}_{k,s} = (k+1)(F_sF_{r+1} - F_{s+1}F_r) + k(F_r - F_s)$$

From (2.9), the first identity is obtained. Similarly, the second identity can be found. □

Taking $k = 1$, the following identities [8] between Leonardo and Fibonacci numbers can be obtained:

$$F_sLe_r - F_rLe_s = (-1)^r(Le_{s-r-1} + 1) + (F_r - F_s)$$

and

$$F_sLe_r + F_rLe_s = Le_{s+r-1} + F_sLe_{r-1} - F_r + 1$$

Proposition 3.10. For non-negative integer s,

$$\sum_{i=0}^{2s}\binom{2s}{i}\mathcal{L}_{k,2i-1} = 5^s(\mathcal{L}_{k,2s-1} + k) - 4^sk$$

and

$$\sum_{i=0}^{2s+1}\binom{2s+1}{i}\mathcal{L}_{k,2i-1} = 5^s(\mathcal{L}_{k,2s-1} + \mathcal{L}_{k,2s+1}) + 2k(5^s - 4^s)$$

where $\mathcal{L}_{k,s}$ is sth generalized Leonardo number.

Proof. Using (2.15),

$$\sum_{i=0}^{2s}\binom{2s}{i}\mathcal{L}_{k,2i-1} = \sum_{i=0}^{2s}\binom{2s}{i}((k+1)F_{2i} - k)$$

From (2.13), the first identity is obtained. Similarly, other identity can be found. □

Taking $k = 1$, the following binomial sums of Leonardo numbers are obtained:

$$\sum_{i=0}^{2s}\binom{2s}{i}Le_{2i-1} = 5^s(Le_{2s-1} + 1) - 4^s$$

and

$$\sum_{i=0}^{2s+1}\binom{2s+1}{i}Le_{2i-1} = 5^s(Le_{2s-1} + Le_{2s+1}) + 2(5^s - 4^s)$$

4. Conclusion

In this study, various identities for generalized Leonardo numbers have been obtained. Additionally, some identities between Fibonacci numbers and generalized Leonardo numbers have been provided. The results obtained in this study are reduced to identities among Leonardo numbers for $k = 1$. In future studies, a new generalization of Leonardo numbers can be defined, and some identities, similar to those provided in this study, can be established.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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