Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1470784
(**2025**) **6**(1) 59 – 74
Research Article

Clique Collocation Method to Solve the Third-Order Multisingular (MS) Functional Differential Equations

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Received: 19 April 2024 **Accepted:** 18 December 2024

Abstract: In this paper, a Clique collocation method is presented to numerically solve the third-order multisingular (MS) functional differential equation. This method convert this equation to a system of the algebraic equations via the collocation points and the matrix relations. Also, the error estimation technique is constituted for the third-order multisingular (MS) functional differential equation. Applications of the Clique collocation method and the error estimation technique are made for three examples. In addition, the comparison is made with another method in the literature. The obtained results are tabulated and visualized to demonstrate the effectiveness of the presented method. Applications of the method and graphics are made by using MATLAB. According to the applications, it is observed that the results have quite decent errors.

Keywords: Clique polynomials, collocation method, error estimation, functional differential equations, singular differential equations.

1. Introduction

Recently, studies on functional differential equations with singular points have been of great importance for researchers. Functional differential equations are used in many applications such as electrodynamics [12], models based on chemical kinetics [31], models of population growth [26], infection models of HIV-1 [27], models of tumor growth [37], B-virus infection hepatitis models [15] and many more [5, 8, 23, 33, 36]. Differential equations with singular points have been used in some important application areas such as oscillating magnetic fields [11], study of thermal explosions model [1], models of the stellar structure [38] and study of the model of isothermal gas spheres [7]. Many researchers have solved functional differential equations using many methods

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2020 *AMS Mathematics Subject Classification*: 65L60, 65L03

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such as one-step implicit methods [3], Taylor polynomial method [35], homotopy analysis method [4], variational iteration method [10], Laguerre matrix method [42], a matrix-collocation method by using Müntz-Legendre polynomials [39], two novel memory-based root-finding approaches [30] and an iterative method [13]. In addition, singularly perturbed differential equations have been solved using many methods such as spline finite difference method [24], the finite difference methods [16, 22], the seventh order numerical method [9], B-spline collocation method [20, 21], Bessel collocation method [40] and Laguerre method [41]. Besides, an approach has been presented to study the existence, uniqueness and stability of the solutions of nonlinear differential equations with infinite delay [6]. The collocation methods are one of the methods that obtain effective results to calculate numerical solutions of differential equations. In the literature, the collocation method has been used to obtain approximate solutions of many differential equations such as the singular-perturbation problem [20], general linear differential-difference equations with variable coefficients [34], the generalized pantograph equations with linear functional argument [35] etc. [29, 40–42]. The clique polynomials were first introduced in [18] and associated with graph theory. Nevertheless, there are many studies in the literature on the numerical solutions of many differential equations using Clique polynomials [2, 14, 17, 19, 25, 28, 43]. Effective results are obtained from these studies. But there is no study in literature yet on the solutions of the third-order multisingular (MS) functional differential equations using Clique polynomials. Hence, the approximate solution of this equation is investigated based on Clique polynomials in this paper.

In this study, we consider the model based on the third-order multisingular (MS) functional differential equations with initial conditions [32]

$$\begin{cases} u^{'''}(s+\theta_1) + \frac{\beta_1}{s}\, u^{''}(s+\theta_2) + \frac{\beta_2}{s^2}\, u^{'}(s+\theta_3) + s\, u(s+\theta_4) = \alpha(s), \\ u(0) = k_1, \quad u^{'}(0) = k_2, \quad u^{''}(0) = k_3. \end{cases} \tag{1}$$

Here, the parameters $\beta_1, \beta_2, \theta_i \ (i = 1,2,3,4),\ k_j \ (j = 1,2,3)$ are the real constant values and $\alpha(s)$ is the continuous function.

Our aim is to obtain the approximate solution of (1) in form of the Clique polynomials

$$u_N(s) = \sum_{n=0}^{N} a_n C_n(s), \tag{2}$$

where $N > 0$ is chosen to be any positive integer. Here, a_n and $C_n(s)$ are, respectively, the unknown coefficients and Clique polynomials described by [19]

$$C_n(s) = \sum_{k=0}^{n} \binom{n}{k} s^k. \tag{3}$$

The recursive formulation of the Clique polynomials is

$$C_{n+1}(s) = (1+s)C_n(s), \quad C_0(s) = 1, \quad C_1(s) = s+1. \tag{4}$$

Let's summarize rest of this paper as follows: The fundamental matrix relations are presented in Section 2. The Clique collocation method is presented in Section 3. The error estimation method is given in Section 4. In Section 5, the applications of the method are made. Also, a comparison is made with another method in the literature. Thus, the obtained results are interpreted. The results of the paper are summarized in Section 6.

2. Fundamental Matrix Relations

Let's start this section by writing the Clique polynomials in matrix form

$$\mathbf{C}_N(s) = \mathbf{S}_N(s)\mathbf{M}_N, \tag{5}$$

where $\mathbf{C}_N(s) = \begin{bmatrix} C_0(s) & C_1(s) & \cdots & C_N(s) \end{bmatrix}$, $\mathbf{S}_N(s) = \begin{bmatrix} 1 & s & s^2 & \cdots & s^N \end{bmatrix}$,

$$\mathbf{M}_N = \begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{N}{0} \\ 0 & \binom{1}{1} & \binom{2}{1} & \cdots & \binom{N}{1} \\ 0 & 0 & \binom{2}{2} & \cdots & \binom{N}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N} \end{bmatrix}.$$

Secondly, we can express the approximate solutions (2) as

$$u_N(s) = \mathbf{C}_N(s)\mathbf{A}_N, \tag{6}$$

where $\mathbf{C}_N(s) = \begin{bmatrix} C_0(s) & C_1(s) & \cdots & C_N(s) \end{bmatrix}$ and $\mathbf{A}_N = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T$.

Using relation (5) in (6), we get

$$u_N(s) = \mathbf{S}_N(s)\mathbf{M}_N\mathbf{A}_N. \tag{7}$$

By taking the derivative of (7), we have

$$u'_N(s) = \mathbf{S}_N(s)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N, \tag{8}$$

where

$$\mathbf{P}_N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Similarly, the second and the third derivative of (7) becomes

$$u''_N(s) = \mathbf{S}_N(s)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N \tag{9}$$

and

$$u_N'''(s) = \mathbf{S}_N(s)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N. \tag{10}$$

By writing $s \to s + \theta_4$ in (7), we obtain the relation

$$u_N(s+\theta_4) = \mathbf{S}_N(s+\theta_4)\mathbf{M}_N\mathbf{A}_N$$

or since $\mathbf{S}_N(s+\theta_4) = \mathbf{S}_N(s)\mathbf{D}_N(\theta_4)$, we can also write it as

$$u_N(s+\theta_4) = \mathbf{S}_N(s)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N, \tag{11}$$

where

$$\mathbf{D}_N(\theta_4) = \begin{bmatrix} \binom{0}{0}(\theta_4)^0 & \binom{1}{0}(\theta_4)^1 & \cdots & \binom{N}{0}(\theta_4)^N \\ 0 & \binom{1}{1}(\theta_4)^0 & \cdots & \binom{N}{1}(\theta_4)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(\theta_4)^0 \end{bmatrix}.$$

Similarly, substituting $s \to s+\theta_3$, $s \to s+\theta_2$ and $s \to s+\theta_1$, respectively, into (8), (9) and (10), we have

$$u_N'(s+\theta_3) = \mathbf{S}_N(s)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N, \tag{12}$$

$$u_N''(s+\theta_2) = \mathbf{S}_N(s)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N \tag{13}$$

and

$$u_N'''(s+\theta_1) = \mathbf{S}_N(s)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N, \tag{14}$$

where

$$\mathbf{D}_N(\theta_3) = \begin{bmatrix} \binom{0}{0}(\theta_3)^0 & \binom{1}{0}(\theta_3)^1 & \cdots & \binom{N}{0}(\theta_3)^N \\ 0 & \binom{1}{1}(\theta_3)^0 & \cdots & \binom{N}{1}(\theta_3)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(\theta_3)^0 \end{bmatrix}, \quad \mathbf{D}_N(\theta_2) = \begin{bmatrix} \binom{0}{0}(\theta_2)^0 & \binom{1}{0}(\theta_2)^1 & \cdots & \binom{N}{0}(\theta_2)^N \\ 0 & \binom{1}{1}(\theta_2)^0 & \cdots & \binom{N}{1}(\theta_2)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(\theta_2)^0 \end{bmatrix},$$

$$\mathbf{D}_N(\theta_1) = \begin{bmatrix} \binom{0}{0}(\theta_1)^0 & \binom{1}{0}(\theta_1)^1 & \cdots & \binom{N}{0}(\theta_1)^N \\ 0 & \binom{1}{1}(\theta_1)^0 & \cdots & \binom{N}{1}(\theta_1)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(\theta_1)^0 \end{bmatrix}.$$

Finally in this section, by writing $s \to 0$ in (7), (8) and (9), we have, respectively

$$u_N(0) = \mathbf{S}_N(0)\mathbf{M}_N\mathbf{A}_N, \tag{15}$$

$$u_N'(0) = \mathbf{S}_N(0)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N \tag{16}$$

and

$$u_N''(0) = \mathbf{S}_N(0)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N. \tag{17}$$

3. Clique Collocation Method

Firstly, we write the relations (11) - (14) instead of (1) and so we have

$$\begin{aligned}\mathbf{S}_N(s)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s}\mathbf{S}_N(s)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{s^2}\mathbf{S}_N(s)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N \\ +s\,\mathbf{S}_N(s)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s).\end{aligned} \tag{18}$$

Secondly, we obtain

$$\begin{aligned}
\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s_0}\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{(s_0)^2}\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N + s_0\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s_0)\\
\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s_1}\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{(s_1)^2}\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N + s_1\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s_1)\\
\vdots\\
\mathbf{S}_N(s_N)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s_N}\mathbf{S}_N(s_N)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{(s_N)^2}\mathbf{S}_N(s_N)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N + s_N\mathbf{S}_N(s_N)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s_N)
\end{aligned} \tag{19}$$

by using the collocation points defined as

$$s_i = a + \frac{b-a}{N}i, \quad i = 0, 1, \dots, N \tag{20}$$

in the range $[a, b]$, where a is a sufficiently small positive number in the range $0 < a < 1$.

Sytem (19) can also be written, briefly, as

$$\mathbf{W}\mathbf{A}_N = \mathbf{G}, \tag{21}$$

where

$$\mathbf{W} = \left(\mathbf{S}\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3 + \mathbf{E}_1\mathbf{S}\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2 + \mathbf{E}_2\mathbf{S}\mathbf{D}_N(\theta_3)\mathbf{P}_N + \mathbf{E}_3\mathbf{S}\mathbf{D}_N(\theta_4)\right)\mathbf{M}_N,$$

$$\mathbf{G} = \begin{bmatrix}\alpha(s_0)\\ \alpha(s_1)\\ \vdots\\ \alpha(s_N)\end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix}\mathbf{S}_N(s_0)\\ \mathbf{S}_N(s_1)\\ \vdots\\ \mathbf{S}_N(s_N)\end{bmatrix}, \quad \mathbf{E}_1 = diag\left(\frac{\beta_1}{s_i}\right), \quad \mathbf{E}_2 = diag\left(\frac{\beta_2}{(s_i)^2}\right), \quad \mathbf{E}_3 = diag\left(s_i\right).$$

As the next step, we write system (15)-(17) instead of any 3 rows of system (21). Thus, we get

$$\begin{aligned}
\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s_0}\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{(s_0)^2}\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N + s_0\mathbf{S}_N(s_0)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s_0)\\
\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s_1}\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{(s_1)^2}\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N + s_1\mathbf{S}_N(s_1)\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s_1)\\
\vdots\\
\mathbf{S}_N(s_{N-3})\mathbf{D}_N(\theta_1)(\mathbf{P}_N)^3\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_1}{s_{N-3}}\mathbf{S}_N(s_{N-3})\mathbf{D}_N(\theta_2)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N + \tfrac{\beta_2}{(s_{N-3})^2}\mathbf{S}_N(s_{N-3})\mathbf{D}_N(\theta_3)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N + s_{N-3}\mathbf{S}_N(s_{N-3})\mathbf{D}_N(\theta_4)\mathbf{M}_N\mathbf{A}_N = \alpha(s_{N-3})\\
\mathbf{S}_N(0)\mathbf{M}_N\mathbf{A}_N = k_1\\
\mathbf{S}_N(0)\mathbf{P}_N\mathbf{M}_N\mathbf{A}_N = k_2\\
\mathbf{S}_N(0)(\mathbf{P}_N)^2\mathbf{M}_N\mathbf{A}_N = k_3.
\end{aligned} \tag{22}$$

Let's note that we select the last 3 rows in the system (22). Finally, we solve the obtained new system and so we calculate the unknown Clique coefficients matrix $\mathbf{A}_N$. Hence, we achieve the Clique polynomial solutions $u_N(s)$ by putting the obtained matrix $\mathbf{A}_N$ into (6).

4. Error Estimation Technique

Let's start this section by defining residual function as

$$R_N(s) = L[u_N(s)] - \alpha(s). \tag{23}$$

Since the Clique polynomial solutions satisfy problem (1), we get

$$\begin{cases} u_N'''(s+\theta_1) + \frac{\beta_1}{s} u_N''(s+\theta_2) + \frac{\beta_2}{s^2} u_N'(s+\theta_3) + s\, u_N(s+\theta_4) = \alpha(s), \\ u_N(0) = k_1, \quad u_N'(0) = k_2, \quad u_N''(0) = k_3. \end{cases} \tag{24}$$

Secondly, we subtract (24) from (1) and so we gain the error problem

$$\begin{cases} e_N'''(s+\theta_1) + \frac{\beta_1}{s} e_N''(s+\theta_2) + \frac{\beta_2}{s^2} e_N'(s+\theta_3) + s\, e_N(s+\theta_4) = -R_N(s), \\ e_N(0) = 0, \quad e_N'(0) = 0, \quad e_N''(0) = 0. \end{cases} \tag{25}$$

Here, $u(s)$, $u_N(s)$ and $e_N(s)$ denote, respectively, the exact solution, the Clique polynomial solution and the actual error function. Also, let's note that $e_N(s) = u(s) - u_N(s)$.

Finally, we solve the system (25) according to Clique collocation method in previous section and thus we gain the estimated error function

$$e_{N,M}(s) = \sum_{n=0}^{M} a_n^* C_n(s), \tag{26}$$

where a_n^* is the unknown coefficients. The error estimation method is important. Because we can calculate the made error if the exact solution of the problem is not known.

5. Applications

In this section, the applications of methods in previous sections are made using MATLAB.

Example 5.1 *Firstly, we consider the model based on the the third-order multisingular (MS) functional differential equations with initial conditions [32] given as*

$$\begin{cases} u'''(s-1) + \frac{1}{s}u''(s+1) + \frac{2}{s^2}u'(s+2) + su(s) = e^{s-1} + \frac{1}{s}e^{s+1} + \frac{2}{s^2}e^{s+2} + se^s, \\ u(0) = 1, \quad u'(0) = 1, \quad u''(0) = 1. \end{cases} \tag{27}$$

Our aim is to obtain Clique polynomial solutions for $N = 3$ as:

$$u_3(s) = \sum_{i=0}^{3} a_i C_i(s), \tag{28}$$

or

$$u_3(s) = \mathbf{S}_3(s)\mathbf{M}_3\mathbf{A}_3, \tag{29}$$

where $\mathbf{S}_3(s) = \begin{bmatrix} 1 & s & s^2 & s^3 \end{bmatrix}$, $\mathbf{A}_3 = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T$ and $\mathbf{M}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

If we use the relation (20), we write the collocation points for $a = 0.01$, $b = 1$ as $s_0 = \frac{1}{100}, s_1 = \frac{17}{50}, s_2 = \frac{67}{100}, s_3 = 1$. Hence, if we utilize the system (21), then we get

$$\mathbf{WA} = \mathbf{G}, \tag{30}$$

where

$$\mathbf{W} = \left(\mathbf{SD}_3(-1)(\mathbf{P}_3)^3 + \mathbf{E}_1\mathbf{SD}_3(1)(\mathbf{P}_3)^2 + \mathbf{E}_2\mathbf{SD}_3(2)\mathbf{P}_3 + \mathbf{E}_3\mathbf{SD}_3(0)\right)\mathbf{M}_3,$$

$$\mathbf{G} = \begin{bmatrix} e^{s_0-1} + \frac{1}{s_0}e^{s_0+1} + \frac{2}{s_0^2}e^{s_0+2} + s_0e^{s_0} \\ e^{s_1-1} + \frac{1}{s_1}e^{s_1+1} + \frac{2}{s_1^2}e^{s_1+2} + s_1e^{s_1} \\ e^{s_2-1} + \frac{1}{s_2}e^{s_2+1} + \frac{2}{s_2^2}e^{s_2+2} + s_2e^{s_2} \\ e^{s_3-1} + \frac{1}{s_3}e^{s_3+1} + \frac{2}{s_3^2}e^{s_3+2} + s_3e^{s_3} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \mathbf{S}_3(s_0) \\ \mathbf{S}_3(s_1) \\ \mathbf{S}_3(s_2) \\ \mathbf{S}_3(s_3) \end{bmatrix}, \quad \mathbf{P}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

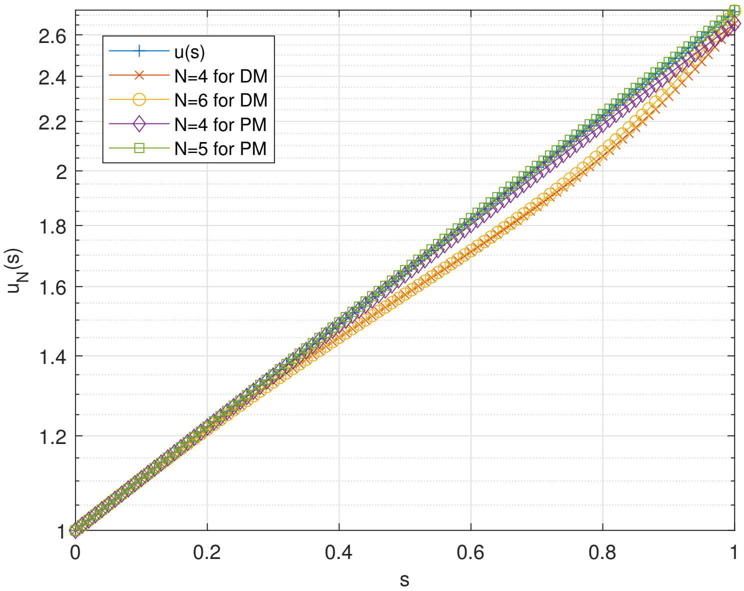
$$\mathbf{D}_3(-1) = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D}_3(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D}_3(2) = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 4 & 12 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{D}_3(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_1 = diag\left(\frac{1}{s_i}\right), \quad \mathbf{E}_2 = diag\left(\frac{2}{(s_i)^2}\right), \quad \mathbf{E}_3 = diag\,(s_i)\,.$$

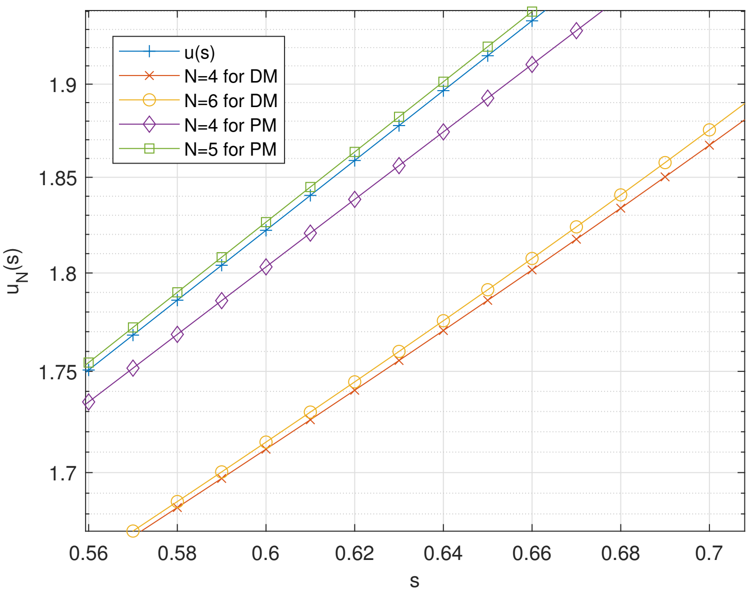
Figure 1 (a) shows the exact solution and the approximate solutions for $N = 4$ and $N = 5$. Also, this figure compares these solutions with the solutions of differential transform method (DM) [32] for $N = 4$ and $N = 6$. Figure 1 (b) shows these functions more closely. Accordingly, the closest result to the exact solution is obtained with our method. In addition, the values of these functions at some s points are compared with the solutions of DM [32] in Table 1.

Figure 2 (a) compares the actual absolute errors of Example 5.1 with the errors of DM [32] for $N = 4$ and $N = 5$. Accordingly, the results of DM [32] for $N = 4$ and $N = 6$ are the same. The result obtained with our method with $N = 4$ is better than these results.

The best result is obtained when $N = 5$ is chosen in our method. In other words, with our method, a more suitable result is obtained with a smaller N value. Figure 2 (b) visualizes the actual absolute errors of Example 5.1 for $N = 4$ and $N = 5$ and the estimated absolute errors of Example 5.1 for $(N, M) = (4, 5)$ and $(N, M) = (5, 6)$. This shows that the actual and estimated absolute error for $N = 4$ and $(N, M) = (4, 5)$ overlap. Also, it can be concluded that the actual



a) The exact solution and the approximate solutions



b) Close angle of solutions

Figure 1: Comparison of solutions of Example 5.1 with DM [32]

Table 1: Comparison of the solutions of Example 5.1 with DM [32]

s_i	Exact solution	DM [32] (N=4)	DM [32] (N=6)	Present Method (N=4)	Present Method (N=5)
0.01	1.01005	1.01005	1.01005	1.01005	1.01005
0.02	1.02020	1.02020	1.02020	1.0202	1.0202
0.03	1.03045	1.03045	1.03045	1.03045	1.03046
0.04	1.04081	1.04080	1.04080	1.0408	1.04081
0.05	1.05127	1.05125	1.05125	1.05125	1.05128
0.06	1.06183	1.06180	1.06180	1.06181	1.06185
0.07	1.07250	1.07246	1.07246	1.07246	1.07252
0.08	1.08328	1.08322	1.09408	1.08322	1.08331
0.09	1.09417	1.09408	1.09408	1.09408	1.09421
0.1	1.10517	1.10504	1.10504	1.10504	1.10521
0.2	1.22140	1.22044	1.22044	1.22046	1.2217
0.3	1.34985	1.34683	1.34685	1.34688	1.35074
0.4	1.49182	1.48515	1.48522	1.48526	1.49361
0.5	1.64872	1.63663	1.63686	1.63684	1.65168
0.6	1.82211	1.80282	1.80341	1.80316	1.82641
0.7	2.01375	1.98556	1.98688	1.98606	2.01938
0.8	2.22554	2.18698	2.18965	2.1877	2.23237
0.9	2.45960	2.40955	2.41451	2.41051	2.46733

and estimated absolute error for $N = 5$ and $(N, M) = (5, 6)$ are very close. Moreover, it can be seen from figures that the error decreases as N increases.

Example 5.2 *Our second model is [32]*

$$\begin{cases} u'''(s-1) + \frac{1}{s}u''(s+1) + \frac{2}{s^2}u'(s+2) + su(s) = s^5 + 45s + 48 + \frac{108}{s} + \frac{64}{s^2}, \\ u(0) = 1, \quad u'(0) = 0, \quad u''(0) = 0. \end{cases} \tag{31}$$

The exact solution of this problem is $1+s^4$. Our aim is to obtain Clique polynomial solutions for $N = 4$ as:

$$u_4(s) = \sum_{i=0}^{4} a_i C_i(s), \tag{32}$$

or

$$u_4(s) = \mathbf{S}_4(s)\mathbf{M}_4\mathbf{A}_4. \tag{33}$$

Utilizing the system (21), we get

$$\mathbf{WA} = \mathbf{G}, \tag{34}$$

where

$$\mathbf{W} = \left(\mathbf{SD}_4(-1)(\mathbf{P}_4)^3 + \mathbf{E}_1\mathbf{SD}_4(1)(\mathbf{P}_4)^2 + \mathbf{E}_2\mathbf{SD}_4(2)\mathbf{P}_4 + \mathbf{E}_4\mathbf{SD}_4(0)\right)\mathbf{M}_4,$$

$$\mathbf{G} = \begin{bmatrix} s_0^5 + 45s_0 + 48 + \frac{108}{s_0} + \frac{64}{s_0^2} \\ s_1^5 + 45s_1 + 48 + \frac{108}{s_1} + \frac{64}{s_1^2} \\ s_2^5 + 45s_2 + 48 + \frac{108}{s_2} + \frac{64}{s_2^2} \\ s_3^5 + 45s_3 + 48 + \frac{108}{s_3} + \frac{64}{s_3^2} \\ s_4^5 + 45s_4 + 48 + \frac{108}{s_4} + \frac{64}{s_4^2} \end{bmatrix}.$$

By writing $s \to 0$ in (7), (8) and (9), we have, respectively

$$u_4(0) = \mathbf{S}_4(0)\mathbf{M}_4\mathbf{A}_4, \tag{35}$$

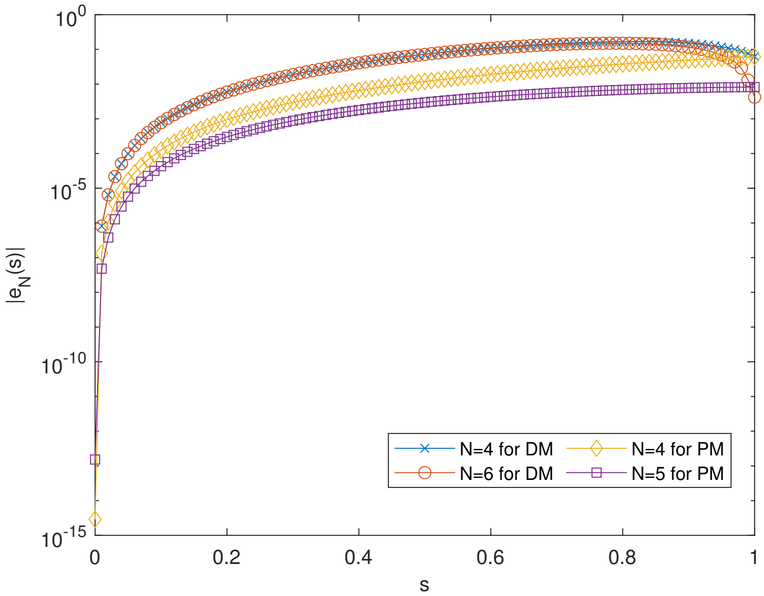
$$u_4'(0) = \mathbf{S}_4(0)\mathbf{P}_4\mathbf{M}_4\mathbf{A}_4 \tag{36}$$

and

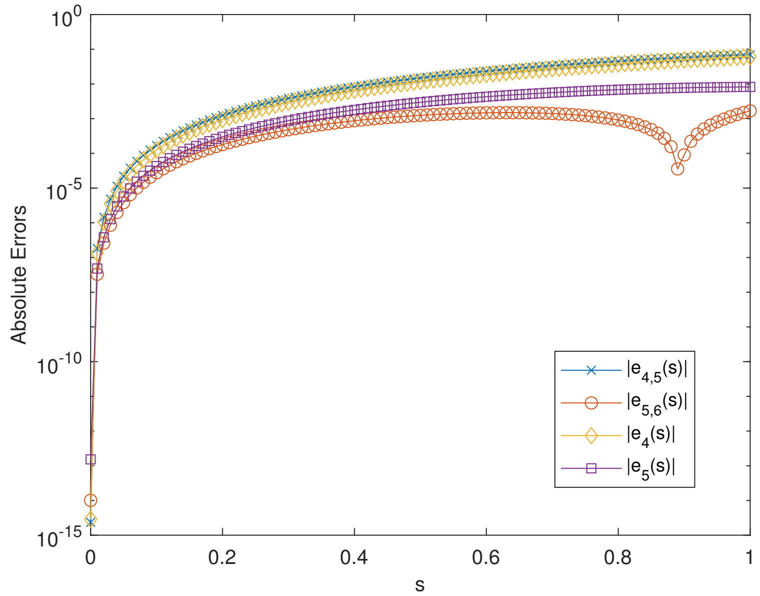
$$u_4'(0) = \mathbf{S}_4(0)(\mathbf{P}_4)^2\mathbf{M}_4\mathbf{A}_4, \tag{37}$$

where $\mathbf{S}_4(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Finally, the approximate solution is obtained $1+s^4$ by solving the system (34) with conditions (35), (36) and (37). This is the exact solution.



a) Actual absolute errors



b) The actual absolute errors and the estimated absolute errors

Figure 2: Comparison of absolute errors of Example 5.1 with DM [32]

Example 5.3 *Finally, we perform the model based on the the third-order multisingular (MS) functional differential equations with initial conditions [32]*

$$\begin{cases} u^{'''}(s-1) + \frac{1}{s}u^{''}(s+1) + \frac{2}{s^2}u^{'}(s+2) + su(s) = s^4 + s + 18 + \frac{30}{s} + \frac{24}{s^2}, \\ u(0) = 1, \quad u^{'}(0) = 0, \quad u^{''}(0) = 0. \end{cases} \tag{38}$$

The exact solution of this problem is $1+s^3$. Our aim is to obtain Clique polynomial solutions for $N = 3$ as:

$$u_3(s) = \sum_{i=0}^{3} a_i C_i(s), \tag{39}$$

or

$$u_3(s) = \mathbf{S}_3(s)\mathbf{M}_3\mathbf{A}_3. \tag{40}$$

Utilizing the system (21), we obtain

$$\mathbf{WA} = \mathbf{G}, \tag{41}$$

where

$$\mathbf{W} = \left(\mathbf{SD}_3(-1)(\mathbf{P}_3)^3 + \mathbf{E}_1\mathbf{SD}_3(1)(\mathbf{P}_3)^2 + \mathbf{E}_2\mathbf{SD}_3(2)\mathbf{P}_3 + \mathbf{E}_3\mathbf{SD}_3(0)\right)\mathbf{M}_3,$$

$$\mathbf{G} = \begin{bmatrix} s_0^4 + s_0 + 48 + \frac{30}{s_0} + \frac{24}{s_0^2} \\ s_1^4 + s_1 + 48 + \frac{30}{s_1} + \frac{24}{s_1^2} \\ s_2^4 + s_2 + 48 + \frac{30}{s_2} + \frac{24}{s_2^2} \\ s_3^4 + s_3 + 48 + \frac{30}{s_3} + \frac{24}{s_3^2} \end{bmatrix}.$$

By writing $s \to 0$ in (7), (8) and (9), we have, respectively

$$u_3(0) = \mathbf{S}_3(0)\mathbf{M}_3\mathbf{A}_3, \tag{42}$$

$$u_3^{'}(0) = \mathbf{S}_3(0)\mathbf{P}_3\mathbf{M}_3\mathbf{A}_3 \tag{43}$$

and

$$u_3^{'}(0) = \mathbf{S}_3(0)(\mathbf{P}_3)^2\mathbf{M}_3\mathbf{A}_3, \tag{44}$$

where $\mathbf{S}_3(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$.

Finally, the approximate solution is obtained $1+s^4$ by solving the system (41) with conditions (42), (43) and (44). This is the exact solution.

6. Conclusions

In this paper, we investigate the approximate solution of the third-order multisingular (MS) functional differential equation via Clique collocation method. In addition to method, we constitute error estimation technique for the problem. Also, we make applications of the Clique collocation method and the error estimation technique for three examples by using MATLAB. Accordingly, we obtain the exact solution in Example 5.2 and Example 5.3. This result demonstrates the advantage of our method. In addition, we compare the results with differential transform method (DM) [32] for Example 5.1. Accordingly, the best result is obtained when $N = 5$ is chosen in our method. In other words, with our method, a more suitable result is obtained with a smaller N value. According to our method, the error decreases as N increases. Moreover, the estimated errors are close to the actual errors, which shows the importance of the error estimation technique. From all numerical results, we conclude that the presented method is efficient and reliable. The presented method can be improved for nonlinear multisingular functional differential equations or multisingular functional differential equations of fractional-order.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Gamze Yıldırım]: Collected the data, contributed to research method or evaluation of data, contributed to completing the research and solving the problem, wrote the manuscript (%50).

Author [Şuayip Yüzbaşı]: Thought and designed the research/problem, contributed to research method or evaluation of data, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] Abd-Elhameed W.M., Doha E.H., Saad A.S., Bassuony M.A., *New Galerkin operational matrices for solving lane-Emden type equations*, Revista Mexicana de Astronomia y Astrofisica, 52(1), 83-92, 2016.

[2] Adel W., Srinivasa K., *A new clique polynomial approach for fractional partial differential equations*, International Journal of Nonlinear Sciences and Numerical Simulation, 24(8), 2839-2851, 2024.

[3] Al-Mutib A.N., *One-step implicit methods or solving delay differential equations*, International Journal of Computer Mathematics, 16(3), 157-168, 1984.

[4] Alomari A.K., Noorani M.S.M., Nazar R., *Solution of delay differential equation by means of homotopy analysis method*, Acta Applicandae Mathematicae, 108, 395-412, 2009.

[5] Baleanu D., Arjunan M.M., Nagaraj M., Suganya S., *Approximate controllability of second-order nonlocal impulsive functional integro-differential systems in Banach spaces*, Bulletin of the Korean Mathematical Society, 55(4), 1065-1092, 2018.

[6] Benaouda M., Sabit S., Günerhan H., Souid M.S., *Modern technique to study Cauchy-type problem of fractional variable order differential equations with infinite delay via phase space*, Modern Physics Letters B, 2450343, 2024.

[7] Boubaker K., Van Gorder R.A., *Application of the BPES to Lane–Emden equations governing polytropic and isothermal gas spheres*, New Astronomy, 17(6), 565-569, 2012.

[8] Caraballo T., Diop M.A., Mane A., *Controllability for neutral stochastic functional integro differential equations with infinite delay*, Applied Mathematics and Nonlinear Sciences, 1(2), 493-506, 2016.

[9] Chakravarthy P.P., Phaneendra K., Reddy Y.N., *A seventh order numerical method for singular perturbation problems*, Applied Mathematics and Computation, 186(1), 860-871, 2007.

[10] Chen X., Wang L., *The variational iteration method for solving a neutral functional-differential equation with proportional delays*, Computers and Mathematics with Applications, 59(8), 2696-2702, 2010.

[11] Dehghan M., Shakeri F., *Solution of an integro-differential equation arising in oscillating magnetic fields using He's homotopy perturbation method*, Progress in Electromagnetics Research, 78, 361-376, 2008.

[12] Dehghan M., Shakeri F., *The use of the decomposition procedure of Adomian for solving a delay differential equation arising in electrodynamics*, Physica Scripta, 78(6), 065004, 2008.

[13] Faheem M., Raza A., Khan A., *Wavelet collocation methods for solving neutral delay differential equations*, International Journal of Nonlinear Sciences and Numerical Simulation, 23(7-8), 1129-1156, 2022.

[14] Ganji R.M., Jafari H., Kgarose M., Mohammadi A., *Numerical solutions of time-fractional Klein-Gordon equations by clique polynomials*, Alexandria Engineering Journal, 60(5), 4563-4571, 2021.

[15] Gourley S.A., Kuang Y., Nagy J.D., *Dynamics of a delay differential equation model of hepatitis B virus infection*, Journal of Biological Dynamics, 2(2), 140-153, 2008.

[16] Habib H.M., El-Zahar E.R., *An algorithm for solving singular perturbation problems with mechanization*, Applied Mathematics and Computation, 188(1), 286-302, 2007.

[17] Heydari M.H., Razzaghi M., *Highly accurate solutions for space–time fractional Schrödinger equations with non-smooth continuous solution using the hybrid clique functions*, Mathematical Sciences, 17(1), 31-42, 2023.

[18] Hoede C., Li X., *Clique polynomials and independent set polynomials of graphs*, Discrete Mathematics, 125(1-3), 219-228, 1994.

[19] Izadi M., Srivastava H.M., *Fractional clique collocation technique for numerical simulations of fractional-order Brusselator chemical model*, Axioms, 11(11), 654, 2022.

[20] Kadalbajoo M.K., Arora P., *B-spline collocation method for the singular-perturbation problem using artificial viscosity*, Computers and Mathematics with Applications, 57(4), 650-663, 2009.

[21] Kadalbajoo M.K., Arora P., *B-splines with artificial viscosity for solving singularly perturbed boundary value problems*, Mathematical and Computer Modelling, 52(5-6), 654-666, 2010.

[22] Kadalbajoo M.K., Patidar K.C., *$\epsilon-$ Uniformly convergent fitted mesh finite difference methods for general singular perturbation problems*, Applied Mathematics and Computation, 179(1), 248-266, 2006.

[23] Klinshov V., Maslennikov O., Nekorkin V., *Jittering regimes of two spiking oscillators with delayed coupling*, Applied Mathematics and Nonlinear Sciences, 1(1), 197-206, 2016.

[24] Kumar M., *A second order spline finite difference method for singular two-point boundary value problems*, Applied Mathematics and Computation, 142(2-3), 283-290, 2003.

[25] Kumbinarasaiah S., Manohara G., *A novel approach for the system of coupled differential equations using clique polynomials of graph*, Partial Differential Equations in Applied Mathematics, 5, 100181, 2022.

[26] Liu X., Ballinger G., *Boundedness for impulsive delay differential equations and applications to population growth models*, Nonlinear Analysis: Theory, Methods and Applications, 53(7-8), 1041-1062, 2003.

[27] Nelson P.W., Perelson A.S., *Mathematical analysis of delay differential equation models of HIV-1 infection*, Mathematical Biosciences, 179(1), 73-94, 2002.

[28] Nirmala A.N., Kumbinarasaiah S., *A novel analytical method for the multi-delay fractional differential equations through the matrix of clique polynomials of the cocktail party graph*, Results in Control and Optimization, 12, 100280, 2023.

[29] Qureshi S., Ramos H., Soomro A., Akinfenwa O.A., Akanbi M.A., *Numerical integration of stiff problems using a new time-efficient hybrid block solver based on collocation and interpolation techniques*, Mathematics and Computers in Simulation, 220, 237-252, 2024.

[30] Qureshi S., Soomro A., Naseem A., Gdawiec K., Argyros I.K., Alshaery A.A., Secer, A., *From Halley to Secant: Redefining root finding with memory-based methods including convergence and stability*, Mathematical Methods in the Applied Sciences, 47(7), 5509-5531, 2024.

[31] Roussel M.R., *The use of delay differential equations in chemical kinetics*, The Journal of Physical Chemistry, 100(20), 8323-8330, 1996.

[32] Sabir Z., Günerhan H., Guirao J.L., *On a new model based on third-order nonlinear multisingular functional differential equations*, Mathematical Problems in Engineering, 2020(1), 1683961, 2020.

[33] Sanz-Serna J.M., Zhu B., *Word series high-order averaging of highly oscillatory differential equations with delay*, Applied Mathematics and Nonlinear Sciences, 4(2), 445-454, 2019.

[34] Sezer M., Akyüz-Daşcıoglu A., *Taylor polynomial solutions of general linear differential–difference equations with variable coefficients*, Applied Mathematics and Computation, 174(2), 1526-1538, 2006.

[35] Sezer M., Akyüz-Daşcıoglu A., *A Taylor method for numerical solution of generalized pantograph equations with linear functional argument*, Journal of Computational and Applied Mathematics, 200(1), 217-225, 2007.

[36] Shvets A., Makaseyev A., *Deterministic chaos in pendulum systems with delay*, Applied Mathematics and Nonlinear Sciences, 4(1), 1-8, 2019.

[37] Villasana M., Radunskaya A., *A delay differential equation model for tumor growth*, Journal of Mathematical Biology, 47, 270-294, 2003.

[38] Wazwaz A.M., *A new method for solving singular initial value problems in the second-order ordinary differential equations*, Applied Mathematics and Computation, 128(1), 45-57, 2002.

[39] Yüzbasi Ş., Gök E., Sezer M., *A numerical method for solving systems of higher order linear functional differential equations*, Open Physics, 14(1), 15-25, 2016.

[40] Yüzbaşı Ş., *A collocation method based on the Bessel functions of the first kind for singular perturbated differential equations and residual correction*, Mathematical Methods in the Applied Sciences, 38(14), 3033-3042, 2015.

[41] Yüzbaşı Ş., *A Laguerre approach for the solutions of singular perturbated differential equations*, International Journal of Computational Methods, 14(4), 1750034, 2017.

[42] Yüzbaşı Ş., Gök E., Sezer M., *Laguerre matrix method with the residual error estimation for solutions of a class of delay differential equations*, Mathematical Methods in the Applied Sciences, 37(4), 453-463, 2014.

[43] Yüzbaşı Ş., Tamar M.E., *A new approaching method for linear neutral delay differential equations by using Clique polynomials*, Turkish Journal of Mathematics, 47(7), 2098-2121, 2023.