Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (XXXX), 1 – 18
DOI : 10.15672/hujms.xx

Research Article

On a nonlinear difference equation of the fourth order solvable in closed form and its solutions

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Abstract

We show that a nonlinear difference equation recently considered in this journal is a special case of a solvable class of nonlinear difference equations and that the difference equation is closely related to a difference equation previously considered in the literature. We give some detailed theoretical explanations for the closed-form formulas for the solutions to the four special cases of the difference equation considered therein without giving any theoretical explanations related to them, and also show that several statements on the long-term behaviour of positive solutions to the difference equation given therein are not true.

Mathematics Subject Classification (2020). 39A20

Keywords. solvable nonlinear difference equation, bilinear difference equation, solutions in closed form

1. Introduction

Let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}$ be the set of whole numbers, $\mathbb{N}_k = \{n \in \mathbb{Z} : n \geq k\}$, $k \in \mathbb{Z}$, $\mathbb{R}$ be the set of reals, and $\mathbb{C}$ be the set of complex numbers. If $s, t \in \mathbb{Z}$, $s \leq t$, then $j = \overline{s, t}$, is a notation which we use for the expression: $j = s, \ldots, t$, $j \in \mathbb{Z}$. If $l \in \mathbb{Z}$, we regard $\prod_{j=l}^{l-1} c_j = 1$, where c_j is a sequence of numbers defined on a domain $I \subset \mathbb{Z}$.

Solvability of difference equations is one of the first problems that attracted attention to scientists. The papers and books [4, 8, 13–16] present some of the oldest results in the research domain. Closed-form formulas for solutions to linear homogeneous difference equations with constant coefficients of small orders were given in [8], whereas a method for solving the equations of any order was given in [4]. For more information on solvability of difference equations see, for instance [5, 10, 17–19].

The bilinear difference equation

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

is transformed to a linear homogeneous difference equation of second order, so it is also solvable (see, e.g., [12, 14]). Some other information and results on the equation and its applications can be found, for example, in [1, 5, 6, 11, 12, 19, 28, 29, 34, 38].

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Received: 19.04.2024; Accepted: 28.05.2024

One of the important things related to the solvable difference equations and systems of difference equations is their direct or potential applicability (see, e.g., [8, 10, 17, 24, 39]). They are also useful in some comparison results [3, 27]. There are also some connections with the numerical mathematics [7].

Investigation of the behaviour of solutions to concrete difference equations and systems is an area of some recent interest. Some of the investigations are devoted to finding invariants and closed-form formulas for the general solutions to the equations and systems. One can consult, for example, the following recent papers: [2, 11, 20–23, 25, 26, 28–38], as well as the papers on these topics quoted therein. These papers present some methods for finding solutions or invariants, based on some scientific investigations.

On the other hand, in the last two decades appear more and more papers which do not give any explanations for the formulas and claims presented therein, many of which are, unfortunately, not true.

The difference equation

$$x_{n+1} = ax_n + \frac{bx_n x_{n-2}}{cx_{n-2} + dx_{n-3}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where the parameters a, b, c, d and the initial values x_{-j}, $j = \overline{0,3}$, are positive numbers, was considered in [9].

Here we show that Eq.(1.2) is a special case of a solvable class of difference equations, and that it is closely related to a difference equation previously considered by one of the authors in [9]. Further, we give some detailed theoretical explanations for the closed-form formulas for the solutions to the four special cases of the difference equation considered therein without giving any theoretical explanations connected to them. Finally, we show that several statements on the long-term behaviour of positive solutions to Eq.(1.2) are not true. In the investigation we use some methods and ideas related to the ones, for example, in [28–30, 34, 38].

2. Analyses, theoretical explanations and counterexamples

Here we analyse and compare the results in [9], with some in the literature, give some detailed theoretical explanations for the closed-form formulas for the solutions to the four special cases of the difference equation considered therein, and give several comments and counterexamples.

2.1. On solvability of Eq.(1.2)

One of the authors in [9], had previously considered the following difference equation:

$$x_{n+1} = ax_n + \frac{bx_n x_{n-3}}{cx_{n-2} + dx_{n-3}}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

where the parameters a, b, c, d and the initial values x_{-j}, $j = \overline{0,3}$, are positive numbers.

If we write Eq.(1.2) in the form

$$x_{n+1} = x_n \frac{(ac+b)x_{n-2} + adx_{n-3}}{cx_{n-2} + dx_{n-3}}, \quad n \in \mathbb{N}_0, \tag{2.2}$$

and Eq.(2.1) in the form

$$x_{n+1} = x_n \frac{acx_{n-2} + (ad+b)x_{n-3}}{cx_{n-2} + dx_{n-3}}, \quad n \in \mathbb{N}_0, \tag{2.3}$$

we see that both equations belong to the same class of difference equations, namely, to the following one

$$x_{n+1} = x_n \frac{\alpha x_{n-2} + \beta x_{n-3}}{\gamma x_{n-2} + \delta x_{n-3}}, \quad n \in \mathbb{N}_0, \tag{2.4}$$

where the parameters $\alpha, \beta, \gamma, \delta$ and the initial values x_{-j}, $j = \overline{0,3}$, are positive numbers, as it was assumed in [9].

Remark 2.1. It is easily seen that the assumption on the positivity of the parameters $\alpha, \beta, \gamma, \delta$ and the initial values x_{-j}, $j = \overline{0,3}$, is, more or less, an artificial assumption. The only reason for posing them, which is justified to some extent, is to avoid dealing with undefined solutions to Eq.(1.2) and Eq.(2.1), that is, to avoid that the denominator $cx_{n-2}+dx_{n-3}$ is equal to zero for some $n \in \mathbb{N}_0$. Hence, we may assume that the parameters and the initial values belong to $\mathbb{R}$ or $\mathbb{C}$, and ignore the solutions which are not defined for all $n \in \mathbb{N}_{-3}$.

Motivated by Eq.(2.1), in [35] we studied a generalization of Eq.(2.4) and proved, among other ones, the following theorem.

Theorem 2.2. *Assume $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0 \neq \gamma^2 + \delta^2$, g is a strictly monotone and continuous function, $g(\mathbb{R}) = \mathbb{R}$ and $g(0) = 0$. Then, the equation*

$$x_{n+1} = g^{-1}\left(g(x_n)\frac{\alpha g(x_{n-2}) + \beta g(x_{n-3})}{\gamma g(x_{n-2}) + \delta g(x_{n-3})}\right), \quad n \in \mathbb{N}_0, \tag{2.5}$$

is solvable in closed form.

Moreover, if $\gamma \neq 0$ and

$$\Delta := (\alpha+\delta)^2 - 4(\alpha\delta - \beta\gamma) \neq 0,$$

then the general solution to Eq.(2.5) is given by

$$x_{3m} = g^{-1}\left(g(x_{-3})\prod_{i=0}^{m} y_{3i}y_{3i-1}y_{3i-2}\right), \tag{2.6}$$

$$x_{3m+1} = g^{-1}\left(g(x_{-2})\prod_{i=0}^{m} y_{3i+1}y_{3i}y_{3i-1}\right), \tag{2.7}$$

$$x_{3m+2} = g^{-1}\left(g(x_{-1})\prod_{i=0}^{m} y_{3i+2}y_{3i+1}y_{3i}\right), \tag{2.8}$$

for $m \in \mathbb{N}_{-1}$, where

$$\begin{aligned}
&y_{3m}y_{3m-1}y_{3m-2}\\
&= \left(\frac{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma}\right)\\
&\times\left(\frac{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma}\right)\\
&\times\left(\frac{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma}\right),
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&y_{3m+1}y_{3m}y_{3m-1}\\
&=\left(\frac{\left(\frac{g(x_{-2})}{g(x_{-3})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+2}-\left(\frac{g(x_{-2})}{g(x_{-3})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+2}}{\left(\frac{g(x_{-2})}{g(x_{-3})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{g(x_{-2})}{g(x_{-3})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{\left(\frac{g(x_{0})}{g(x_{-1})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{g(x_{0})}{g(x_{-1})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_{0})}{g(x_{-1})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m}-\left(\frac{g(x_{0})}{g(x_{-1})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{\left(\frac{g(x_{-1})}{g(x_{-2})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{g(x_{-1})}{g(x_{-2})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_{-1})}{g(x_{-2})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m}-\left(\frac{g(x_{-1})}{g(x_{-2})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}\tag{2.10}$$

$$\begin{aligned}
&y_{3m+2}y_{3m+1}y_{3m}\\
&=\left(\frac{\left(\frac{g(x_{-1})}{g(x_{-2})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+2}-\left(\frac{g(x_{-1})}{g(x_{-2})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+2}}{\left(\frac{g(x_{-1})}{g(x_{-2})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{g(x_{-1})}{g(x_{-2})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{\left(\frac{g(x_{-2})}{g(x_{-3})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+2}-\left(\frac{g(x_{-2})}{g(x_{-3})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+2}}{\left(\frac{g(x_{-2})}{g(x_{-3})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{g(x_{-2})}{g(x_{-3})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{\left(\frac{g(x_{0})}{g(x_{-1})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{g(x_{0})}{g(x_{-1})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_{0})}{g(x_{-1})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m}-\left(\frac{g(x_{0})}{g(x_{-1})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}\tag{2.11}$$

for $m\in\mathbb{N}_0$, *and*

$$\lambda_1=\frac{\alpha+\delta+\sqrt{\Delta}}{2\gamma}\quad \textit{and}\quad \lambda_2=\frac{\alpha+\delta-\sqrt{\Delta}}{2\gamma}.$$

We will not repeat the proof of the theorem given in [35]. However, we will mention the key points of the proof of the theorem for the completeness and benefit of the reader.

From Eq.(2.5) and the monotonicity of the function g, it is immediately seen that the following relation holds

$$g(x_{n+1})=g(x_n)\frac{\alpha g(x_{n-2})+\beta g(x_{n-3})}{\gamma g(x_{n-2})+\delta g(x_{n-3})},\quad n\in\mathbb{N}_0.\tag{2.12}$$

The change of variables

$$y_n=\frac{g(x_n)}{g(x_{n-1})},\quad n\in\mathbb{N}_{-2},$$

transforms Eq.(2.12) to

$$y_{n+1}=\frac{\alpha y_{n-2}+\beta}{\gamma y_{n-2}+\delta},\tag{2.13}$$

for $n\in\mathbb{N}_0$.

Eq.(2.13) is a difference equation with interlacing indices. For a precise definition, some discussions on such difference equations and systems, and some example, see [31,34].

Hence, the sequences

$$z_m^{(j)}=y_{3m-j},\quad m\in\mathbb{N}_0,\ j=\overline{0,2},$$

satisfy the relation

$$z_{m+1}^{(j)}=\frac{\alpha z_m^{(j)}+\beta}{\gamma z_m^{(j)}+\delta},\tag{2.14}$$

for $m\in\mathbb{N}_0$, $j=\overline{0,2}$, that is, they are three solutions to the bilinear difference equation

$$z_{m+1}=\frac{\alpha z_m+\beta}{\gamma z_m+\delta},\quad m\in\mathbb{N}_0.$$

As we have already mentioned in the first section, it is well known that the difference equation is solvable in closed form, and that it can be solved by several methods (see, for instance, [1, 5, 10, 12–14, 17, 19]).

Let $\gamma \neq 0$ and

$$z_m^{(j)} = \frac{u_{m+1}^{(j)}}{u_m^{(j)}} - \frac{\delta}{\gamma}, \quad m \in \mathbb{N}_0,\ j = \overline{0,2}, \tag{2.15}$$

(if $\gamma = 0$ then the equations in (2.14) are linear in m), then from (2.14) we get

$$\gamma^2 u_{m+2}^{(j)} - \gamma(\alpha+\delta)u_{m+1}^{(j)} + (\alpha\delta - \beta\gamma)u_m^{(j)} = 0, \tag{2.16}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

Eq.(2.16) is a linear homogeneous difference equation with constant coefficients of second order. So, we can find closed-form formulas for the general solution to the equation by using the formulas by De Moivre and D. Bernoulli ([4, 8]), from which the formulas for the general solutions to Eq.(2.14) are obtained, and consequently the formulas for solutions to Eq.(2.13), from which the general solution to Eq.(2.5) is found.

So, in [35], we proved that Eq.(1.2) is also solvable. This shows that Eqs.(1.2) and (2.1) could have been studied together.

Remark 2.3. Note that in [9] are presented some closed-form formulas for the following four special cases of Eq.(1.2):

$$x_{n+1} = x_n + \frac{x_n x_{n-2}}{x_{n-2} + x_{n-3}}, \tag{2.17}$$

$$x_{n+1} = x_n + \frac{x_n x_{n-2}}{x_{n-2} - x_{n-3}}, \tag{2.18}$$

$$x_{n+1} = x_n - \frac{x_n x_{n-2}}{x_{n-2} + x_{n-3}}, \tag{2.19}$$

$$x_{n+1} = x_n - \frac{x_n x_{n-2}}{x_{n-2} - x_{n-3}}, \tag{2.20}$$

for $n \in \mathbb{N}_0$.

It was not explained therein why these very special cases are chosen for presenting their general solutions. Note that Theorem 2.2, in fact, shows that the equation in these four cases is not distinguished in any way, to be considered separately in an investigation.

Remark 2.4. The closed-form formulas for solutions to Eqs.(2.17)-(2.20) in [9] can be obtained from Theorem 2.2. Note that Eq.(1.2) is a special case of Eq.(2.5) with

$$g(x) = x, \quad \alpha = ac + b, \quad \beta = ad, \quad \gamma = c \quad \text{and} \quad \delta = d.$$

This means that Theorem 2.2 gives a theoretical explanation for all the formulas given in [9].

3. On the solutions to Eqs.(2.17)-(2.20) in [9]

The formulas for solutions to Eqs.(2.17)-(2.20) given in [9] are presented in terms of a specially chosen sequence. Now we will explain how the representations of the solutions can be obtained from Theorem 2.2. Before this, we need to mention a known sequence, which essentially appears in the representations.

Recall that the sequence $(f_n)_{n\in\mathbb{N}}$ satisfying the difference equation

$$f_{n+2} = f_{n+1} + f_n, \quad n \in \mathbb{N}, \tag{3.1}$$

and the initial conditions

$$f_1 = 1, \quad f_2 = 1,$$

is called the Fibonacci sequence. Some properties of the sequence can be found, for example, in [39] (see, also [12]). From the De Moivre formula given in [8] we have

$$f_n = \frac{t_1^n - t_2^n}{\sqrt{5}}, \quad n \in \mathbb{N}, \tag{3.2}$$

where

$$t_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad t_2 = \frac{1-\sqrt{5}}{2}. \tag{3.3}$$

Formula (3.2) can be found in D. Bernoulli's paper [4]. It is a special case of the generalized Fibonacci sequence (see [28, 38]). Since (3.1) implies

$$f_n = f_{n+2} - f_{n+1},$$

we see that f_n can be also calculated for $n \le 0$, and that formula (3.2) holds for every $n \in \mathbb{Z}$.

Note also that from (3.3) and some simple calculations we get

$$t_1^2 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad t_2^2 = \frac{3-\sqrt{5}}{2}. \tag{3.4}$$

Example 3.1. Consider Eq.(2.17). Note that in this case we have $a = b = c = d = 1$, which implies that $\alpha = 2$, $\beta = \gamma = \delta = 1$. Hence, by using Theorem 2.2, we have that the general solution to Eq.(2.17) is

$$\begin{aligned} x_{3m} =& x_{-3}\prod_{i=0}^{m}\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_0}{x_{-1}}-\lambda_1+1)\lambda_2^{i+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+1)\lambda_1^{i}-(\frac{x_0}{x_{-1}}-\lambda_1+1)\lambda_2^{i}}-1\right)\\ &\times\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+1)\lambda_2^{i+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+1)\lambda_1^{i}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+1)\lambda_2^{i}}-1\right)\\ &\times\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+1)\lambda_2^{i+1}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+1)\lambda_1^{i}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+1)\lambda_2^{i}}-1\right), \end{aligned} \tag{3.5}$$

$$\begin{aligned} x_{3m+1} =& x_{-2}\prod_{i=0}^{m}\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+1)\lambda_1^{i+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+1)\lambda_2^{i+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+1)\lambda_2^{i+1}}-1\right)\\ &\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_0}{x_{-1}}-\lambda_1+1)\lambda_2^{i+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+1)\lambda_1^{i}-(\frac{x_0}{x_{-1}}-\lambda_1+1)\lambda_2^{i}}-1\right)\\ &\times\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+1)\lambda_2^{i+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+1)\lambda_1^{i}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+1)\lambda_2^{i}}-1\right), \end{aligned} \tag{3.6}$$

$$\begin{aligned} x_{3m+2} =& x_{-1}\prod_{i=0}^{m}\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+1)\lambda_1^{i+2}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+1)\lambda_2^{i+2}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+1)\lambda_2^{i+1}}-1\right)\\ &\times\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+1)\lambda_1^{i+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+1)\lambda_2^{i+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+1)\lambda_2^{i+1}}-1\right)\\ &\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_0}{x_{-1}}-\lambda_1+1)\lambda_2^{i+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+1)\lambda_1^{i}-(\frac{x_0}{x_{-1}}-\lambda_1+1)\lambda_2^{i}}-1\right), \end{aligned} \tag{3.7}$$

for $m \in \mathbb{N}_{-1}$, where

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{3-\sqrt{5}}{2}.$$

Note that

$$\lambda_1 = t_1^2, \quad \lambda_2 = t_2^2, \tag{3.8}$$

and

$$1-\lambda_1 = -t_1, \quad 1-\lambda_2 = -t_2. \tag{3.9}$$

Employing (3.8), (3.9) and the fact

$$t_1 t_2 = -1,$$

we have

$$\begin{aligned} v_i^{(j)} :=& \Big(\frac{x_{-j}}{x_{-j-1}} - \lambda_2 + 1\Big)\lambda_1^i - \Big(\frac{x_{-j}}{x_{-j-1}} - \lambda_1 + 1\Big)\lambda_2^i \\ =& \frac{(x_{-j} + (1-\lambda_2)x_{-j-1})\lambda_1^i - (x_{-j} + (1-\lambda_1)x_{-j-1})\lambda_2^i}{x_{-j-1}} \\ =& \frac{x_{-j}(\lambda_1^i - \lambda_2^i) + x_{-j-1}((1-\lambda_2)\lambda_1^i - (1-\lambda_1)\lambda_2^i)}{x_{-j-1}} \\ =& \frac{x_{-j}(t_1^{2i} - t_2^{2i}) + x_{-j-1}(t_1^{2i-1} - t_2^{2i-1})}{x_{-j-1}} \\ =& \frac{(x_{-j}f_{2i} + x_{-j-1}f_{2i-1})\sqrt{5}}{x_{-j-1}}, \end{aligned} \tag{3.10}$$

for $i \in \mathbb{N}_0$ and $j = \overline{0,2}$.

Using (3.10), and then (3.1), we get

$$\begin{aligned} v_{i+1}^{(j)} - v_i^{(j)} =& \frac{(x_{-j}f_{2i+2} + x_{-j-1}f_{2i+1})\sqrt{5}}{x_{-j-1}} - \frac{(x_{-j}f_{2i} + x_{-j-1}f_{2i-1})\sqrt{5}}{x_{-j-1}} \\ =& \frac{(x_{-j}f_{2i+1} + x_{-j-1}f_{2i})\sqrt{5}}{x_{-j-1}}, \end{aligned} \tag{3.11}$$

for $i \in \mathbb{N}_0$ and $j = \overline{0,2}$.

By using (3.10) and (3.11) in (3.5)-(3.7), we get

$$x_{3m} = x_{-3}\prod_{i=0}^{m}\left(\frac{x_0 f_{2i+1} + x_{-1}f_{2i}}{x_0 f_{2i} + x_{-1}f_{2i-1}}\right)\left(\frac{x_{-1}f_{2i+1} + x_{-2}f_{2i}}{x_{-1}f_{2i} + x_{-2}f_{2i-1}}\right)\left(\frac{x_{-2}f_{2i+1} + x_{-3}f_{2i}}{x_{-2}f_{2i} + x_{-3}f_{2i-1}}\right), \tag{3.12}$$

$$x_{3m+1} = x_{-2}\prod_{i=0}^{m}\left(\frac{x_0 f_{2i+1} + x_{-1}f_{2i}}{x_0 f_{2i} + x_{-1}f_{2i-1}}\right)\left(\frac{x_{-1}f_{2i+1} + x_{-2}f_{2i}}{x_{-1}f_{2i} + x_{-2}f_{2i-1}}\right)\left(\frac{x_{-2}f_{2i+3} + x_{-3}f_{2i+2}}{x_{-2}f_{2i+2} + x_{-3}f_{2i+1}}\right), \tag{3.13}$$

$$x_{3m+2} = x_{-1}\prod_{i=0}^{m}\left(\frac{x_0 f_{2i+1} + x_{-1}f_{2i}}{x_0 f_{2i} + x_{-1}f_{2i-1}}\right)\left(\frac{x_{-1}f_{2i+3} + x_{-2}f_{2i+2}}{x_{-1}f_{2i+2} + x_{-2}f_{2i+1}}\right)\left(\frac{x_{-2}f_{2i+3} + x_{-3}f_{2i+2}}{x_{-2}f_{2i+2} + x_{-3}f_{2i+1}}\right), \tag{3.14}$$

for $m \in \mathbb{N}_{-1}$.

In this way, we explained how the formulas in Theorem 5.1 in [9] can be obtained in a natural way. Note also that we proved that the formulas also hold for $m = -1$, which was not noticed in [9].

Example 3.2. Consider Eq.(2.18). Note that in this case we have

$$a = b = c = 1 \quad \text{and} \quad d = -1,$$

which implies that

$$\alpha = 2, \quad \beta = \delta = -1 \quad \text{and} \quad \gamma = 1.$$

Hence, by using Theorem 2.2, we have that the general solution to Eq.(2.18) is

$$\begin{aligned}
x_{3m} =& x_{-3}\prod_{i=0}^{m}\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_0}{x_{-1}}-\lambda_1-1)\lambda_2^{i+1}}{(\frac{x_0}{x_{-1}}-\lambda_2-1)\lambda_1^{i}-(\frac{x_0}{x_{-1}}-\lambda_1-1)\lambda_2^{i}}+1\right)\\
&\times\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1-1)\lambda_2^{i+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2-1)\lambda_1^{i}-(\frac{x_{-1}}{x_{-2}}-\lambda_1-1)\lambda_2^{i}}+1\right)\\
&\times\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1-1)\lambda_2^{i+1}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2-1)\lambda_1^{i}-(\frac{x_{-2}}{x_{-3}}-\lambda_1-1)\lambda_2^{i}}+1\right), \qquad (3.15)\\
x_{3m+1} =& x_{-2}\prod_{i=0}^{m}\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2-1)\lambda_1^{i+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1-1)\lambda_2^{i+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1-1)\lambda_2^{i+1}}+1\right)\\
&\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_0}{x_{-1}}-\lambda_1-1)\lambda_2^{i+1}}{(\frac{x_0}{x_{-1}}-\lambda_2-1)\lambda_1^{i}-(\frac{x_0}{x_{-1}}-\lambda_1-1)\lambda_2^{i}}+1\right)\\
&\times\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1-1)\lambda_2^{i+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2-1)\lambda_1^{i}-(\frac{x_{-1}}{x_{-2}}-\lambda_1-1)\lambda_2^{i}}+1\right), \qquad (3.16)\\
x_{3m+2} =& x_{-1}\prod_{i=0}^{m}\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2-1)\lambda_1^{i+2}-(\frac{x_{-1}}{x_{-2}}-\lambda_1-1)\lambda_2^{i+2}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1-1)\lambda_2^{i+1}}+1\right)\\
&\times\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2-1)\lambda_1^{i+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1-1)\lambda_2^{i+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1-1)\lambda_2^{i+1}}+1\right)\\
&\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_0}{x_{-1}}-\lambda_1-1)\lambda_2^{i+1}}{(\frac{x_0}{x_{-1}}-\lambda_2-1)\lambda_1^{i}-(\frac{x_0}{x_{-1}}-\lambda_1-1)\lambda_2^{i}}+1\right), \qquad (3.17)
\end{aligned}$$

for $m \in \mathbb{N}_{-1}$, where

$$\lambda_1 = \frac{1+\sqrt{5}}{2} = t_1$$

and

$$\lambda_2 = \frac{1-\sqrt{5}}{2} = t_2.$$

Employing the facts

$$t_1t_2 = -1 \quad \text{and} \quad t_1 + t_2 = 1,$$

we have

$$\begin{aligned}
w_i^{(j)} :=&\Big(\frac{x_{-j}}{x_{-j-1}}-\lambda_2-1)\lambda_1^i-(\frac{x_{-j}}{x_{-j-1}}-\lambda_1-1\Big)\lambda_2^i\\
=&\frac{(x_{-j}-(1+\lambda_2)x_{-j-1})\lambda_1^i-(x_{-j}-(1+\lambda_1)x_{-j-1})\lambda_2^i}{x_{-j-1}}\\
=&\frac{x_{-j}(t_1^i-t_2^i)-x_{-j-1}((1+t_2)t_1^i-(1+t_1)t_2^i)}{x_{-j-1}}\\
=&\frac{x_{-j}(t_1^i-t_2^i)-x_{-j-1}(t_1^{i-2}-t_2^{i-2})}{x_{-j-1}}\\
=&\frac{(x_{-j}f_i-x_{-j-1}f_{i-2})\sqrt{5}}{x_{-j-1}},
\end{aligned} \tag{3.18}$$

for $i\in\mathbb{N}_0$ and $j=\overline{0,2}$.

Using (3.18), and then (3.1), we get

$$\begin{aligned}
w_{i+1}^{(j)}+w_i^{(j)} =&\frac{(x_{-j}f_{i+1}-x_{-j-1}f_{i-1})\sqrt{5}}{x_{-j-1}}+\frac{(x_{-j}f_i-x_{-j-1}f_{i-2})\sqrt{5}}{x_{-j-1}}\\
=&\frac{(x_{-j}f_{i+2}-x_{-j-1}f_i)\sqrt{5}}{x_{-j-1}},
\end{aligned} \tag{3.19}$$

for $i\in\mathbb{N}_0$ and $j=\overline{0,2}$.

By using (3.18) and (3.19) in (3.15)-(3.17), we get

$$x_{3m} =x_{-3}\prod_{i=0}^{m}\left(\frac{x_0f_{i+2}-x_{-1}f_i}{x_0f_i-x_{-1}f_{i-2}}\right)\left(\frac{x_{-1}f_{i+2}-x_{-2}f_i}{x_{-1}f_i-x_{-2}f_{i-2}}\right)\left(\frac{x_{-2}f_{i+2}-x_{-3}f_i}{x_{-2}f_i-x_{-3}f_{i-2}}\right), \tag{3.20}$$

$$x_{3m+1} =x_{-2}\prod_{i=0}^{m}\left(\frac{x_0f_{i+2}-x_{-1}f_i}{x_0f_i-x_{-1}f_{i-2}}\right)\left(\frac{x_{-1}f_{i+2}-x_{-2}f_i}{x_{-1}f_i-x_{-2}f_{i-2}}\right)\left(\frac{x_{-2}f_{i+3}-x_{-3}f_{i+1}}{x_{-2}f_{i+1}-x_{-3}f_{i-1}}\right), \tag{3.21}$$

$$x_{3m+2} =x_{-1}\prod_{i=0}^{m}\left(\frac{x_0f_{i+2}-x_{-1}f_i}{x_0f_i-x_{-1}f_{i-2}}\right)\left(\frac{x_{-1}f_{i+3}-x_{-2}f_{i+1}}{x_{-1}f_{i+1}-x_{-2}f_{i-1}}\right)\left(\frac{x_{-2}f_{i+3}-x_{-3}f_{i+1}}{x_{-2}f_{i+1}-x_{-3}f_{i-1}}\right), \tag{3.22}$$

for $m\in\mathbb{N}_{-1}$.

In this way, we explained how the formulas in Theorem 5.2 in [9] can be obtained in a natural way. Note also that we proved that the formulas also hold for $m=-1$, which was not noticed in [9].

Example 3.3. Consider Eq.(2.19). Note that in this case we have $a=c=d=1$ and $b=-1$, which implies that $\alpha=0$, $\beta=\gamma=\delta=1$. Hence, by using Theorem 2.2, we have that the general solution to Eq.(2.19) is

$$\begin{aligned}
x_{3m} =&x_{-3}\prod_{i=0}^{m}\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_0}{x_{-1}}-\lambda_1+1)\lambda_2^{i+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+1)\lambda_1^{i}-(\frac{x_0}{x_{-1}}-\lambda_1+1)\lambda_2^{i}}-1\right)\\
&\times\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+1)\lambda_2^{i+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+1)\lambda_1^{i}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+1)\lambda_2^{i}}-1\right)\\
&\times\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+1)\lambda_2^{i+1}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+1)\lambda_1^{i}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+1)\lambda_2^{i}}-1\right), \qquad (3.23)\\
x_{3m+1} =&x_{-2}\prod_{i=0}^{m}\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+1)\lambda_1^{i+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+1)\lambda_2^{i+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+1)\lambda_2^{i+1}}-1\right)\\
&\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_0}{x_{-1}}-\lambda_1+1)\lambda_2^{i+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+1)\lambda_1^{i}-(\frac{x_0}{x_{-1}}-\lambda_1+1)\lambda_2^{i}}-1\right)\\
&\times\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+1)\lambda_2^{i+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+1)\lambda_1^{i}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+1)\lambda_2^{i}}-1\right), \qquad (3.24)\\
x_{3m+2} =&x_{-1}\prod_{i=0}^{m}\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+1)\lambda_1^{i+2}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+1)\lambda_2^{i+2}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+1)\lambda_2^{i+1}}-1\right)\\
&\times\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+1)\lambda_1^{i+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+1)\lambda_2^{i+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+1)\lambda_2^{i+1}}-1\right)\\
&\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2+1)\lambda_1^{i+1}-(\frac{x_0}{x_{-1}}-\lambda_1+1)\lambda_2^{i+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+1)\lambda_1^{i}-(\frac{x_0}{x_{-1}}-\lambda_1+1)\lambda_2^{i}}-1\right), \qquad (3.25)
\end{aligned}$$

for $m\in\mathbb{N}_{-1}$, where

$$\lambda_1=\frac{1+\sqrt{5}}{2}=t_1 \quad\text{and}\quad \lambda_2=\frac{1-\sqrt{5}}{2}=t_2. \qquad (3.26)$$

Employing (3.26), the facts $t_1t_2=-1$ and $t_1+t_2=1$, we have

$$\begin{aligned}
v_i^{(j)} :=&\Big(\frac{x_{-j}}{x_{-j-1}}-\lambda_2+1\Big)\lambda_1^i-\Big(\frac{x_{-j}}{x_{-j-1}}-\lambda_1+1\Big)\lambda_2^i\\
=&\frac{(x_{-j}+(1-\lambda_2)x_{-j-1})\lambda_1^i-(x_{-j}+(1-\lambda_1)x_{-j-1})\lambda_2^i}{x_{-j-1}}\\
=&\frac{x_{-j}(t_1^i-t_2^i)+x_{-j-1}((1-t_2)t_1^i-(1-t_1)t_2^i)}{x_{-j-1}}\\
=&\frac{x_{-j}(t_1^i-t_2^i)+x_{-j-1}(t_1^{i+1}-t_2^{i+1})}{x_{-j-1}}\\
=&\frac{(x_{-j}f_i+x_{-j-1}f_{i+1})\sqrt{5}}{x_{-j-1}}, \qquad (3.27)
\end{aligned}$$

for $i\in\mathbb{N}_0$ and $j=\overline{0,2}$.

Using first the relation in (3.27), and after that the recursive relation in (3.1), it follows that

$$\begin{aligned} v_{i+1}^{(j)} - v_i^{(j)} =& \frac{(x_{-j}f_{i+1} + x_{-j-1}f_{i+2})\sqrt{5}}{x_{-j-1}} - \frac{(x_{-j}f_i + x_{-j-1}f_{i+1})\sqrt{5}}{x_{-j-1}} \\ =& \frac{(x_{-j}f_{i-1} + x_{-j-1}f_i)\sqrt{5}}{x_{-j-1}}, \end{aligned} \tag{3.28}$$

for $i \in \mathbb{N}_0$ and $j = \overline{0,2}$.

Further, if we use the relations (3.27) and (3.28) in formulas (3.23)-(3.25), it follows that

$$\begin{aligned} x_{3m} =& x_{-3}\prod_{i=0}^{m}\left(\frac{x_0f_{i-1}+x_{-1}f_i}{x_0f_i+x_{-1}f_{i+1}}\right)\left(\frac{x_{-1}f_{i-1}+x_{-2}f_i}{x_{-1}f_i+x_{-2}f_{i+1}}\right)\left(\frac{x_{-2}f_{i-1}+x_{-3}f_i}{x_{-2}f_i+x_{-3}f_{i+1}}\right), \\ x_{3m+1} =& x_{-2}\prod_{i=0}^{m}\left(\frac{x_0f_{i-1}+x_{-1}f_i}{x_0f_i+x_{-1}f_{i+1}}\right)\left(\frac{x_{-1}f_{i-1}+x_{-2}f_i}{x_{-1}f_i+x_{-2}f_{i+1}}\right)\left(\frac{x_{-2}f_i+x_{-3}f_{i+1}}{x_{-2}f_{i+1}+x_{-3}f_{i+2}}\right), \\ x_{3m+2} =& x_{-1}\prod_{i=0}^{m}\left(\frac{x_0f_{i-1}+x_{-1}f_i}{x_0f_i+x_{-1}f_{i+1}}\right)\left(\frac{x_{-1}f_i+x_{-2}f_{i+1}}{x_{-1}f_{i+1}+x_{-2}f_{i+2}}\right)\left(\frac{x_{-2}f_i+x_{-3}f_{i+1}}{x_{-2}f_{i+1}+x_{-3}f_{i+2}}\right), \end{aligned}$$

for $m \in \mathbb{N}_{-1}$.

From these relations, after canceling the same factors we get

$$\begin{aligned} x_{3m} =& x_{-3}\left(\frac{x_0f_{-1}+x_{-1}f_0}{x_0f_m+x_{-1}f_{m+1}}\right)\left(\frac{x_{-1}f_{-1}+x_{-2}f_0}{x_{-1}f_m+x_{-2}f_{m+1}}\right)\left(\frac{x_{-2}f_{-1}+x_{-3}f_0}{x_{-2}f_m+x_{-3}f_{m+1}}\right), \\ x_{3m+1} =& x_{-2}\left(\frac{x_0f_{-1}+x_{-1}f_0}{x_0f_m+x_{-1}f_{m+1}}\right)\left(\frac{x_{-1}f_{-1}+x_{-2}f_0}{x_{-1}f_m+x_{-2}f_{m+1}}\right)\left(\frac{x_{-2}f_0+x_{-3}f_1}{x_{-2}f_{m+1}+x_{-3}f_{m+2}}\right), \\ x_{3m+2} =& x_{-1}\left(\frac{x_0f_{-1}+x_{-1}f_0}{x_0f_m+x_{-1}f_{m+1}}\right)\left(\frac{x_{-1}f_0+x_{-2}f_1}{x_{-1}f_{m+1}+x_{-2}f_{m+2}}\right)\left(\frac{x_{-2}f_0+x_{-3}f_1}{x_{-2}f_{m+1}+x_{-3}f_{m+2}}\right), \end{aligned}$$

for $m \in \mathbb{N}_{-1}$.

Finally, by using the facts

$$f_{-1} = 1, \quad f_0 = 0 \quad \text{and} \quad f_1 = 1,$$

we get

$$x_{3m} = \frac{x_{-3}x_{-2}x_{-1}x_0}{(x_0f_m + x_{-1}f_{m+1})(x_{-1}f_m + x_{-2}f_{m+1})(x_{-2}f_m + x_{-3}f_{m+1})}, \tag{3.29}$$

$$x_{3m+1} = \frac{x_{-3}x_{-2}x_{-1}x_0}{(x_0f_m + x_{-1}f_{m+1})(x_{-1}f_m + x_{-2}f_{m+1})(x_{-2}f_{m+1} + x_{-3}f_{m+2})}, \tag{3.30}$$

$$x_{3m+2} = \frac{x_{-3}x_{-2}x_{-1}x_0}{(x_0f_m + x_{-1}f_{m+1})(x_{-1}f_{m+1} + x_{-2}f_{m+2})(x_{-2}f_{m+1} + x_{-3}f_{m+2})}, \tag{3.31}$$

for $m \in \mathbb{N}_{-1}$.

In this way, we explained how the formulas in Theorem 5.3 in [9] can be obtained in a natural way. Note also that we proved that the formulas also hold for $m = -1$, which was not noticed in [9].

Example 3.4. Consider Eq.(2.20). Note that in this case we have $a = c = 1$ and $b = d = -1$, which implies that $\alpha = 0$, $\beta = \delta = -1$ and $\gamma = 1$. Hence, by using Theorem

2.2, we have that the general solution to Eq.(2.20) is

$$\begin{aligned}
x_{3m} =& x_{-3}\prod_{i=0}^{m}\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_0}{x_{-1}}-\lambda_1-1)\lambda_2^{i+1}}{(\frac{x_0}{x_{-1}}-\lambda_2-1)\lambda_1^{i}-(\frac{x_0}{x_{-1}}-\lambda_1-1)\lambda_2^{i}}+1\right)\\
&\times\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1-1)\lambda_2^{i+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2-1)\lambda_1^{i}-(\frac{x_{-1}}{x_{-2}}-\lambda_1-1)\lambda_2^{i}}+1\right)\\
&\times\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1-1)\lambda_2^{i+1}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2-1)\lambda_1^{i}-(\frac{x_{-2}}{x_{-3}}-\lambda_1-1)\lambda_2^{i}}+1\right),\qquad (3.32)\\
x_{3m+1} =& x_{-2}\prod_{i=0}^{m}\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2-1)\lambda_1^{i+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1-1)\lambda_2^{i+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1-1)\lambda_2^{i+1}}+1\right)\\
&\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_0}{x_{-1}}-\lambda_1-1)\lambda_2^{i+1}}{(\frac{x_0}{x_{-1}}-\lambda_2-1)\lambda_1^{i}-(\frac{x_0}{x_{-1}}-\lambda_1-1)\lambda_2^{i}}+1\right)\\
&\times\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1-1)\lambda_2^{i+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2-1)\lambda_1^{i}-(\frac{x_{-1}}{x_{-2}}-\lambda_1-1)\lambda_2^{i}}+1\right),\qquad (3.33)\\
x_{3m+2} =& x_{-1}\prod_{i=0}^{m}\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2-1)\lambda_1^{i+2}-(\frac{x_{-1}}{x_{-2}}-\lambda_1-1)\lambda_2^{i+2}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1-1)\lambda_2^{i+1}}+1\right)\\
&\times\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2-1)\lambda_1^{i+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1-1)\lambda_2^{i+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1-1)\lambda_2^{i+1}}+1\right)\\
&\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2-1)\lambda_1^{i+1}-(\frac{x_0}{x_{-1}}-\lambda_1-1)\lambda_2^{i+1}}{(\frac{x_0}{x_{-1}}-\lambda_2-1)\lambda_1^{i}-(\frac{x_0}{x_{-1}}-\lambda_1-1)\lambda_2^{i}}+1\right),\qquad (3.34)
\end{aligned}$$

for $m\in\mathbb{N}_{-1}$, where

$$\lambda_1=\frac{-1+i\sqrt{3}}{2}=\varepsilon\quad\text{and}\quad\lambda_2=\frac{-1-i\sqrt{3}}{2}=\bar{\varepsilon}.$$

We also have

$$1+\lambda_1=-\lambda_1^2\quad\text{and}\quad 1+\lambda_2=-\lambda_2^2.\qquad (3.35)$$

Employing (3.35) and the facts $\lambda_1\lambda_2=1$ and $\lambda_1^3=\lambda_2^3=1$, we have

$$\begin{aligned}
w_i^{(j)} :=&\Big(\frac{x_{-j}}{x_{-j-1}}-\lambda_2-1)\lambda_1^i-(\frac{x_{-j}}{x_{-j-1}}-\lambda_1-1\Big)\lambda_2^i\\
=&\frac{(x_{-j}-(1+\lambda_2)x_{-j-1})\lambda_1^i-(x_{-j}-(1+\lambda_1)x_{-j-1})\lambda_2^i}{x_{-j-1}}\\
=&\frac{x_{-j}(\lambda_1^i-\lambda_2^i)-x_{-j-1}((1+\lambda_2)\lambda_1^i-(1+\lambda_1)\lambda_2^i)}{x_{-j-1}}\\
=&\frac{x_{-j}(\lambda_1^i-\lambda_2^i)+x_{-j-1}(\lambda_1^{i+1}-\lambda_2^{i+1})}{x_{-j-1}}\qquad (3.36)\\
=&i\sqrt{3}\frac{x_{-j}g_i+x_{-j-1}g_{i+1}}{x_{-j-1}},\qquad (3.37)
\end{aligned}$$

for $i\in\mathbb{N}_0$ and $j=\overline{0,2}$, where

$$g_n=\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2},\quad n\in\mathbb{N}_0.\qquad (3.38)$$

Using (3.35), (3.36) and (3.38), we get

$$\begin{aligned} w_{i+1}^{(j)}+w_i^{(j)} =& \frac{x_{-j}(\lambda_1^{i+1}-\lambda_2^{i+1})+x_{-j-1}(\lambda_1^{i+2}-\lambda_2^{i+2})}{x_{-j-1}} \\ &+\frac{x_{-j}(\lambda_1^{i}-\lambda_2^{i})+x_{-j-1}(\lambda_1^{i+1}-\lambda_2^{i+1})}{x_{-j-1}} \\ =&-\frac{x_{-j}(\lambda_1^{i+2}-\lambda_2^{i+2})+x_{-j-1}(\lambda_1^{i+3}-\lambda_2^{i+3})}{x_{-j-1}} \\ =&-i\sqrt{3}\frac{x_{-j}g_{i+2}+x_{-j-1}g_{i+3}}{x_{-j-1}}, \end{aligned} \tag{3.39}$$

for $i\in\mathbb{N}_0$ and $j=\overline{0,2}$.

By using (3.37) and (3.39) in (3.32)-(3.34), we get

$$x_{3m}=x_{-3}(-1)^{m+1}\prod_{i=0}^{m}\left(\frac{x_0g_{i+2}+x_{-1}g_{i+3}}{x_0g_i+x_{-1}g_{i+1}}\right)\left(\frac{x_{-1}g_{i+2}+x_{-2}g_{i+3}}{x_{-1}g_i+x_{-2}g_{i+1}}\right)\left(\frac{x_{-2}g_{i+2}+x_{-3}g_{i+3}}{x_{-2}g_i+x_{-3}g_{i+1}}\right), \tag{3.40}$$

$$x_{3m+1}=x_{-2}(-1)^{m+1}\prod_{i=0}^{m}\left(\frac{x_0g_{i+2}+x_{-1}g_{i+3}}{x_0g_i+x_{-1}g_{i+1}}\right)\left(\frac{x_{-1}g_{i+2}+x_{-2}g_{i+3}}{x_{-1}g_i+x_{-2}g_{i+1}}\right)\left(\frac{x_{-2}g_{i+3}+x_{-3}g_{i+4}}{x_{-2}g_{i+1}+x_{-3}g_{i+2}}\right), \tag{3.41}$$

$$x_{3m+2}=x_{-1}(-1)^{m+1}\prod_{i=0}^{m}\left(\frac{x_0g_{i+2}+x_{-1}g_{i+3}}{x_0g_i+x_{-1}g_{i+1}}\right)\left(\frac{x_{-1}g_{i+3}+x_{-2}g_{i+4}}{x_{-1}g_{i+1}+x_{-2}g_{i+2}}\right)\left(\frac{x_{-2}g_{i+3}+x_{-3}g_{i+4}}{x_{-2}g_{i+1}+x_{-3}g_{i+2}}\right), \tag{3.42}$$

for $m\in\mathbb{N}_{-1}$.

In this way, we explained how the formulas in Theorem 5.4 in [9] can be obtained in a natural way. Note also that we proved that the formulas also hold for $m=-1$, which was not noticed in [9].

3.1. On the statements on the behaviour of solutions to Eq.(1.2) in [9]

Now we conduct some analyses of the statements in [9] on the behavior of solutions to Eq.(1.2).

On page 482 in [9] it says that Eq.(1.2) has a unique equilibrium point $\bar{x}$ which satisfies the relation

$$\bar{x}=a\bar{x}+\frac{b\bar{x}^2}{c\bar{x}+d\bar{x}} \tag{3.43}$$

or

$$\bar{x}^2(1-a)(c+d)=b\bar{x}^2, \tag{3.44}$$

and if it is further assumed

$$(1-a)(c+d)\neq b,$$

that the unique equilibrium point is $\bar{x}=0$.

However, note that the relations in (3.43) and (3.44) are not equivalent. Namely, the right-hand side of the relation in (3.43) is not defined for $\bar{x}=0$, whereas both sides in (3.44) are. Hence, $\bar{x}=0$ cannot be an equilibrium of Eq.(1.2). For the same reason the linearized equation in [9] is not correct. Moreover, the following statement, that is, Theorem 2.1 in [9] makes no sense:

Statement 1. *Assume that*

$$b(c+3d)<(1-a)(c+d)^2.$$

Then the equilibrium point of Eq.(1.2) *is locally asymptotically stable.*

On page 483 in [9] it says that the following statement holds.

Statement 2. *The equilibrium point* $\bar{x}$ *of Eq.*(1.2) *is global attractor if* $c(1-a) \neq b$.

However, since the equilibrium is not correctly determined, the statement also makes no sense. Besides, the 'proof' given therein incorrectly applies Theorem B quoted therein.

The example which follows, demonstrates the extent of the error in the previous statement.

Example 3.5. Consider the Eq.(1.2), with the parameters a, b, c, d satisfying the conditions

$$\min\{ac+b+d, c\} > 0, \tag{3.45}$$

$$(ac+b+d)^2 > 4bd, \tag{3.46}$$

$$ac+b-d+\sqrt{(ac+b+d)^2-4bd} > 2c, \tag{3.47}$$

$$c(1-a) \neq b. \tag{3.48}$$

The associated characteristic polynomial to the corresponding linear difference equation in (2.16) is

$$p_2(\lambda) = c^2\lambda^2 - c(ac+b+d)\lambda + bd,$$

and its roots are

$$\lambda_1 = \frac{ac+b+d+\sqrt{(ac+b+d)^2-4bd}}{2c},$$

and

$$\lambda_2 = \frac{ac+b+d-\sqrt{(ac+b+d)^2-4bd}}{2c}.$$

Employing the formulas in (2.6)-(2.11), we have

$$x_{3m} = x_{-3}\prod_{i=0}^{m} y_{3i}y_{3i-1}y_{3i-2} \tag{3.49}$$

$$x_{3m+1} = x_{-2}\prod_{i=0}^{m} y_{3i+1}y_{3i}y_{3i-1}, \tag{3.50}$$

$$x_{3m+2} = x_{-1}\prod_{i=0}^{m} y_{3i+2}y_{3i+1}y_{3i}, \tag{3.51}$$

for $m \in \mathbb{N}_0$, where

$$\begin{aligned} &y_{3m}y_{3m-1}y_{3m-2} \\ &= \left(\frac{\left(\frac{x_0}{x_{-1}}-\lambda_2+\frac{d}{c}\right)\lambda_1^{m+1}-\left(\frac{x_0}{x_{-1}}-\lambda_1+\frac{d}{c}\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}}-\lambda_2+\frac{d}{c}\right)\lambda_1^{m}-\left(\frac{x_0}{x_{-1}}-\lambda_1+\frac{d}{c}\right)\lambda_2^{m}}-\frac{d}{c}\right) \\ &\times\left(\frac{\left(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{d}{c}\right)\lambda_1^{m+1}-\left(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{d}{c}\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{d}{c}\right)\lambda_1^{m}-\left(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{d}{c}\right)\lambda_2^{m}}-\frac{d}{c}\right) \\ &\times\left(\frac{\left(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{d}{c}\right)\lambda_1^{m+1}-\left(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{d}{c}\right)\lambda_2^{m+1}}{\left(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{d}{c}\right)\lambda_1^{m}-\left(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{d}{c}\right)\lambda_2^{m}}-\frac{d}{c}\right), \end{aligned} \tag{3.52}$$

$$
\begin{aligned}
&y_{3m+1}y_{3m}y_{3m-1}\\
&=\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{d}{c})\lambda_1^{m+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{d}{c})\lambda_2^{m+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{d}{c})\lambda_1^{m+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{d}{c})\lambda_2^{m+1}}-\frac{d}{c}\right)\\
&\quad\times\left(\frac{(\frac{x_{0}}{x_{-1}}-\lambda_2+\frac{d}{c})\lambda_1^{m+1}-(\frac{x_{0}}{x_{-1}}-\lambda_1+\frac{d}{c})\lambda_2^{m+1}}{(\frac{x_{0}}{x_{-1}}-\lambda_2+\frac{d}{c})\lambda_1^{m}-(\frac{x_{0}}{x_{-1}}-\lambda_1+\frac{d}{c})\lambda_2^{m}}-\frac{d}{c}\right)\\
&\quad\times\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{d}{c})\lambda_1^{m+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{d}{c})\lambda_2^{m+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{d}{c})\lambda_1^{m}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{d}{c})\lambda_2^{m}}-\frac{d}{c}\right),
\end{aligned}
\tag{3.53}
$$

$$
\begin{aligned}
&y_{3m+2}y_{3m+1}y_{3m}\\
&=\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{d}{c})\lambda_1^{m+2}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{d}{c})\lambda_2^{m+2}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{d}{c})\lambda_1^{m+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{d}{c})\lambda_2^{m+1}}-\frac{d}{c}\right)\\
&\quad\times\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{d}{c})\lambda_1^{m+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{d}{c})\lambda_2^{m+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{d}{c})\lambda_1^{m+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{d}{c})\lambda_2^{m+1}}-\frac{d}{c}\right)\\
&\quad\times\left(\frac{(\frac{x_{0}}{x_{-1}}-\lambda_2+\frac{d}{c})\lambda_1^{m+1}-(\frac{x_{0}}{x_{-1}}-\lambda_1+\frac{d}{c})\lambda_2^{m+1}}{(\frac{x_{0}}{x_{-1}}-\lambda_2+\frac{d}{c})\lambda_1^{m}-(\frac{x_{0}}{x_{-1}}-\lambda_1+\frac{d}{c})\lambda_2^{m}}-\frac{d}{c}\right),
\end{aligned}
\tag{3.54}
$$

for $m \in \mathbb{N}_0$.

Now note that conditions (3.45) and (3.46) imply

$$\lambda_1 > |\lambda_2|. \tag{3.55}$$

By using (3.47) and (3.55), we have

$$
\begin{aligned}
&\lim_{m\to+\infty}\frac{(\frac{x_{0}}{x_{-1}}-\lambda_2+\frac{d}{c})\lambda_1^{m+1}-(\frac{x_{0}}{x_{-1}}-\lambda_1+\frac{d}{c})\lambda_2^{m+1}}{(\frac{x_{0}}{x_{-1}}-\lambda_2+\frac{d}{c})\lambda_1^{m}-(\frac{x_{0}}{x_{-1}}-\lambda_1+\frac{d}{c})\lambda_2^{m}}-\frac{d}{c}\\
&=\lim_{m\to+\infty}\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{d}{c})\lambda_1^{m+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{d}{c})\lambda_2^{m+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{d}{c})\lambda_1^{m}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{d}{c})\lambda_2^{m}}-\frac{d}{c}\\
&=\lim_{m\to+\infty}\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{d}{c})\lambda_1^{m+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{d}{c})\lambda_2^{m+1}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{d}{c})\lambda_1^{m}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{d}{c})\lambda_2^{m}}-\frac{d}{c}\\
&=\lambda_1-\frac{d}{c}=\frac{ac+b-d+\sqrt{(ac+b+d)^2-4bd}}{2c}>1,
\end{aligned}
\tag{3.56}
$$

when

$$\frac{x_{-i}}{x_{-(i+1)}} \neq \lambda_2-\frac{d}{c}=\frac{ac+b-d-\sqrt{(ac+b+d)^2-4bd}}{2c}, \quad i=\overline{0,2}. \tag{3.57}$$

Let

$$\min_{i=\overline{0,3}} x_{-i} > 0$$

and (3.57) holds. Then the relations in (3.49)-(3.54) and (3.56) yield

$$\lim_{n\to+\infty} x_n = +\infty.$$

Thus, this is a counterexample to Statement 2, even if we neglect the wrong equilibrium obtained in [9].

Remark 3.6. If $\min\{a,b,c,d\}>0$, as it was the case in [9], then the condition (3.45) is automatically satisfied. Since

$$(ac+b+d)^2-4bd=(ac)^2+2ac(b+d)+(b-d)^2>0$$

we see that the condition (3.46) is also satisfied. Hence, in this case we can only assume that the conditions (3.47) and (3.48) hold.

If $bd<0$, then $(ac+b+d)^2-4bd>0$, so the condition (3.46) is satisfied. The same holds if $bd=0$ and $ac+b+d\neq 0$.

Remark 3.7. If $\max\{b,d\}<0$, then the situation is more complex. Let $t:=ac$ and

$$p_2(t):=(t+b+d)^2-4bd=t^2+2(b+d)t+(b-d)^2.$$

Then the zeros of the polynomial $p_2(t)$ are

$$t_1=-(b+d)+2\sqrt{bd} \quad \text{and} \quad t_2=-(b+d)-2\sqrt{bd}.$$

Note that

$$t_1=|b|+|d|+2\sqrt{|b||d|}=(\sqrt{|b|}+\sqrt{|d|})^2$$

and

$$t_2=|b|+|d|-2\sqrt{|b||d|}=(\sqrt{|b|}-\sqrt{|d|})^2.$$

Hence, if

$$(\sqrt{|b|}-\sqrt{|d|})^2\le ac\le(\sqrt{|b|}+\sqrt{|d|})^2$$

condition (3.46) does not hold.

Further, note that if

$$(\sqrt{|b|}-\sqrt{|d|})^2\le ac\le |b|+|d|, \tag{3.58}$$

the condition (3.45) does not hold, since $|b|+|d|=-b-d$. Since the inequality $|b|+|d|\le(\sqrt{|b|}+\sqrt{|d|})^2$ always holds, we see that if (3.58) holds, then the conditions (3.45) and (3.46) do not hold.

Remark 3.8. Paper [9] is, unfortunately, one of many recent papers with wrong results or results which trivially or very easily follow from known ones. From time to time, we have analyzed some of such papers in detail and give number of comments and theoretical explanations for wrong or problematic results of various types (see, e.g., [29, 31, 35]).

Remark 3.9. The only correct result in [9] on the behavior of solutions to Eq.(1.2), is the following results on the boundedness (see Theorem 4.1 in [9]):

Every (positive) solution of Eq.(1.2) *is bounded if*

$$a+\frac{b}{c}<1. \tag{3.59}$$

However, not only that it is a trivial consequence of the obvious inequality

$$x_{n+1}=ax_n+\frac{bx_nx_{n-2}}{cx_{n-2}+dx_{n-3}}\le x_n\Big(a+\frac{bx_{n-2}}{cx_{n-2}}\Big)=x_n\Big(a+\frac{b}{c}\Big),$$

but it was even not noticed that the claim holds under the condition $a+\frac{b}{c}\le 1$, and that from (3.59) each positive solution to Eq.(1.2) geometrically/exponentially converges to zero, since

$$0<x_{n+1}\le x_0\Big(a+\frac{b}{c}\Big)^{n+1},$$

for every $n\in\mathbb{N}_0$.

For a related result on the boundedness character of a concrete difference equation see, e.g., [27].

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